Increasing Binary Trees and the (α,β)-Eulerian Polynomials

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Abstract

In light of the grammar given by Ji for the (α,β)-Eulerian polynomials introduced by Carlitz and Scoville, we provide a labeling scheme for increasing binary trees. In this setting, we obtain a combinatorial interpretation of the γ-coefficients of the α-Eulerian polynomials in terms of forests of planted 0-1-2-plane trees, which specializes to a combinatorial interpretation of the γ-coefficients of the derangement polynomials in the same vein. By means of a decomposition of an increasing binary tree into a forest, we find combinatorial interpretations of the sums involving two identities of Ji, one of which can be viewed as (α,β)-extensions of the formulas of Petersen and Stembridge.

Keywords: Context-free grammars, grammatical labelings, increasing binary trees, (α,β)-Eulerian polynomials, γ-positivity.

AMS Classification: 05A15, 05A19

1 Introduction

The objective of this paper is to explore a labeling scheme for increasing binary trees as an alternative combinatorial interpretation of the (α,β)-Eulerian polynomials introduced by Carlitz and Scoville [2], which can be regarded as an extension of the bivariate Eulerian polynomials. A grammatical treatment of these polynomials has been given by Ji, where a labeling scheme for permutations is presented. Employing the grammatical calculus, Ji obtained (α,β)-extensions of the formulas of Petersen and Stembridge.

We begin with a combinatorial setting of the (α,β)-Eulerian polynomials in terms of increasing binary trees. Based on an equivalent definition of Ji relying on the number of left-to-right minima and the number of right-to-left minima of a permutation, we identify two special leaves of an increasing binary tree, called the a-leaf and the b-leaf. Then we move on to define the α-vertices and the β-vertices, and add the α-labels and the β-labels to certain internal vertices, while adopting the (x,y)-labeling for the leaves, as given in [3] for the bivariate Eulerian polynomials.

Observe that the two special leaves (the a-leaf and the b-leaf) can be considered as two poles to stretch a binary tree aligned on a horizontal line, which is reminiscent of the decomposition of a doubly rooted tree into a linear order of rooted trees in Joyal's proof of Cayley's formula [11]. More precisely, with these two special vertices at disposal, an increasing binary tree can be decomposed into a forest of planted increasing binary trees. Such a decomposition gives rise to a combinatorial interpretation of the γ-coefficients of the α-Eulerian polynomials in terms of forests of planted 0-1-2-plane trees. An interpretation in the permutation setting has been given by Ji-Lin [10] by devising a group action.

The idea of the labeling scheme for the (α,β)-Eulerian polynomials can be adapted to a grammar of Dumont related to the derangement polynomials. In this setting, we are led to a combinatorial interpretation of the γ-coefficients of the derangement polynomials and the q-derangement polynomials (with respect to the number of cycles), in terms of forests of planted increasing 0-1-2-plane trees, where the exponents of q are connected with the number components of a forest. This topic has been extensively studied, see, for example, [12]–[14], [16]–[18].

The grammatical labelings of increasing binary trees make it possible to give combinatorial interpretations of the sums involving the identities of Ji. We first realize that the number of interior peaks of a permutation can be read off from a labeling of increasing binary trees. For the rest, the decomposition of an increasing binary tree is the key ingredient all along.

2 The (α,β)-Eulerian polynomials

For $n \geq 1$, let $[n] = \{1,2,\ldots,n\}$. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ of $[n]$, we patch a zero both at the beginning and at the end, that is, $\sigma_0 = \sigma_{n+1} = 0$. An index i $(0 \leq i \leq n-1)$ is called an ascent if $\sigma_i < \sigma_{i+1}$. An index i $(1 \leq i \leq n)$ is called a descent if $\sigma_i > \sigma_{i+1}$. Let $\mathrm{asc}(\sigma)$ and des(σ) denote the number of ascents and the number of descents of σ, respectively.

