On the Bilateral Series ${}_2\psi_2$

Vincent Y. B. Chen[1], William Y. C. Chen[2] and Nancy S. S. Gu[3]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071
People's Republic of China

Email: [1]ybchen@mail.nankai.edu.cn, [2]chen@nankai.edu.cn, [3]gu@nankai.edu.cn

Abstract

We obtain a formula which reduces the evaluation of a ${}_2\psi_2$ series to two ${}_2\phi_1$ series. In some sense, this identity may be considered as a companion of Slater's formulas. We also find that a two-term ${}_2\psi_2$ summation formula due to Slater can be derived from a unilateral summation formula of Andrews by bilateral extension and parameter augmentation.

Keywords: basic hypergeometric series, bilateral series, bilateral extension, parameter augmentation, q-Gauss summation.

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1 Introduction

It is well known that many bilateral basic hypergeometric identities can be derived from unilateral identities. Using Cauchy's method [5, 15, 20, 21] one may obtain bilateral basic hypergeometric identities from terminating unilateral identities. Starting with nonterminating unilateral basic hypergeometric series, Chen and Fu [8] developed a method to construct semi-finite forms by shifting the summation index by m. Then the bilateral summations are consequences of the semi-finite forms by letting m tend to infinity. We call this method bilateral extension. In this paper we use bilateral extensions of a ${}_3\phi_2$ series and an identity of Andrews [2] to study the bilateral series ${}_2\psi_2$:

$$ {}_2\psi_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; q, z\right]. \tag{1.1} $$

The above ${}_2\psi_2$ series is closely related to the question of finding a q-extension of Dougall's bilateral hypergeometric series summation formula [10]:

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} = \frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, \tag{1.2}$$

where $\mathrm{Re}(c+d-a-b-1)>0$, $(a)_k = a(a+1)\cdots(a+k-1)$, $k=1,2,\cdots$, $(a)_0 = 1$ and $(a)_k = (-1)^k/(1-a)_{-k}$ when k is a negative integer.

Bailey [6] first suggested that there did not exist any q-extension of (1.2). Since (1.2) is an extension of the Gauss ${}_2F_1$ summation formula, one naturally expects that a q-analogue of (1.2) should be concerned with the following series:

$${}_2\psi_2 \left[\begin{matrix} a, & b \\ c, & d \end{matrix} ; q, \frac{cd}{abq} \right]. \tag{1.3}$$

Clearly, when c or d equals q, (1.3) reduces to the q-Gauss sum [13, Appendix II.8]:

$${}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} ; q, \frac{c}{ab} \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \qquad |c/ab| < 1. \tag{1.4}$$

Even for the above series (1.3), Gasper [12] pointed out that one could not use analytic continuation to derive an infinite product representation.

On the other hand, many results on the bilateral ${}_2\psi_2$ series (1.1) have been obtained. In [6], Bailey found several transformation formulas for the ${}_2\psi_2$ series (1.1). Later, Slater obtained a general transformation formula for an ${}_r\psi_r$ series in [23] based on Sears' transformation on the ${}_{r+s+1}\phi_{r+s}$ series in [22] subject to suitable substitutions and the following relation

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=-\infty}^{-1} f(-n-1) \tag{1.5}$$

to combine two unilateral series to form a bilateral series.

Gasper and Rahman [13] have shown that based on Slater's transformation formula, one could obtain two expansions of an ${}_r\psi_r$ series in terms of r ${}_r\phi_{r-1}$ series [13, Eq. (5.4.4), (5.4.5)]. When $r=2$, they become

