THE e-POSITIVITY OF MULTIVARIATE k-ORDER EULERIAN POLYNOMIALS

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ABSTRACT. Inspired by the recent work of Chen and Fu on the e-positivity of trivariate second-order Eulerian polynomials, we show the e-positivity of a family of multivariate k-order Eulerian polynomials. A combinatorial interpretation of the coefficients of the e-positive expansion is also established. In particular, we give a grammatical proof of the fact that the joint distribution of the ascent, descent and j-plateau statistics over k-Stirling permutations are symmetric distribution. By using Chen-Fu's transformation, a symmetric expansion of trivariate Schett polynomial is also established.

Keywords: Eulerian polynomials, e-Positivity, Plane trees, Schett polynomials

1. INTRODUCTION

Let n be a positive integer and let $\mathrm{X}_n = \{x_1, x_2, \ldots, x_n\}$ be a set of commuting variables. Define

$$S_n(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{k=0}^{n}(-1)^k e_k x^{n-k}.$$

Then $e_0 = 1$ and

$$e_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

is the *kth elementary symmetric function* associated with X_n. A function $f(x_1, x_2, \cdots) \in \mathbb{R}[x_1, x_2, \cdots]$ is said to be *symmetric* if it is invariant under any permutation of its indeterminates. We say that a symmetric function is *e-positive* if it can be written as a nonnegative linear combination of elementary symmetric functions. Recently, Chen and Fu [4] discovered the e-positivity of the trivariate second-order Eulerian polynomials [4]. As a continuation, in this paper we shall show the e-positivity of a family of multivariate k-order Eulerian polynomials.

Let $[n] = \{1, 2, \ldots, n\}$. The *Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}$ is the number of ways to partition $[n]$ into k non-empty blocks. The *second-order Eulerian polynomials* are defined by

$$\sum_{k=0}^{\infty}\left\{ {n+k \atop k} \right\} x^k = \frac{C_n(x)}{(1-x)^{2n+1}}.$$

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In [9], Gessel and Stanley found that $C_n(x)$ is the descent polynomial of Stirling permutations of order n. Below are the polynomials $C_n(x)$ for $n \leqslant 5$:

$$\begin{aligned}
C_1(x) &= x, \\
C_2(x) &= x + 2x^2, \\
C_3(x) &= x + 8x^2 + 6x^3, \\
C_4(x) &= x + 22x^2 + 58x^3 + 24x^4, \\
C_5(x) &= x + 52x^2 + 328x^3 + 444x^4 + 120x^5.
\end{aligned}$$

A *Stirling permutation* of order n is a permutation of $\{1, 1, 2, 2, \ldots, n, n\}$ such that for each i, $1 \leqslant i \leqslant n$, all entries between the two occurrences of i are larger than i. Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. In this paper, we always set $\sigma_0 = \sigma_{2n+1} = 0$. Following [1, 9], for $0 \leqslant i \leqslant 2n$, we say that an index i is a *descent* (resp. *ascent*, *plateau*) of σ if $\sigma_i > \sigma_{i+1}$ (resp. $\sigma_i < \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$). Let $\operatorname{des}(\sigma), \operatorname{asc}(\sigma)$ and $\operatorname{plat}(\sigma)$ be the number of descents, ascents and plateaus of σ, respectively. The notion a plateau introduced by Bóna was named as repetition by Dumont [7]).

The *trivariate second-order Eulerian polynomials* are defined as follows:

$$C_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\operatorname{asc}(\sigma)} y^{\operatorname{des}(\sigma)} z^{\operatorname{plat}(\sigma)}.$$

It is now well known that

$$C_{n+1}(x, y, z) = xyz \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) C_n(x, y, z),\ C_0(x, y, z) = 1. \tag{1}$$

As pointed out by Chen and Fu [4], the recursion (1) first appeared in the work of Dumont [7, p. 317], which implies that $C_n(x, y, z)$ is symmetric in the variables x, y and z. The symmetry of $C_n(x, y, z)$ was rediscovered by Janson [11, Theorem 2.1] by constructing an urn model. In [10], Haglund and Visontai introduced a refinement of the polynomial $C_n(x, y, z)$ by indexing each ascent, descent and plateau by the value where they appear. The symmetry of $C_n(x, y, z)$ is follows from the symmetry of the recursion [10, Eq. (17)].