Carlitz and Scoville [2] introduced an extension of the bivariate Eulerian polynomials, denoted by $A_n(x,y|\alpha,\beta)$, which are called the (α,β)-Eulerian polynomials by Ji [9]. They are defined by

$$A_n(x,y|\alpha,\beta) = \sum_{\sigma\in S_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} \alpha^{\mathrm{lrmax}(\sigma)-1} \beta^{\mathrm{rlmax}(\sigma)-1}, \tag{2.1}$$

where S_{n+1} is the set of permutations of $[n+1]$, lrmax(σ) and rlmax(σ) denote the number of left-to-right maxima and the number of right-to-left maxima of σ, respectively.

By taking complementation of a permutation and exchanging the roles of x and y, Ji [9] presented an equivalent definition

$$A_n(x,y|\alpha,\beta) = \sum_{\sigma\in S_{n+1}} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1}, \tag{2.2}$$

where lrmin(σ) and rlmin(σ) denote the number of left-to-right minima and the number of right-to-left minima of σ, respectively. The initial values of $A_n(x,y|\alpha,\beta)$ are given below,

$$\begin{aligned} A_0(x,y|\alpha,\beta) &= 1,\\ A_1(x,y|\alpha,\beta) &= x\beta + y\alpha,\\ A_2(x,y|\alpha,\beta) &= xy\alpha + xy\beta + 2xy\alpha\beta + x^2\beta^2 + y^2\alpha^2. \end{aligned}$$

Ji [9] found a context-free grammar for the (α,β)-Eulerian polynomials, which can be paraphrased as

$$G = \{a \to \alpha a y,\ b \to \beta b x,\ x \to xy,\ y \to xy\}. \tag{2.3}$$

By providing a grammatical labeling for permutations, it has been shown that the (α,β)-Eulerian polynomials can be generated by the above grammar.

Theorem 2.1 (Ji)**.** *Let D denote the formal derivative of the grammar G. For $n \geq 0$, we have*

$$D^n(ab) = ab A_n(x,y|\alpha,\beta). \tag{2.4}$$

As is well-known, permutations are in one-to-one correspondence with increasing binary trees, or complete increasing binary trees, when the external leaves are used to so that very internal vertex has exactly two children. This correspondence enables us to define the (α,β)-Eulerian polynomials in terms of increasing binary trees.

2.1 The (a,b,α,β)-labeling

Here is a labeling scheme for increasing binary trees, called the (a,b,α,β)-labeling.

Let $n \geq 1$, and let T be an increasing binary tree on $[n]$, where $n \geq 1$. Consider the left child of the root. If it is a leaf, we call it the leftmost leaf of T. If not, we restrict to the left subtree of T and continue to seek the leftmost leaf. Eventually, we end up with the leftmost leaf of T. The rightmost leaf is defined in the same way. Now we label leftmost leaf of T by a and label the rightmost leaf of T by b.

Next, the α-vertices and the β-vertices are defined as follows. Each vertex on the path from the root to the a-leaf (other than the root and the leftmost leaf) is labeled by α, which we call an a-vertex. Each vertex on the path from the root to the b-leaf (other than the root and the rightmost leaf) is labeled by β, which we call a β-vertex. The rest of the leaves are labeled like the usual (x, y)-labeling, that is, a left leaf is labeled by x and a right leaf is labeled by y. It can be readily seen that a pair of sibling leaves labeled by x and y correspond to an interior peak of a permutation. For example, Figure 1 demonstrates an (a, b, α, β)-labeling of an increasing binary tree on [9], where the corresponding permutation reads

$$8\,4\,9\,6\,1\,2\,5\,3\,7.$$

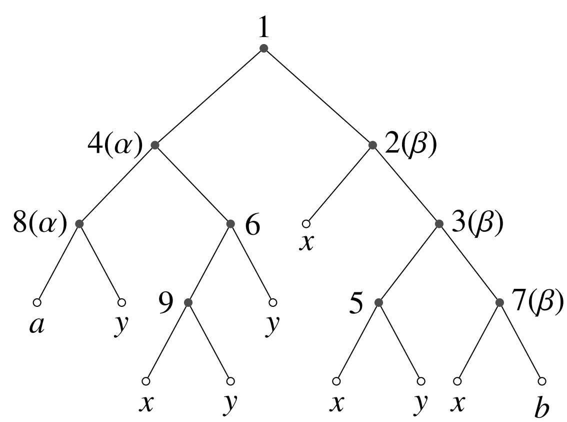

Figure 1: An example for the (a, b, α, β)-labeling.