$$\begin{aligned} {}_2\psi_2 \left[\begin{matrix} a, & b \\ c, & d \end{matrix} ; q, z \right] &= \frac{a(q, qa/b, c/a, d/a, az, q/az, qb, 1/b; q)_\infty}{(a/b, qb/a, c, d, q/a, q/b, z, q/z; q)_\infty} \\ &\times {}_2\phi_1 \left[\begin{matrix} qa/c, & qa/d \\ & qa/b \end{matrix} ; q, \frac{cd}{abz} \right] + \mathrm{idem}(a; b) \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} {}_2\psi_2 \left[\begin{matrix} a, & b \\ c, & d \end{matrix} ; q, z \right] &= \frac{q}{c} \frac{(q, c/a, c/b, abz/dq, dq^2/abz, q/d; q)_\infty}{(c, c/d, q/a, q/b, abz/cd, qcd/abz; q)_\infty} \\ &\times {}_2\phi_1 \left[\begin{matrix} qa/c, & qb/c \\ & qd/c \end{matrix} ; q, z \right] + \mathrm{idem}(c; d), \end{aligned} \tag{1.7}$$

where the symbol "idem$(a;b)$" after an expression means that the preceding expression is repeated with a and b interchanged.

Setting $d=q$, (1.6) reduces to a three-term transformation formula [13, Appendix III.32] for the ${}_2\phi_1$ series:

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix};q,z\right]=\frac{(b,c/a,az,q/az;q)_\infty}{(c,b/a,z,q/z;q)_\infty}{}_2\phi_1\left[\begin{matrix} a, & aq/c \\ & aq/b \end{matrix};q,\frac{cq}{abz}\right]+\mathrm{idem}(a;b). \quad (1.8) $$

However, it should be noted that when c or d equals q, (1.7) does not lead to any nontrivial identity.

The first result of this paper is to give a new formula for the ${}_2\psi_2$ series (1.1) in terms of two ${}_2\phi_1$ series which is different from Slater's formulas (1.6) and (1.7). It reduces to a different three-term transformation formula (2.4) when $c=q$ compared with the three-term transformation formula (1.8) deduced by Slater's transformation. Moreover, this identity may be considered as a companion of Slater's formulas (1.6) and (1.7). Note that Slater's formulas do not seem to imply the special cases that can be deduced from our formula except for Ramanujan's ${}_1\psi_1$ summation formula [13, Appendix II.29]. As a consequence, our formula yields a two-term closed product form for the ${}_2\psi_2$ series:

$$ \begin{aligned} {}_2\psi_2\left[\begin{matrix} b, & c \\ aq/b, & aq/c \end{matrix};q,-\frac{aq}{bc}\right] &= \frac{(-b,aq/bc,-q/b,b/a,q;q)_\infty(aq^2/c^2;q^2)_\infty}{(aq/c,-1,q/c,q/b,-aq/bc;q)_\infty(b^2/a;q^2)_\infty} \\ &\quad+\frac{(aq/bc,b,-aq/b,-b/a,q;q)_\infty(aq^2/c^2;q^2)_\infty}{(aq/b,aq/c,-1,-aq/bc,q/c;q)_\infty(b^2/a;q^2)_\infty}. \end{aligned} \quad (1.9) $$

For comparison, we recall the known formula for the well-poised ${}_2\psi_2$ series [13, Appendix II.30]:

$$ {}_2\psi_2\left[\begin{matrix} b, & c \\ aq/b, & aq/c \end{matrix};q,-\frac{aq}{bc}\right]=\frac{(aq/bc;q)_\infty(aq^2/b^2,aq^2/c^2,q^2,aq,q/a;q^2)_\infty}{(aq/b,aq/c,q/b,q/c,-aq/bc;q)_\infty}. \quad (1.10) $$

Let us turn our attention back to Dougall's formula. As pointed out by Askey [4], Bailey seemed to have been partly right concerning his opinion towards the q-extension of Dougall's formula. According to Askey [4], in certain sense the following q-extension of Cauchy's beta integral was similar to a q-extension of Dougall's formula:

$$ \int_{-\infty}^{\infty}\frac{(ct,-dt;q)_\infty}{(at,-bt;q)_\infty}\mathrm{d}_qt=2\frac{(1-q)(c/a,d/b,-c/b,-d/a,ab,q/ab;q)_\infty(q^2;q^2)_\infty^2}{(cd/abq,q;q)_\infty(a^2,q^2/a^2,b^2,q^2/b^2;q^2)_\infty}. \quad (1.11) $$