For an alphabet A, let $\mathbb{Q}[[A]]$ be the ring of the rational commutative ring of formal power series in monomials formed from letters in A. Following Chen [2], a *context-free grammar* over A is a function $G : A \to \mathbb{Q}[[A]]$ that replaces a letter in A with an element of $\mathbb{Q}[[A]]$. The formal derivative D_G is a linear operator defined with respect to the grammar G. In other words, D_G is the unique derivation satisfying $D_G(x) = G(x)$ for $x \in A$. Recently, two methods are developed in the theory of context-free grammar, i.e., grammatical labeling and change of the grammars. A *grammatical labeling* is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar (see [3]). A *change of grammars* is a substitution method in which the original grammars are replaced with functions of other grammars. In particular, the following type of change of grammars can be used to study the γ-positivity and partial γ-positivity of several enumerative polynomials (see [13, 14]):

$$\begin{cases} u = xy, \\ v = x + y. \end{cases}$$

Let G be the following grammar

$$G = \{x \to xyz, y \to xyz, z \to xyz\}. \tag{2}$$

It has been shown by Dumont [7], Chen etal. [5], Haglund-Visontai [10] that

$$D_G^n(x) = C_n(x, y, z).$$

Very recently, Chen and Fu [4] introduced a new type of change of grammars:

$$\begin{cases} u = x + y + z, \\ v = xy + yz + zx, \\ w = xyz, \end{cases} \tag{3}$$

which stimulates the proposal of a novel approach for studying symmetric functions.

A *rooted tree* of order n with the vertices labelled $1, 2, \ldots, n$, is an increasing tree if the node labelled 1 is distinguished as the root, and for each $2 \leqslant i \leqslant n$, the labels of the nodes in the unique path from the root to the node labelled i form an increasing sequence. In this paper, the *degree* of a vertex in a rooted tree is meant to be the number of its children (sometimes called outdegree). A *plane tree* is a rooted tree in which the children of each vertex are linearly ordered (from left to right, say). A 3-*ary increasing plane tree* on $[n]$ is an increasing plane tree for which each vertex has at most three children. It follows from (2) and (3) that $D_G(u) = 3w,\ D_G(v) = 2uw,\ D_G(w) = vw$. Let

$$H = \{u \to 3w, v \to 2uw,\ w \to vw\}.$$

By using the grammar H, Chen and Fu [4] obtained the following result.

Theorem 1. *For $n \geqslant 1$, one has*

$$C_n(x, y, z) = \sum_{k \geqslant 1} (xyz)^k \sum_{j \geqslant 0} \gamma_{n,k,j} (xy + yz + zx)^j (x + y + z)^{2n+1-2j-3k},$$

where the coefficient $\gamma_{n,k,j}$ equals the number of 3-ary increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices.

Corollary 2. *For $n \geqslant 1$, one has*

$$C_n(x) = \sum_{k \geqslant 1} x^k \sum_{j \geqslant 0} \gamma_{n,k,j} (1 + 2x)^j (2 + x)^{2n+1-2j-3k}.$$

As a natural extension of (3), it is natural to introduce the following definition.

Definition 3. *Let G be the grammar defined by*

$$G = \{x_1 \to f_1(x_1, x_2, \ldots, x_n), x_2 \to f_2(x_1, x_2, \ldots, x_n), \cdots, x_n \to f_n(x_1, x_2, \ldots, x_n)\}.$$

Suppose that $f_i(x_1, x_2, \ldots, x_n)$ are symmetric function for all $1 \leqslant i \leqslant n$. The Chen-Fu's transformation of G is defined by $u_j = g_j(e_1, e_2, \ldots, e_n)$, where $1 \leqslant j \leqslant n$ and e_ℓ is the ℓth elementary symmetric function associated with $\{x_1, x_2, \ldots, x_n\}$ and $g_j(x_1, x_2, \ldots, x_n)$ is a function.

Let j^k denote k times of the letter j. A k-Stirling permutation of order n is a multiset permutation of $\{1^k, 2^k, \ldots, n^k\}$ with the property that all elements between two occurrences of i are at least i, where $i \in [n]$. Let $\mathcal{Q}_n(k)$ be the set of k-Stirling permutations of order n. Following [18, p. 657], an *k-ary tree T* is either empty, or else one specially designated vertex is called the root of T and the remaining vertices (excluding the root) are put into a (weak) ordered partition $(T_1, \ldots, T_k)$ of exactly k disjoint (possibly empty) sets $T_1, \ldots, T_k$, each of which is an k-ary tree. A bijection between $\mathcal{Q}_n(k)$ and the set of $(k+1)$-ary increasing trees was independently established by Gessel [15] and Janson-Kuba [12, Theorem 1].