The following theorem shows that the (α,β)-Eulerian polynomials have a combinatorial interpretation in terms of increasing binary trees. For an increasing binary tree T, we use $w(T)$ denote the weight of T with respect to the (a, b, α, β)-labeling, that is, the product of the grammatical labels. For instance, the weight of the increasing binary in Figure 1 equals $abx^4y^4\alpha^2\beta^3$.

In view of the correspondence between permutations and increasing binary trees, we see that the α-vertices together with the root are the left-to-right minima of the corresponding permutation, whereas the β-vertices together with the root are the right-to-left minima of the corresponding permutation. Recall that the (x, y)-labeling of an increasing binary tree for the bivariate Eulerian polynomials given in [3] can be described as follows. A leaf is called an x-leaf if it is a left child, and it is labeled by x, whereas a y-leaf is referred to a leaf that is a right child, and is labeled by y. Observe that the (x,y)-labels are exactly in agreement with the labeling, provided that a is set to x and b is set to y. Thus we arrive at the following assertion.

Theorem 2.2. *For $n \geq 1$, we have*

$$abA_n(x,y|\alpha,\beta) = \sum_T w(T), \tag{2.5}$$

where the sum ranges over the set of increasing binary trees on $[n+1]$ with the (a,b,α,β)-labeling.

2.2 A decomposition

The (a,b,α,β)-labeling leads us to consider a decomposition of an increasing binary tree into a forest of planted increasing binary trees, which is conducive to divide the set of increasing binary trees into classes relative to the labeling scheme. In the usual sense, by a planted increasing binary tree we mean an increasing tree structure consisting of a single root or a root with an increasing binary tree as a subtree.

We define the (a,b)-decomposition of an increasing binary tree T on $[n] = \{1,2,\ldots,n\}$ to be a forest of increasing binary trees, which are the planted increasing binary trees rooted at the α-vertices and the β-vertices. The resulting forest is called the supporting forest of T. For an increasing binary tree T on $[n]$, its supporting forest is on the set $[2,n] = \{2,3,\ldots,n\}$.

If we arrange the components of a supporting forest in the increasing order of their roots, then Figure 2 is an exhibition of the supporting forest of the increasing tree in Figure 1.

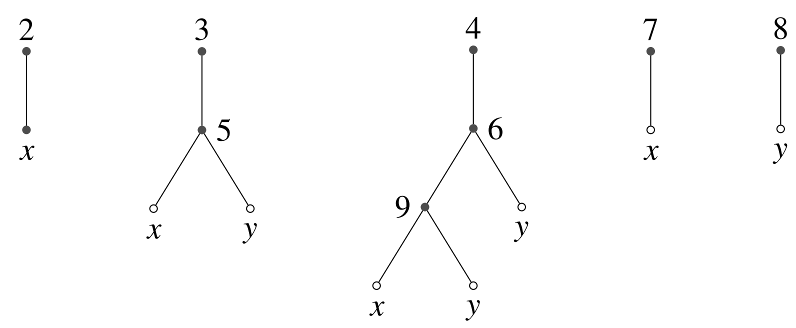

Figure 2: A supporting forest on $[2,n]$ with inherited labels.

The structure of a supporting forest can be employed to divide the set of increasing binary trees on $[2,n]$ into classes whose total weight can be readily characterized. To this end, we define the weight of a supporting forest by the following rules. First, we suppress the leaf of a single root.

1. A single root is assigned the weight $x\beta + y\alpha$.

2. A root with a child has weight $\alpha + \beta$.

3. Any leaf has the weight (or label) inherited from the original increasing binary tree.

The updated labeling of a supporting forest is illustrated in Figure 3.