In fact, this integral can be recast as a two-term summation formula for the ${}_2\psi_2$ series (1.3):

$$ \begin{aligned} &\frac{(c,-d;q)_\infty}{(a,-b;q)_\infty}{}_2\psi_2\left[\begin{matrix} a, & -b \\ c, & -d \end{matrix};q,q\right]+\frac{(-c,d;q)_\infty}{(-a,b;q)_\infty}{}_2\psi_2\left[\begin{matrix} -a, & b \\ -c, & d \end{matrix};q,q\right] \\ &\quad=2\frac{(1-q)(c/a,d/b,-c/b,-d/a,ab,q/ab;q)_\infty(q^2;q^2)_\infty^2}{(cd/abq,q;q)_\infty(a^2,q^2/a^2,b^2,q^2/b^2;q^2)_\infty}. \end{aligned} \quad (1.12) $$

As observed by Ismail and Rahman [14], the above two-term summation formula is a special of a transformation formula due to Slater [23]. When $r = 2$, by substitutions and the q-Gauss sum (1.4), Slater's general transformation on the ${}_r\psi_r$ series reduces to the following two-term summation formula:

$$\frac{(c/ef, qef/c, q, q/a, q/b, c/a, c/b; q)_\infty}{(e, f, q/e, q/f, c/ab; q)_\infty} = \frac{q}{e}\frac{(c/qf, q^2f/c, e/a, e/b, qc/e, q^2/e; q)_\infty}{(e, q/e, e/f, qf/e; q)_\infty}$$
$$\times {}_2\psi_2\left[\begin{matrix} e/c, & e/q \\ e/a, & e/b \end{matrix}; q, q\right] + \mathrm{idem}(e; f). \tag{1.13}$$

The second result of this paper is concerned with the above two-term summation formula (1.13) for ${}_2\psi_2$. Andrews [2] established a three-term transformation formula which is the key to proving many of Ramanujan's identities for partial θ-functions. In view of the symmetry in this formula, he obtained a generalization of Ramanujan's ${}_1\psi_1$ summation:

$$d\sum_{n=0}^{\infty}\frac{(q/bc, acdf; q)_n}{(ad, df; q)_{n+1}}(bd)^n - c\sum_{n=0}^{\infty}\frac{(q/bd, acdf; q)_n}{(ac, cf; q)_{n+1}}(bc)^n$$
$$= d\frac{(q, qd/c, c/d, abcd, acdf, bcdf; q)_\infty}{(ac, ad, bc, bd, cf, df; q)_\infty}, \qquad |bc|, |bd| < 1. \tag{1.14}$$

Using the approach of parameter augmentation developed by Chen and Liu [9], we find that the two-term summation formula (1.13) for ${}_2\psi_2$ series is a consequence of the above identity (1.14) of Andrews by bilateral extension and parameter augmentation.

As is customary, we employ the notation and terminology of basic hypergeometric series in [13]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k) \text{ and } (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \text{ for } n \in \mathbb{Z}.$$

For convenience, we shall adopt the following notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n,$$

where n is an integer or infinity. In particular, for a nonnegative integer k, we have

$$(a; q)_{-k} = \frac{1}{(aq^{-k}; q)_k}. \tag{1.15}$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}; q, z\right] = \sum_{k=0}^{\infty}\frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k, \tag{1.16}$$

while the bilateral basic hypergeometric series ${}_r\psi_s$ is defined by

$${}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}; q, z\right] = \sum_{k=-\infty}^{\infty}\frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r} z^k. \tag{1.17}$$

2 An Expansion Formula for the ${}_2\psi_2$ Series

In this section, we derive a representation for the ${}_2\psi_2$ series (1.1) in terms of two ${}_2\phi_1$ series. This formula can be considered as a companion of Slater's formulas (1.6) and (1.7). We also present some consequences including a two-term infinite product representation for the sum of a well-poised ${}_2\psi_2$ series (1.9).