In the next section, we first present the e-positive expansion of multivariate k-order Eulerian polynomials by using Chen-Fu's transformation of grammars. We then present a similar expansion of trivariate Schett polynomials.

2. Main results

In this section, we always let $\mathrm{X}_{k+1} = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$ and let e_i be the ith elementary symmetric function associated with X_{k+1}. In particular,

$$e_0 = 1,\ e_1 = x_1 + x_2 + \cdots + x_{k+1},\ e_{k+1} = x_1 x_2 \cdots x_{k+1}.$$

2.1. **Multivariate k-order Eulerian polynomials.**

Let k be a given positive integer, and let $\sigma \in \mathcal{Q}_n(k)$. The ascents, descents and plateaux of σ of are defined as before, where we always set $\sigma_0 = \sigma_{kn+1} = 0$. More precisely, an index i is called an ascent (resp. descent, plateau) of σ if $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$). It is clear that $\operatorname{asc}(\sigma) + \operatorname{des}(\sigma) + \operatorname{plat}(\sigma) = kn + 1$. As a natural refinement of ascents, descents and plateaux, Janson and Kuba [12] introduced the following definition, and related the distribution of j-ascents, j-descents and j-plateaux in k-Stirling permutations with certain parameters in $(k+1)$-ary increasing trees.

Definition 4 ([12])**.** *An index i is called a j-plateau (resp. j-descent, j-ascent) if i is a plateau (resp. descent, ascent) and there are exactly $j-1$ indices $\ell < i$ such that $a_\ell = a_i$.*

Let $\operatorname{plat}_j(\sigma)$ be the number of j-plateaux of σ. For $\sigma \in \mathcal{Q}_n(k)$, it is clear that $\operatorname{plat}_j(\sigma) \leqslant k-1$.

Example 5. *Consider the 4-Stirling permutation $\sigma = 111223333221$. The set of 1-plateaux is given by $\{1, 4, 6\}$, the set of 2-plateaux is given by $\{2, 7\}$, and the set of 3-plateaux is given by $\{8, 10\}$. Thus $\operatorname{plat}_1(\sigma) = 3$ and $\operatorname{plat}_2(\sigma) = \operatorname{plat}_3(\sigma) = 2$.*

The *multivariate k-order Eulerian polynomials* $C_n(x_1, \ldots, x_{k+1})$ are defined by

$$C_n(x_1, x_2, \ldots, x_{k+1}) = \sum_{\sigma \in \mathcal{Q}_n(k)} {x_1}^{\operatorname{plat}_1(\sigma)} {x_2}^{\operatorname{plat}_2(\sigma)} \cdots {x_{k-1}}^{\operatorname{plat}_{k-1}(\sigma)} {x_k}^{\operatorname{des}(\sigma)} {x_{k+1}}^{\operatorname{asc}(\sigma)}.$$

In particular, when $x_1 = z,\ x_2 = \cdots = x_{k-1} = 0,\ x_k = y$ and $x_{k+1} = x$, the polynomials $C_n(x_1, x_2, \ldots, x_{k+1})$ reduce to $C_n(x, y, z)$. The following lemma is fundamental.

Lemma 6. *Let G_1 be the grammar defined by*

$$G_1 = \{x_1 \to e_{k+1},\ x_2 \to e_{k+1}, \ldots,\ x_{k+1} \to e_{k+1}\},$$

where $e_{k+1} = x_1 x_2 \cdots x_{k+1}$. For $n \geqslant 1$, one has $D^n_{G_1}(x_1) = C_n(x_1, x_2, \ldots, x_{k+1})$.

Proof. We shall show that the grammar G_1 can be used to generate k-Stirling permutations. We first introduce a grammatical labeling of $\sigma\in\mathcal{Q}_n(k)$ as follows:

(L_1) If i is an ascent, then put a superscript label x_{k+1} right after σ_i;
(L_2) If i is a descent, then put a superscript label x_k right after σ_i;
(L_3) If i is a j-plateau, then put a superscript label x_j right after σ_i.