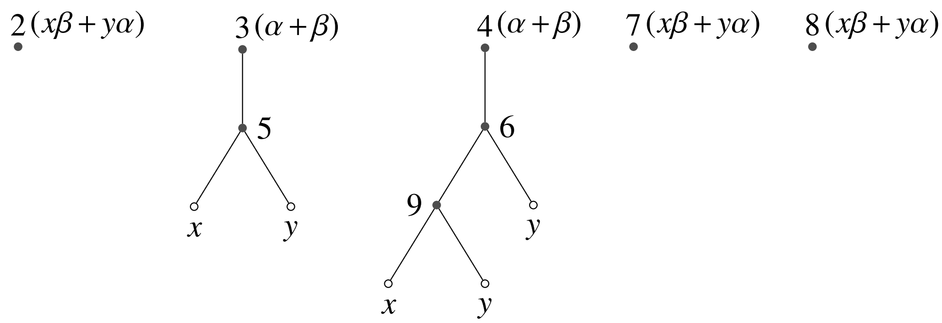

Figure 3: A supporting forest with updated labels.

Since the root of a component of a supporting forest can be either an α-vertex or a β-vertex, we are led to the following expansion, where the underlying set of the supporting forests has been rescaled down to $[n]$.

Theorem 2.3. *For $n \geq 0$, $A_n(x,y|\alpha,\beta)$ equals the total weight of supporting forests on* $[n]$.

Now we further classify supporting forests via a group action. We say that two supporting forests are in the same class if one can be obtained by swapping a leaf with its non-leaf sibling. Therefore, such a class of supporting forests can be represented by a forest of planted 0-1-2 plane trees (without external leaves), bearing the following labeling rules:

1. A single root is endowed with a weight $x\beta + y\alpha$.

2. A root with a child has weight $\alpha + \beta$.

3. A degree one non-root vertex (a nonroot vertex with exactly one child) has weight $x + y$.

4. A leaf has weight xy.

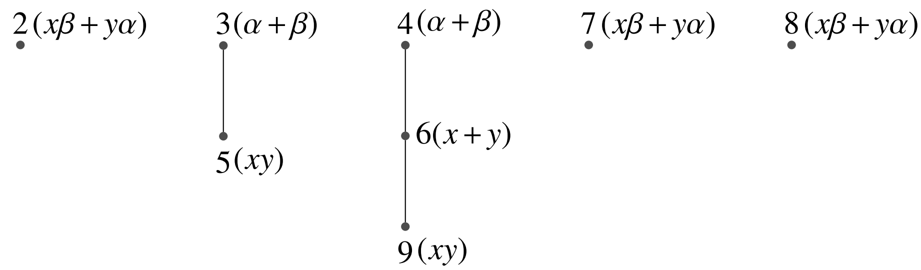

Figure 4: A forest of planted 0-1-2-plane trees.

Figure 4 is an illustration of a forest of planted 0-1-2-plane trees. The above classification implies the following expansion of the (α,β)-Eulerian polynomials.

Theorem 2.4. *For $n \geq 0$, $A_n(x,y|\alpha,\beta)$ equals the total weight of forests of planted 0-1-2-plane trees on $[n]$.*

2.3 The α-Eulerian polynomials

When $\alpha=\beta$, the (α,β)-Eulerian polynomials are called the α-Eulerian polynomials in [9], denoted by $A_n(x,y|\alpha)$. The labeling scheme of the corresponding trees is so called (a,b,α)-labeling. That is, all b-vertices are labeled by α as well. By a transformation of grammars, it is easy to see that these polynomials are γ-positive. Ji and Lin [10] provided a combinatorial proof of the γ-coefficients by via a group action on permutations. With the help of the (a,b,α)-labeling, we obtain an alternative combinatorial interpretation of the γ-coefficients in terms of forests of planted 0-1-2-plane trees.

Setting $\alpha=\beta$, the previous weight assignment reduces to the following rules for the α-Eulerian polynomials. For a forest F of planted increasing 0-1-2-plane trees on $[n+1]$, we have the following rules:

1. A single root has weight $\alpha(x+y)$.
2. Other roots have weight 2α.
3. If a non-root vertex has only one child, it has weight $x+y$.
4. A leaf has weight xy.

Theorem 2.5. *For $n \geq 1$, the α-Eulerian polynomial $A_n(x,y|\alpha)$ has the γ-expansion*

$$\sum_F w(F),$$

where the sum ranges over forest F of planted 0-1-2-plane trees on $[n]$.