Theorem 2.1 *We have*

$$
\begin{aligned}
{}_2\psi_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; q, z\right] &= \frac{(c/b, abz/d, dq/abz, q/d, q; q)_\infty}{(c, az/d, q/a, q/b, cd/abz; q)_\infty}\, {}_2\phi_1\left[\begin{matrix} cd/abz, & d/a \\ & dq/az \end{matrix}; q, \frac{bq}{d}\right] \\
&\quad - \frac{(cq/d, b, d/a, az/q, q^2/az, q/d, q; q)_\infty}{(d/q, c, bq/d, az/d, dq/az, q^2/d, q/a; q)_\infty}\, {}_2\phi_1\left[\begin{matrix} aq/d, & bq/d \\ & cq/d \end{matrix}; q, z\right], \quad (2.1)
\end{aligned}
$$

where $|cd/ab| < |z| < 1$ *and* $|bq/d| < 1$.

Proof. We start with a three-term transformation of ${}_3\phi_2$ series [13, Appendix III.33]:

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}; q, \frac{de}{abc}\right] &= \frac{(e/b, e/c, cq/a, q/d; q)_\infty}{(e, cq/d, q/a, e/bc; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} c, & d/a, & cq/e \\ & cq/a, & bcq/e \end{matrix}; q, \frac{bq}{d}\right] \\
&\quad - \frac{(q/d, eq/d, b, c, d/a, de/bcq, bcq^2/de; q)_\infty}{(d/q, e, bq/d, cq/d, q/a, e/bc, bcq/e; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} aq/d, & bq/d, & cq/d \\ & q^2/d, & eq/d \end{matrix}; q, \frac{de}{abc}\right],
\end{aligned}
$$

where $|bq/d|, |de/abc| < 1$.

Shifting the index of summation on the left hand side of the above identity by m such that the new sum runs from $-m$ to infinity, and then replacing a, b, d, e by aq^{-m}, bq^{-m}, dq^{-m}, eq^{-m}, respectively, we get

$$
\begin{aligned}
\sum_{k=-m}^{\infty} \frac{(a, b, cq^m; q)_k}{(q^{m+1}, d, e; q)_k}\left(\frac{de}{abc}\right)^k &= \frac{(cq/e, q/d, q; q)_m}{(c, q/a, q/b; q)_m}\frac{(e/b, e/c, cq^{1+m}/a, q^{1+m}/d; q)_\infty}{(e, cq^{1+m}/d, q^{1+m}/a, e/bc; q)_\infty} \\
&\quad \times {}_3\phi_2\left[\begin{matrix} c, & d/a, & cq^{1+m}/e \\ & cq^{1+m}/a, & bcq/e \end{matrix}; q, \frac{bq}{d}\right] - \frac{(bcq^2/de, q/d, q; q)_m}{(q^2/d, q/a, c; q)_m}\frac{(q^{1+m}/d, eq/d, b; q)_\infty}{(d/q, e, bq/d; q)_\infty} \\
&\quad \times \frac{(c, d/a, de/bcq, bcq^{2+m}/de; q)_\infty}{(cq^{1+m}/d, q^{1+m}/a, e/bc, bcq/e; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} aq/d, & bq/d, & cq^{1+m}/d \\ & q^{2+m}/d, & eq/d \end{matrix}; q, \frac{de}{abc}\right], \quad (2.2)
\end{aligned}
$$

where $|bq/d|, |de/abc| < 1$.

Setting $m \to \infty$ in (2.2) and assuming $|c| < 1$, Tannery's theorem [7] enables us to interchange the limit and the summation. This gives

$$
\begin{aligned}
{}_2\psi_2\left[\begin{matrix} a, & b \\ d, & e \end{matrix}; q, \frac{de}{abc}\right] &= \frac{(cq/e, q/d, q, e/b, e/c; q)_\infty}{(c, q/a, q/b, e, e/bc; q)_\infty}\, {}_2\phi_1\left[\begin{matrix} c, & d/a \\ & bcq/e \end{matrix}; q, \frac{bq}{d}\right] \\
&\quad - \frac{(bcq^2/de, q/d, q, eq/d, b, d/a, de/bcq; q)_\infty}{(q^2/d, q/a, d/q, e, bq/d, e/bc, bcq/e; q)_\infty}\, {}_2\phi_1\left[\begin{matrix} aq/d, & bq/d \\ & eq/d \end{matrix}; q, \frac{de}{abc}\right], \quad (2.3)
\end{aligned}
$$

where $|bq/d|, |c|, |de/abc| < 1$.