The weight of σ is defined as the product of the labels, that is

$$w(\sigma)={x_1}^{\mathrm{plat}_1(\sigma)}{x_2}^{\mathrm{plat}_2(\sigma)}\cdots {x_{k-1}}^{\mathrm{plat}_{k-1}(\sigma)}{x_k}^{\mathrm{des}(\sigma)}{x_{k+1}}^{\mathrm{asc}(\sigma)}.$$

Recall that we always set $\sigma_0=\sigma_{kn+1}=0$. Thus the index 0 is always an ascent and the index kn is always a descent. For $n=1$, we have $\mathcal{Q}_1(k)=\{{}^{x_{k+1}}1^{x_1}1^{x_2}1^{x_3}\cdots 1^{x_k}\}$. The are $k+1$ elements in $\mathcal{Q}_2(k)$ and they can be labeled as follows, respectively:

$$^{x_{k+1}}1^{x_1}1^{x_2}\cdots 1^{x_{k-1}}1^{x_{k+1}}2^{x_1}2^{x_2}\cdots 2^{x_{k-1}}2^{x_k},$$

$$^{x_{k+1}}1^{x_1}1^{x_2}\cdots 1^{x_{k-2}}1^{x_{k+1}}2^{x_1}2^{x_2}\cdots 2^{x_{k-1}}2^{x_k}1^{x_k},$$

$$\cdots$$

$$^{x_{k+1}}2^{x_1}2^{x_2}\cdots 2^{x_{k-1}}2^{x_k}1^{x_1}1^{x_2}\cdots 1^{x_{k-1}}1^{x_k}.$$

Note that $D_{G_1}(x_1)=e_{k+1}$ and $D_{G_2}^2(x_1)=e_ke_{k+1}$. Then the weight of the element in $\mathcal{Q}_1(k)$ is given by $D_{G_1}(x_1)$, and the sum of weights of the elements in $\mathcal{Q}_2(k)$ is given by $D_{G_1}^2(x)$. Hence the result holds for $n=1,2$. We proceed by induction on n. Suppose we get all labeled permutations in $\mathcal{Q}_{n-1}(k)$, where $n\geqslant 3$. Let σ' be obtained from $\sigma\in\mathcal{Q}_{n-1}(k)$ by inserting the string $nn\cdots n$ with length k. Then the changes of labeling are illustrated as follows:

$$\cdots\sigma_i^{x_j}\sigma_{i+1}\cdots\mapsto\cdots\sigma_i^{x_{k+1}}n^{x_1}n^{x_2}\cdots n^{x_k}\sigma_{i+1}\cdots;$$

$$\sigma^{x_k}\mapsto\sigma^{x_{k+1}}n^{x_1}n^{x_2}\cdots n^{x_k};\ {}^{x_{k+1}}\sigma\mapsto^{x_{k+1}}n^{x_1}n^{x_2}\cdots n^{x_k}\sigma.$$

In each case, the insertion of the string $nn\cdots n$ corresponds to one substitution rule in G_1. Then the action of D_{G_1} on the set of weights of elements in $\mathcal{Q}_{n-1}(k)$ gives the set of weights of all elements in $\mathcal{Q}_n(k)$. Therefore, we get a grammatical interpretation of $C_n(x_1,x_2,\ldots,x_{k+1})$, and this completes the proof. □

From the symmetry of the grammar G_1 and the fact that $D_{G_1}(x_1)=e_{k+1}$, one can immediately get the following result, which has been obtained by Janson and Kuba [12, Theorem 2, Theorem 8] by using an urn model for the exterior leaves.

Corollary 7. *The multivariate k-order Eulerian polynomial $C_n(x_1,x_2,\ldots,x_{k+1})$ is symmetric in the variables $x_1,\ x_2,\ldots,\ x_{k+1}$.*

A $(k+1)$-ary increasing plane tree on $[n]$ is an increasing plane tree for which each vertex has degree at most $k+1$. We can now conclude the following result.