2.4 The derangement polynomials

As a special case of the γ-expansion of the α-Eulerian polynomials, we come to the γ-expansion of the derangement polynomials.

Given a permutation σ in the cycle notation, for an index $1 \le i \le n$, we call it an excedance if $\sigma(i) > i$, or a drop if $\sigma(i) < i$, or a fixed point if $\sigma(i) = i$. Denote by $\mathrm{exc}(\sigma)$, $\mathrm{drop}(\sigma)$ and $\mathrm{cyc}(\sigma)$ the number of excedances, the number of drops and the number of cycles of σ, respectively. Let D_n be the set of permutations without fixed points. Then the derangement polynomials are defined by

$$d_n(x) = \sum_{\sigma \in D_n} x^{\mathrm{exc}(\sigma)},$$

see [1].

We can rely on the structure of forests of planted 0-1-2-plane trees to give a combinatorial interpretation of the γ-coefficients of the derangement polynomials, along with the q-analogue with respect to the number of cycles. Let

$$d_n(x,y,q) = \sum_{\sigma \in D_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} q^{\mathrm{cyc}(\sigma)}.$$

Notice that a permutation without fixed points corresponds to a complete increasing binary tree without β-vertices whose left child is a leaf. A planted increasing binary tree is said to be fully planted if the root has a child that is not a leaf. By relabeling the root 1 by β and setting $\alpha = 1$ in the (a, b, α, β)-labeling, an increasing binary tree can be decomposed into a forest of fully planted increasing binary trees for which the root of each tree is labeled by β. Then we take group action on a fully planted increasing 0-1-2-plane tree as follows. We label the root of each component by q. If a non-root vertex has degree one, then label it by $x + y$. A leaf is labeled by xy. Then the weight of a forest F of fully planted increasing 0-1-2-plane trees is defined to be the product of all the grammatical labels of F, denoted by $w(F)$. Then we get the following γ-expansion.

Theorem 2.6. *For $n \ge 1$, we have*

$$d_n(x,y,q) = \sum_F w(F), \tag{2.6}$$

where F ranges over forests of fully planted increasing 0-1-2-plane trees on $[n]$.

To conclude this section, we note that as for a γ-expansion, we mean an expansion in $x + y$ and xy.

3 A labeling scheme for interior peaks

In this section, we give two labeling schemes of increasing binary trees in connection with interior peaks of a permutation, and we obtain expansion theorems for summations of Ji [9] related to the (α,β)-Eulerian polynomials. As a result, we find combinatorial proofs of two identities of Ji.

Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, we patch a zero at the beginning and at the end of the permutation. We call an index i a left peak, if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ with $1 \leq i \leq n-1$. We call an index i an interior peak, if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ while $2 \leq i \leq n-1$. We follow the notation $M(\sigma)$ in [4] for the number of interior peaks of σ.

If we use a and b to label the left-most leaf and the rightmost leaf, respectively, and obey the rules for the (x,y)-labeling for the rest of the leaves, then we get a labeling for the interior peaks, and we call it the (a,b,x,y,M)-labeling. In other words, a pair of sibling leaves labeled by x and y corresponds to an interior peak of the permutation. A leaf whose sibling is not a leaf is labeled by x (y, resp.) corresponding to an ascent (descent, resp.) of the permutation. Observe that the labels a and b play the role of preventing the first position and the last position being counted as interior peaks. For $n = 2$, below are the two increasing binary trees with the (a,x,y,M)-labeling:

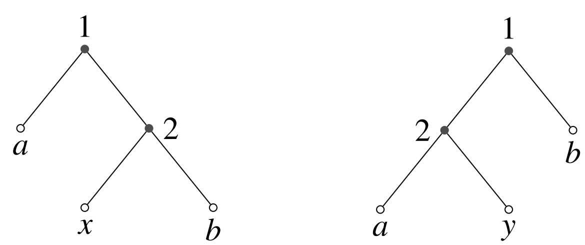

Figure 5: The (a,b,x,y,M)-labeling.

Besides the above labeling, we also need the (a,x,y,z)-labeling of increasing binary trees with the z-leaves corresponding to fixed points of permutations, see [5]:

1. If a β-vertex has a left child, then this child is labeled by z, signifying a fixed point.
2. The rest of the leaves are labeled in the same manner as for the (α,β)-Eulerian polynomials.