By the substitutions $c \to de/abz$ and $e \to c$ in (2.3), we get the desired formula. ∎

Note that Theorem 2.1 may be considered as a bilateral extension of the following three-term transformation formula [13, Appendix III.31]

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & d \end{matrix}; q, z\right] = \frac{(abz/d, q/d; q)_\infty}{(az/d, q/a; q)_\infty} {}_2\phi_1\left[\begin{matrix} d/a, & dq/abz \\ & dq/az \end{matrix}; q, \frac{bq}{d}\right]
$$

$$
- \frac{(b, d/a, az/q, q^2/az, q/d; q)_\infty}{(d/q, bq/d, az/d, dq/az, q/a; q)_\infty} {}_2\phi_1\left[\begin{matrix} aq/d, & bq/d \\ & q^2/d \end{matrix}; q, z\right], \quad (2.4)
$$

where $|bq/d|, |z| < 1$. It is clear that (2.4) is a special case of (2.1) for $c = q$.

Since Slater's formula (1.7) and our formula (2.1) deal with the same series, we are naturally led to an identity on ${}_2\phi_1$ series. The right hand sides of (1.7) and (2.1) give rise to the following identity by replacing a, b, c, z by d/b, dz/q, adz/c, bq/c, respectively,

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right] = \frac{(abz/c, q/c; q)_\infty}{(az/c, q/a; q)_\infty} {}_2\phi_1\left[\begin{matrix} cq/abz, & c/a \\ & cq/az \end{matrix}; q, \frac{bq}{c}\right]
$$

$$
+ \left(\frac{q(1-a)(b, q/z, d/aq, aq^2/d, cq/adz, adz/c, q/c; q)_\infty}{d(d, c/az, 1/a, aq/c, dz/c, cq/dz, q/d; q)_\infty} \right.
$$

$$
\left. + \frac{(azq/c, dz/q, b, d/c, cq/d, q^2/dz, a; q)_\infty}{(d/q, z, c, q^2/d, aq/c, dz/c, cq/dz; q)_\infty} \right) {}_2\phi_1\left[\begin{matrix} q/b, & z \\ & azq/c \end{matrix}; q, \frac{bq}{c}\right]. \quad (2.5)
$$

It is worth noting that the parameter d occurs only in the factors of the second term on the right hand side of (2.5). Hence the sum of the two products in the parentheses does not depend on d. This fact does not seem to be obvious by direct verification. Setting $d = aq$, it follows that

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right] = \frac{(abz/c, q/c; q)_\infty}{(az/c, q/a; q)_\infty} {}_2\phi_1\left[\begin{matrix} cq/abz, & c/a \\ & cq/az \end{matrix}; q, \frac{bq}{c}\right]
$$

$$
+ \frac{(az, b, c/a, q/az; q)_\infty}{(z, c, q/a, c/az; q)_\infty} {}_2\phi_1\left[\begin{matrix} q/b, & z \\ & azq/c \end{matrix}; q, \frac{bq}{c}\right]. \quad (2.6)
$$

From Heine's transformation [13, Appendix III.1]

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}; q, b\right], \quad (2.7)
$$

it is easily seen that (2.6) is equivalent to (2.4) by the substitution $c \to d$.

Corollary 2.2 *We have*

$$
{}_2\psi_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; q, z\right] = \frac{(abz/d, c/b, dq/abz, q/d, q; q)_\infty}{(c, az/d, q/a, q/b, cd/abz; q)_\infty} {}_2\phi_1\left[\begin{matrix} cd/abz, & d/a \\ & dq/az \end{matrix}; q, \frac{bq}{d}\right]
$$

$$
+ \frac{(d/a, b, az, q/az, q; q)_\infty}{(d, c, d/az, z, q/a; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & azq/d \end{matrix}; q, \frac{bq}{d}\right], \quad (2.8)
$$

where $|cd/ab| < |z| < 1$ *and* $|bq/d| < 1$.