Theorem 8. *For $n\geqslant 2$ and $k\geqslant n-2$, we have*

$$C_n(x_1,x_2,\ldots,x_{k+1})=\sum\gamma(n;i_1,i_2,\ldots,i_n)e_{k-n+2}^{i_n}e_{k-n+3}^{i_{n-1}}\cdots e_k^{i_2}e_{k+1}^{i_1}, \tag{4}$$

where the summation is over all sequences $(i_1,i_2,\ldots,i_n)$ of nonnegative integers such that $i_1+i_2+\cdots+i_n=n$, $1\leqslant i_1\leqslant n-1$, $i_n=0$ or $i_n=1$. When $i_n=1$, one has $i_1=n-1$. The *number $\gamma(n; i_1, i_2, \ldots, i_n)$ equals the number of $(k+1)$-ary increasing plane trees with i_j degree $j-1$ vertices for all $1 \leqslant j \leqslant n$.*

Proof. Let G_1 be the grammar given in Lemma 6. We first consider a change of the grammar G_1. Note that

$$D_{G_1}(x_1) = e_{k+1},\ D_{G_1}(e_i) = (k-i+2)e_{i-1}e_{k+1} \text{ for } 1 \leqslant i \leqslant k+1.$$

Let G_2 be the grammar defined by

$$G_2 = \{x_1 \to e_{k+1},\ e_i \to (k-i+2)e_{i-1}e_{k+1} \text{ for } 1 \leqslant i \leqslant k+1\}, \tag{5}$$

Note that

$$\begin{aligned}
&D_{G_2}(x_1) = e_{k+1},\ D^2_{G_2}(x_1) = e_k e_{k+1},\ D^3_{G_2}(x_1) = e_k^2 e_{k+1} + 2e_{k-1}e_{k+1}^2,\\
&D^4_{G_2}(x_1) = e_k^3 e_{k+1} + 8e_{k-1}e_k e_{k+1}^2 + 6e_{k-2}e_{k+1}^3,\\
&D^5_{G_2}(x_1) = e_k^4 e_{k+1} + 22e_k^2 e_{k-1}e_{k+1}^2 + 16e_{k-1}^2 e_{k+1}^3 + 42e_{k-2}e_k e_{k+1}^3 + 24e_{k-3}e_{k+1}^4.
\end{aligned}$$

In general, we assume that

$$D^n_{G_2}(x_1) = \sum \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1}. \tag{6}$$

Note that

$$\begin{aligned}
&D^{n+1}_{G_2}(x_1)\\
&= D_{G_2}\left(\sum \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1}\right)\\
&= \sum n i_n \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+1} e_{k-n+2}^{i_n-1} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1+1} +\\
&\sum (n-1) i_{n-1} \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n+1} e_{k-n+3}^{i_{n-1}-1} \cdots e_k^{i_2} e_{k+1}^{i_1+1} + \cdots +\\
&\sum 2 i_2 \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_{k-1}^{i_3+1} e_k^{i_2-1} e_{k+1}^{i_1+1} +\\
&\sum i_1 \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2+1} e_{k+1}^{i_1}.
\end{aligned}$$

Then the expansion (6) holds for $n+1$. Combining Lemma 6 and (6), we get (4). By induction, one can easily deduce that in the right hand side of (6), one has $i_1+i_2+\cdots+i_n = n$, $1 \leqslant i_1 \leqslant n-1$, $i_n = 1$ or $i_n = 0$.

Along the same lines as the proof of [4, Theorem 4.1], by using (5), we can now deduce the combinatorial interpretation of $\gamma(n; i_1, i_2, \ldots, i_n)$. Let T be a $(k+1)$-ary increasing plane tree on $[n]$. The labeling of T is given by labeling a degree i vertex by e_{k-i+1} for all $0 \leqslant i \leqslant k+1$. In particular, label a leaf by e_{k+1} and label a degree $k+1$ vertex by 1. Let T' be a $(k+1)$-ary increasing plane tree on $[n+1]$ by adding $n+1$ to T as a leaf. We can add $n+1$ to T only as a child of a vertex v that is not of degree $k+1$. For $1 \leqslant i \leqslant k+1$, if the vertex v is a degree $k-i+1$ vertex with label e_i, there are $k-i+2$ cases to attach $n+1$ (from left to right, say). In either case, in T', the vertex v becomes a degree $k-i+2$ with label e_{i-1} and $n+1$ becomes a leaf with label e_{k+1}. Hence the insertion of $n+1$ corresponds to the substitution rule $e_i \to (k-i+2)e_{i-1}e_{k+1}$. Therefore, $D_{G_2}(x_1)$ equals the sum of the weights of $(k+1)$-ary increasing plane trees on $[n]$. This completes the proof. □