As will be seen, the label z signifies a fixed point of a permutation. Thus a permutation without fixed points corresponds to an increasing binary tree without z-leaves.

With the aid of the above labeling schemes, we are in a position to give combinatorial expansions of summations of Ji involving identities related to the (α,β)-Eulerian polynomials. First, let us consider the (α,β)-extension of Stembridge's identity [9, Theorem 1.9].

Theorem 3.1 (Ji)**.** *For $n \geq 1$,*

$$\sum_{\sigma\in S_n} (xy)^{M(\sigma)} \left(\frac{x+y}{2}\right)^{n-2M(\sigma)-1} \alpha^{\text{lrmin}(\sigma)-1}\beta^{\text{rlmin}(\sigma)-1}$$
$$= \sum_{\sigma\in S_n} x^{\text{des}(\sigma)} y^{n-\text{des}(\sigma)-1} \left(\frac{\alpha+\beta}{2}\right)^{\text{lrmin}(\sigma)+\text{rlmin}(\sigma)-2}. \tag{3.1}$$

The case for $n = 1$ is trivial, so we assume that $n \geq 2$. To provide a combinatorial interpretation of the above relation, we shall give expansions of both sides in terms of forests of plane 0-1-2 plane trees, and will show that these two expansions are equinumerous, that is, these come to the same total weights.

To reformulate the above relation in terms of trees, let $\mathcal{B}_n$ denote the set of increasing binary trees on $[n]$. Given $T \in \mathcal{B}_n$ endowed with the (a,b,x,y,M)-labeling, let $M(T)$ denote the number of vertices of T having two leaf children. We adopt the notations xleaf and yleaf as used in [3] for the number of x-leaves and the number of y-leaves. Moreover, we write $N_\alpha(T)$ and $N_\beta(T)$ for the number of α-vertices and the number of β-vertices of T, respectively.

The relation 3.1 can be split into two parts. As for the left side, we have the following relation, where the set of forests of planted 0-1-2-plane trees on $[2,n]$ is denoted by $\mathcal{P}_n$.

Theorem 3.2. *For $n \geq 1$, we have*

$$\sum_{T\in\mathcal{B}_n} (xy)^{M(T)} \left(\frac{x+y}{2}\right)^{n-2M(T)-1} \alpha^{N_\alpha(T)}\beta^{N_\beta(T)} = \sum_{F\in\mathcal{P}_n} w(F), \tag{3.2}$$

where the sum ranges over the set of forests of planted 0-1-2-plane trees on $[2,n]$ with the following labeling rules and $w(F)$ stands for the weight of F:

1. *A single root is labeled by $(x+y)\frac{\alpha+\beta}{2}$.*
2. *The root of a component with at least two internal vertices is labeled by $\alpha+\beta$.*
3. *A degree one vertex is labeled by $x+y$.*
4. *A pair of proper leaves are labeled by x and y like the (x,y)-labeling.*

Proof. We begin with representing the sum on the left side over permutations in terms of a sum over increasing binary trees. Let $\mathcal{B}_n$ be the set of increasing binary trees on $[n]$, and let $T \in \mathcal{B}_n$. We say that a leaf is proper if it is neither an a-leaf nor a b-leaf. In view of the (a,b,x,y,M)-labeling, $M(\sigma)$ corresponds to the number of internal vertices having two proper leaf children, whereas $n - M(\sigma) - 1$ equals the number of internal vertices having exactly one proper leaf child. Consequently, we are supposed to label T by the following rules, which we call the modified (a,b,x,y,M)-labeling.

1. An internal vertex on the path from the root to the a-leaf (other than the root) is labeled by α. An internal vertex on the path from the root to the b-leaf (other than the root) is labeled by β.

2. Label the leftmost leaf by a and label the rightmost leaf by b.

3. For a pair of proper sibling leaves, we label the left leaf by x and the right leaf by y.

4. For a leaf whose sibling is not a proper leaf, we label it by $(x+y)/2$, no matter whether it is on the left or on the right.

For example, for the tree in Figure 1, the modified (a,b,x,y,M)-labeling is demonstrated in Figure 6.