Proof. By Heine's transformation (2.7), the second term on the right hand side of (2.1) equals

$$\begin{aligned}
&-\frac{(cq/d, b, d/a, az/q, q^2/az, q/d, q; q)_\infty}{(d/q, c, bq/d, az/d, dq/az, q^2/d, q/a; q)_\infty} {}_2\phi_1\left[\begin{matrix} aq/d, & bq/d \\ & cq/d \end{matrix}; q, z\right] \\
&= -\frac{(b, d/a, az/q, q^2/az, q/d, q, azq/d; q)_\infty}{(d/q, c, az/d, dq/az, q^2/d, q/a, z; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & azq/d \end{matrix}; q, \frac{bq}{d}\right] \\
&= -\frac{(d/a, b, az, q/az, q; q)_\infty}{(d, c, d/az, z, q/a; q)_\infty} \frac{(1-az/q)(1-q/d)(1-d/az)}{(1-d/q)(1-az/d)(1-q/az)} \\
&\quad \times {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & azq/d \end{matrix}; q, \frac{bq}{d}\right] \\
&= \frac{(d/a, b, az, q/az, q; q)_\infty}{(d, c, d/az, z, q/a; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & azq/d \end{matrix}; q, \frac{bq}{d}\right]. \qquad (2.9)
\end{aligned}$$

∎

Remark 2.3 Corollary 2.2 can also be obtained from the following three-term transformation formula [13, Appendix III.34]

$$\begin{aligned}
{}_3\phi_2\left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}; q, \frac{de}{abc}\right] &= \frac{(e/b, e/c; q)_\infty}{(e, e/bc; q)_\infty} {}_3\phi_2\left[\begin{matrix} d/a, & b, & c \\ & d, & bcq/e \end{matrix}; q, q\right] \\
&+ \frac{(d/a, b, c, de/bc; q)_\infty}{(d, e, bc/e, de/abc; q)_\infty} {}_3\phi_2\left[\begin{matrix} e/b, & e/c, & de/abc \\ & de/bc, & eq/bc \end{matrix}; q, q\right].
\end{aligned}$$

Shifting the summation index by m *on the left hand side and replacing* a, c, d, e *by* aq^{-m}, cq^{-m}, dq^{-m}, eq^{-m}, *respectively, we are led to* (2.8) *by taking the limit* $m \to \infty$ *and making suitable substitutions.*

As a consequence of Corollary 2.2, we may deduce the following expansion of a ${}_2\psi_2$ series in terms of a ${}_2\phi_1$ series [11, Eq. (3.13.1.7)]. Setting $z = q/a$ in (2.8), the second summation on the right hand side vanishes. It follows from (2.7) that

$${}_2\psi_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; q, \frac{q}{a}\right] = \frac{(c/b, d/b, bq/a, q; q)_\infty}{(c, d, q/a, q/b; q)_\infty} {}_2\phi_1\left[\begin{matrix} bq/c, & bq/d \\ & bq/a \end{matrix}; q, \frac{cd}{bq}\right], \qquad (2.10)$$

which was originally derived from a double sum transformation formula of Slater, see [11, Section 3.13].

Corollary 2.4 *We have*

$$\begin{aligned}
{}_2\psi_2\left[\begin{matrix} b, & c \\ aq/b, & aq/c \end{matrix}; q, -\frac{aq}{bc}\right] &= \frac{(-b, aq/bc, -q/b, b/a, q; q)_\infty (aq^2/c^2; q^2)_\infty}{(aq/c, -1, q/c, q/b, -aq/bc; q)_\infty (b^2/a; q^2)_\infty} \\
&+ \frac{(aq/bc, b, -aq/b, -b/a, q; q)_\infty (aq^2/c^2; q^2)_\infty}{(aq/b, aq/c, -1, -aq/bc, q/c; q)_\infty (b^2/a; q^2)_\infty}, \qquad (2.11)
\end{aligned}$$

where $|aq/bc| < 1$.