By using $D_{G_2}^{n+1}(x_1)=D_{G_2}\left(D_{G_2}^{n}(x_1)\right)$, it is routine to verify that

$$\begin{aligned}
&\gamma(n+1;1,n,0\ldots,0)=\gamma(n;1,n-1,0,\ldots,0)=1,\\
&\gamma(n+1;n,0,\ldots,0,1)=n\gamma(n;n-1,0,\ldots,0,1)=n!,\\
&\gamma(n+1;i_1,i_2,\ldots,i_n,0)=i_1\gamma(n;i_1,i_2-1,i_3,\ldots,i_n)+\\
&\sum_{j=2}^{n-1}j(i_j+1)\gamma(n;i_1-1,i_2,\ldots,i_{j-1},i_j+1,i_{j+1}-1,i_{j+2}\ldots,i_n).
\end{aligned}$$

Note that $\gamma(3;2,0,1,0,\ldots,0)=2,\ \gamma(4;2,1,1,0,\ldots,0)=8$ and

$$\gamma(n+1;2,n-2,1,0,\ldots,0)=2\gamma(n;2,n-3,1,0,\ldots,0)+2(n-1)\gamma(n;1,n-1,0,\ldots,0).$$

Then by induction, one can derive that

$$\gamma(n;2,n-3,1,0,\ldots,0)=2^n-2n \text{ for } n\geqslant 3.$$

Let $C_n(x)=\sum_{j=1}^{n}C(n,j)x^j$. Following [9], the *second-order Eulerian numbers* $C(n,j)$ satisfy the recurrence relation

$$C(n+1,j)=jC(n,j)+(2n+2-j)C(n,j-1),$$

with the initial conditions $C(1,1)=1$ and $C(1,j)=0$ if $j\neq 1$. Based on empirical evidence, we propose the following.

Conjecture 9. *When $n\geqslant 2$ and $k\geqslant n-2$, the second-order Eulerian number $C(n,j)$ equals the the number of $(k+1)$-ary increasing plane trees with j leaves. In other words, one has*

$$C(n,j)=\sum_{i_1+i_2+\cdots+i_{n-1}=n-j}\gamma(n;j,i_2,\ldots,i_{n-1},i_n).$$

Let G_3 be the grammar defined by

$$G_3=\{I\rightarrow qIe_{k+1},\ e_i\rightarrow(k-i+2)e_{i-1}e_{k+1} \text{ for } 1\leqslant i\leqslant k+1\},$$

where q is a constant. Below are the $D_{G_3}^n(I)$ for $n\leqslant 4$:

$$\begin{aligned}
&D_{G_3}(I)=qIe_{k+1},\ D_{G_3}^2(I)=qIe_ke_{k+1}+q^2Ie_{k+1}^2,\\
&D_{G_3}^3(I)=qI(e_k^2e_{k+1}+2e_{k-1}e_{k+1}^2)+3q^2Ie_ke_{k+1}^2+q^3Ie_{k+1}^3.
\end{aligned}$$

For $n\geqslant 1$, we define

$$D_{G_3}^n(I)=I\sum_{i=0}^{n-1}q^{n-i}f_{n,i}. \tag{7}$$

It is evident that $f_{n,i}$ is a function of $e_{k-i+1},e_{k-i+2},\ldots,e_{k+1}$. Thus we can write $D_{G_3}^n(I)$ as follows:

$$D_{G_3}^n(I)=I\sum_{i=0}^{n-1}q^{n-i}f_{n,i}(e_{k-i+1},e_{k-i+2},\ldots,e_{k+1}).$$

In particular, $f_{n,0}=1,\ f_{1,0}=f_{2,1}=1$. By using the Leibniz rule, we get

$$D_{G_3}^{n+1}(I)=q\sum_{k=0}^{n}\binom{n}{k}D_{G_3}^k(I)D_{G_3}^{n-k}(e_{k+1}).$$

Recall that an unordered forest of increasing plane trees is an unordered forest of plane trees with the same type of labeling. By using the same labeling of $(k+1)$-ary increasing plane tree that given in the proof of Theorem 8, it is routine to verify that $f_{n,i}(e_{k-i+1}, e_{k-i+2}, \ldots, e_{k+1})$ is the generating polynomial for unordered forests of increasing plane trees on the vertex set $[n]$ with $n-i$ trees, in which each degree $k-i+1$ vertex with label e_i. In particular, for $n \geqslant 1$, one has

$$f_{n,n-1}(e_{k-n+2}, e_{k-n+3}, \ldots, e_{k+1}) = D_{G_2}^{n}(x_1).$$

It should be noted that a variation of $\sum_{i=0}^{n-1} q^{n-i} f_{n,i}$ has been introduced by Pétréolle and Sokal [16] in the name of generic Lah polynomials.