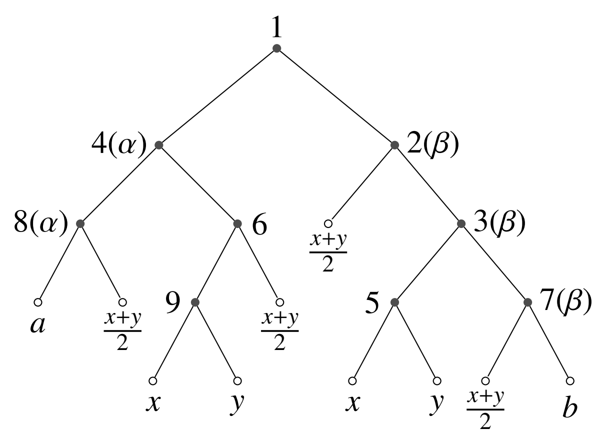

Figure 6: The modified (a,b,x,y,M)-labeling.

We now process to compute the sum of weights over $\mathcal{B}_n$ by utilizing supporting forests. The power of 2 suggests that we should restore a tree structure from a supporting forest. Let F be a supporting forest, that is, a forest of planted increasing binary trees on $[n]$. Let us characterize the set of trees T in $\mathcal{B}_n$ with supporting forest F on $[2,n]$. There are two choices for a planted increasing binary tree in F to belong to the left side (with the root being an α-vertex) or the right side (with the root being a β-vertex).

For a single root, it may originate from an α-vertex in T or a β-vertex in T. These two cases lead to the sum of weights

$$\frac{x+y}{2}\alpha + \frac{x+y}{2}\beta = (x+y)\frac{\alpha+\beta}{2}.$$

For a component of F containing at least two internal vertices, its root may originate from an α-vertex or a β-vertex, so the sum of weights equals $\alpha+\beta$.

Moreover, we can take a group action by swapping a proper leaf with its sibling that is not a leaf. Keep in mind that the a-leaf and the b no longer appear in F. Let orb(F) denote the orbit of F under this group action. Then let us compute the sum of weights of T with a supporting forest in orb(F). This quantity can be derived from a labeling of a representative of orb(F), that is a forest P of planted 0-1-2-plane trees.

Note that a proper left leaf with weight $(x+y)/2$ is paired with a proper right leaf with weight $(x+y)/2$, summing to a weight $x+y$. The above considerations suggest that we should comply with the injunction as stated in the theorem. This completes the proof. ∎

Let us now turn to the sum on the right side of (3.1). A modification of the (a,b,α,β)-labeling is needed. In this case, both the α-vertices and the β-vertices are labeled by $(\alpha+\beta)/2$. For example, Figure 7 gives the modified labeling for the two trees in $\mathcal{B}_2$.

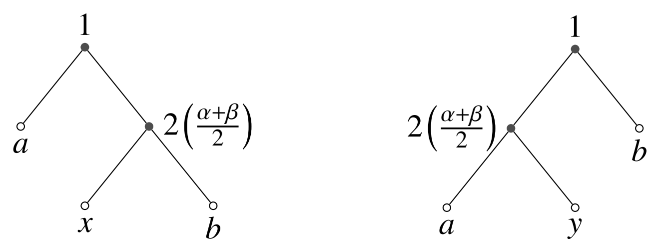

Figure 7: The modified (a,b,α,β)-labeling.

At this point, the sum on the right of (3.1) can be recast in terms of the modified labeling for increasing binary trees in $\mathcal{B}_n$, and we are left with the task to establish the following relation. Bear in mind that xleaf(T) and yleaf(T) refer to the number of x-leaves and the number of y-leaves with respect to the modified labeling for which the leftmost leaf and the rightmost leaf are labeled by a and b rather than x and y.

Theorem 3.3. *For $n \geq 1$,*

$$\sum_{T\in\mathcal{B}_n} x^{\mathrm{xleaf}(T)} y^{\mathrm{yleaf}(T)} \left(\frac{\alpha+\beta}{2}\right)^{N_\alpha(T)+N_\beta(T)} = \sum_{F\in\mathcal{P}_n} w(F), \tag{3.3}$$

where the sum ranges over $\mathcal{P}_n$ as in Theorem 3.2 and ditto the weight.