Proof. Setting $c = cq/a$, $d = cq/b$, and $z = -cq/ab$ in (2.8), we find that the summations on the right hand side of the identity are both equal to

$$\sum_{k=0}^{\infty} \frac{(c^2q^2/a^2b^2;q^2)_k}{(q^2;q^2)_k} \left(\frac{b^2}{c}\right)^k, \tag{2.12}$$

which can be summed by the Cauchy q-binomial theorem [13, Appendix II.3]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z| < 1. \tag{2.13}$$

Thus the following relation holds

$$\begin{aligned} {}_2\psi_2\left[\begin{matrix} a, & b \\ cq/a, & cq/b \end{matrix}; q, -\frac{cq}{ab}\right] &= \frac{(-b, cq/ab, -q/b, b/c, q;q)_\infty (cq^2/a^2;q^2)_\infty}{(cq/a, -1, q/a, q/b, -cq/ab;q)_\infty (b^2/c;q^2)_\infty} \\ &\quad + \frac{(cq/ab, b, -cq/b, -b/c, q;q)_\infty (cq^2/a^2;q^2)_\infty}{(cq/b, cq/a, -1, -cq/ab, q/a;q)_\infty (b^2/c;q^2)_\infty}. \end{aligned}$$

The proof is thus completed by interchanging a and c. ∎

Combining (2.11) and (1.10), we are led to the following identity

$$\begin{aligned} &(-b, -q/b, b/a, aq/b;q)_\infty + (b, q/b, -b/a, -aq/b;q)_\infty \\ &\quad = \frac{2(aq, q/a, b^2/a, aq^2/b^2;q^2)_\infty}{(q;q^2)_\infty^2}. \end{aligned} \tag{2.14}$$

To restate the above identity in a symmetric form, we replace a by b/a in (2.14).

Theorem 2.5 *We have*

$$(a, -b, q/a, -q/b;q)_\infty + (-a, b, -q/a, q/b;q)_\infty = \frac{2(ab, q^2/ab, aq/b, bq/a;q^2)_\infty}{(q;q^2)_\infty^2}. \tag{2.15}$$

More identities on sums of infinite products have been found by Bailey [5] and Slater [24–26].

While no attempt will be made to derive a closed product formula for the series (1.3), we obtain a formula involving a product and a summation which has the advantage that it reduces to the q-Gauss summation (1.4) when $c = q$ or $d = q$. Combining Corollary 2.2 and Cauchy's q-binomial theorem (2.13), we deduce

Corollary 2.6

$$\begin{aligned} {}_2\psi_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; q, \frac{cd}{abq}\right] &= \frac{(c/b, c/q, q^2/c, q/d;q)_\infty}{(c, c/bq, q/a, q/b;q)_\infty} \sum_{k=0}^{\infty} \frac{(d/a;q)_k}{(bq^2/c;q)_k} \left(\frac{bq}{d}\right)^k \\ &\quad + \frac{(c/a, d/a, b, cd/bq, bq^2/cd, q;q)_\infty}{(c, d, bq/c, bq/d, q/a, cd/abq;q)_\infty}, \end{aligned} \tag{2.16}$$

where $|bq/d|, |cd/abq| < 1$.

3 A Two-term Summation Formula for ${}_2\psi_2$

In this section, we show that a two-term summation formula for the ${}_2\psi_2$ series (1.13) due to Slater can be derived from an identity of Andrews (1.14) by bilateral extension and parameter augmentation.

We recall that the q-difference operator, or Euler derivative, is defined as

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}. \tag{3.1}$$

The q-shift operator η in the literature [1, 19] is defined as follows:

$$\eta\{f(a)\} = f(aq) \quad \text{and} \quad \eta^{-1}\{f(a)\} = f(aq^{-1}), \tag{3.2}$$

which was introduced by Rogers in [16–18].