2.2. **Trivariate Schett polynomials.**

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A value $i \in [n]$ is called a *cycle peak* of π if $\pi^{-1}(i) < i > \pi(i)$. Let $\mathrm{cpk}^{\mathrm{o}}(\pi)$ and $\mathrm{cpk}^{\mathrm{e}}(\pi)$ denote the number of odd and even cycle peaks of π, respectively.

Let D_J be the derivative operator, acting on three commuting variables $\{x, y, z\}$, that is given by

$$\mathrm{D}_J = yz\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}.$$

Following an approach due to Schett for computing the Taylor expansion coefficients of Jacobian elliptic functions [17], Dumont [6] gave a connection between the coefficients of the expansion of $\mathrm{D}_J^n(x)$ and cycle peak statistics. Let $\{s_{n,i,j}\}_{i,j\geqslant 0}$ be the arrays defined by

$$\mathrm{D}_J^{2n}(x) = \sum_{i,j\geqslant 0} s_{2n,i,j} x^{2i+1} y^{2j} z^{2n-2i-2j},$$

$$\mathrm{D}_J^{2n+1}(x) = \sum_{i,j\geqslant 0} s_{2n+1,i,j} x^{2i} y^{2j+1} z^{2n-2i-2j+1}.$$

Dumont [6] found that

$$s_{n,i,j} = \#\{\pi \in \mathfrak{S}_n \mid \mathrm{cpk}^{\mathrm{o}}(\pi) = i, \mathrm{cpk}^{\mathrm{e}}(\pi) = j\}.$$

The reader is referred to [14] for more details on this topic.

Following [8, Section 4], the *trivariate Schett polynomials* $S_n := S_n(x,y,z)$ are defined by the following induction relation:

$$S_1 = 2xyz,\ S_n = yz\frac{\partial}{\partial x}S_{n-1} + xz\frac{\partial}{\partial y}S_{n-1} + xy\frac{\partial}{\partial z}S_{n-1} \text{ for } n \geqslant 2.$$

Let

$$S(u;x,y,z) = \sum_{n=1}^{\infty} S_n(x,y,z)\frac{u^n}{n!}.$$

Dumont [8, Eq. (4.7)] found that $S := S(u;x,y,z)$ is the solution of the differential equation:

$$\frac{\mathrm{d}S}{\mathrm{d}u} = 2\sqrt{(x^2+S)(y^2+S)(z^2+S)},\ S(0;x,y,z) = 0.$$

As pointed out by Dumont [8], the polynomials $S_n(x,y,z)$ can be used to compute the square of the Jacobian elliptic function $\mathrm{sn}(u,k)$.

In the following, we shall present a symmetric expansion of the Schett polynomials. Let G_4 be the grammar defined by

$$G_4 = \{x \rightarrow yz,\ y \rightarrow xz,\ z \rightarrow xy\}.$$

Then we have

$$S_{n+1}(x,y,z) = D_{G_4}^n(2xyz) = 2D_{G_4}^n(xyz).$$

Let $u = x^2 + y^2 + z^2,\ v = x^2y^2 + y^2z^2 + x^2z^2$ and $w = xyz$. Then

$$D_{G_4}(u) = 6w,\ D_{G_4}(w) = v,\ D_{G_4}(v) = 4uw.$$

Define

$$G_5 = \{w \rightarrow v,\ v \rightarrow 4uw,\ u \rightarrow 6w\}.$$

Note that

$$S_{n+1}(x,y,z) = 2D_{G_5}^n(w).$$

By induction, it is routine to deduce the following result.

Theorem 10. *For $n \geqslant 1$, there exist nonnegative integers $\xi_{n,k,j}$ such that*

$$S_n(x,y,z) = 2\sum_{k\geqslant 0}(xyz)^k \sum_{j\geqslant 0}\xi_{n,k,j}(x^2y^2 + y^2z^2 + z^2x^2)^j(x^2+y^2+z^2)^{\lfloor \frac{n+2-4j-3k}{2} \rfloor}.$$

It should be noted that $\xi_{n,k,j}$ equals the number of weighted 3-ary increasing plane trees on $[n]$. We leave the precise definition to the interested readers, because it can be performed completely similarly as in [4].

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