Proof. As before, we first consider the supporting forest of a tree in $\mathcal{B}_n$, and consider which trees in $\mathcal{B}_n$ share the same supporting forest F. Let T be an increasing binary tree in $\mathcal{B}_n$ with the supporting forest F.

For a single root in F, it may originate from an α-vertex with a right leaf child labeled by y, or a β-vertex with a left leaf child labeled by x. Given that all the α-vertices and β-vertices are labeled by $(\alpha+\beta)/2$. Hence the two cases contribute a total weight of $(x+y)\frac{\alpha+\beta}{2}$, in accordance with the labeling of F.

For a component of F containing at least two internal vertices, its root may originate from an α-vertex or a β-vertex. Thus we get a total weight of

$$\frac{\alpha+\beta}{2}+\frac{\alpha+\beta}{2}=\alpha+\beta,$$

which coincides with the label of the root for F.

For other leaves of T, we consider the group action that swaps a proper leaf with its sibling that is an internal vertex. More specifically, a proper x-leaf is paired with a proper y-leaf, giving a total weight of $x+y$. This group action gives rise to an orbit of F, which can be represented by a forest of planted 0-1-2-plane trees with weights defined by the labeling scheme described in the theorem. This completes the proof. ∎

We finish with a combinatorial proof of the following identity due to Ji, where D_n denotes the set of derangements of $[n]$.

Theorem 3.4 (Ji). *For $n \geq 1$,*

$$\sum_{\sigma\in S_{n+1}}(-1)^{\mathrm{des}(\sigma)}\left(\frac{1}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2}=\sum_{\sigma\in D_n}(-1)^{\mathrm{exc}(\sigma)}. \tag{3.4}$$

Proof. Let T be a tree in $\mathcal{B}_{n+1}$ on $\{0,1,\ldots,n\}$ with the (a,b,x,y,M)-labeling. Let F be the supporting forest of T. Consider the set of trees that share the same supporting forest as T. First, we observe a cancellation property. Note that F is a forest of planted increasing binary trees on $[n]$. Assume that T is endowed with the (x,y)-labeling for the Eulerian polynomials, that is, a left leaf is labeled by x and a right leaf is labeled by y. In the end, we set $x=-1$ and $y=1$.

We claim that a cancellation occurs when F contains a single root. If F contains a single root, then T has either an α-vertex with a y-leaf or a β-vertex with an x-leaf. These two possibilities create a pair of trees with the same supporting forest and opposite signs, here the sign of T is determined by the parity of the number of x-leaves. Moreover, such a pair of trees possess the same quantity

$$\mathrm{lrmin}(T)+\mathrm{rlmin}(T)-2,$$

and hence we are led to a cancellation in the sum on the left of (3.4), which implies that the sum can be reduced to T whose supporting forests are fully planted.

Observe that the (x, y)-labels of T are carried over to the forest F. This means that if two trees have the same supporting forest (without single roots), then they must have the same sign. Now we proceed to compute the left side of (3.4) by classifying the supporting forests. Clearly, a supporting forest of k components generates 2^k trees in $\mathcal{B}_{n+1}$.

On the other hand, a supporting forest F can be viewed as an increasing binary tree on $[n]$ by gluing the component together. Finally, it remains to make use of the fact that the labels carried over are precisely the same as the labels for the derangement polynomials with respect to the (a, x, y, z)-labeling, except the rightmost y-leaf. A detailed description of the (a, x, y, z)-labeling can be found in [5]. Thus we may consider the x-labels representing the number of excedances and the y-labels representing the number of drops. Finally, a special attention has to be paid to the leftmost y-leaf of T in $\mathcal{B}_{n+1}$. Since y is set to 1 at last, we should have no worries. This completes the proof. ∎

To conclude, we remark that the above combinatorial argument yields a refinement of (3.4) by restricting the sum to

$$\mathrm{lrmin}(\sigma) + \mathrm{rlmin}(\sigma) - 2 = k.$$

Then the sum of the right side can be restricted to derangements with k cycles.

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