In [19], Roman combined q-differential operator and the q-shift operator to built an operator which was denoted by θ in [9]:

$$\theta = \eta^{-1} D_q. \tag{3.3}$$

In [9], Chen and Liu introduced the operator:

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}, \tag{3.4}$$

and proved the following basic relations:

$$E(b\theta)\left\{(at;q)_\infty\right\} = (at, bt; q)_\infty, \tag{3.5}$$

$$E(b\theta)\left\{(as, at;q)_\infty\right\} = \frac{(as, at, bs, bt; q)_\infty}{(abst/q;q)_\infty}, \qquad |abst/q| < 1. \tag{3.6}$$

The procedure to apply the operator $E(b\theta)$ in order to derive a new identity is called parameter augmentation.

The following theorem is equivalent to Slater's formula (1.13), as pointed out by Ismail and Rahman [14]. We proceed to demonstrate how to derive it from the identity (1.14) of Andrews by bilateral extension and parameter augmentation.

Theorem 3.1 *We have*

$$\begin{aligned} &{}_2\psi_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; q, \frac{cd}{abq}\right] - \frac{\alpha}{q}\frac{(q/c, q/d, \alpha/a, \alpha/b; q)_\infty}{(q/a, q/b, \alpha/c, \alpha/d; q)_\infty}{}_2\psi_2\left[\begin{matrix} aq/\alpha, & bq/\alpha \\ cq/\alpha, & dq/\alpha \end{matrix}; q, \frac{cd}{abq}\right] \\ &\quad = \frac{(\alpha, q/\alpha, cd/\alpha q, \alpha q^2/cd, q, c/a, c/b, d/a, d/b; q)_\infty}{(c/\alpha, \alpha q/c, d/\alpha, \alpha q/d, c, d, q/a, q/b, cd/abq; q)_\infty}, \end{aligned} \tag{3.7}$$

where $|cd/abq| < 1$.

Proof. Shifting the index of summation by m and then replacing a, b, f by aq^{-m}, bq^m, fq^{-m} in (1.14), respectively, we obtain

$$\frac{d(q^{1-m}/bc, acdfq^{-2m};q)_m\,(bdq^m)^m}{(1-adq^{-m})(1-dfq^{-m})(adq^{1-m}, dfq^{1-m};q)_m}\sum_{k=-m}^{\infty}\frac{(q/bc, acdfq^{-m};q)_k}{(adq, dfq;q)_k}\,(bdq^m)^k$$
$$-\frac{c(q^{1-m}/bd, acdfq^{-2m};q)_m\,(bcq^m)^m}{(1-acq^{-m})(1-cfq^{-m})(acq^{1-m}, cfq^{1-m};q)_m}\sum_{k=-m}^{\infty}\frac{(q/bd, acdfq^{-m};q)_k}{(acq, cfq;q)_k}\,(bcq^m)^k$$
$$=\frac{d(q, qd/c, c/d, abcd, acdfq^{-2m}, bcdf;q)_\infty}{(acq^{-m}, adq^{-m}, bcq^m, bdq^m, cfq^{-m}, dfq^{-m};q)_\infty}. \tag{3.8}$$

Letting $m\to\infty$ in (3.8) and employing Tannery's theorem, we get

$$\frac{c(bc;q)_\infty}{(1/ad, 1/df;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{(q/bc;q)_k}{(adq, dfq;q)_k}\left(-abcd^2f\right)^k q^{\binom{k}{2}}$$
$$-\frac{d(bd;q)_\infty}{(1/ac, 1/cf;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{(q/bd;q)_k}{(acq, cfq;q)_k}\left(-abc^2df\right)^k q^{\binom{k}{2}}$$
$$=\frac{acd^2f(q, qd/c, c/d, abcd, acdf, bcdf, q/acdf;q)_\infty}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df;q)_\infty}. \tag{3.9}$$

Now, (3.9) can be written as

$$\frac{c}{(1/ad, 1/df;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{(bcq^{-k};q)_\infty}{(adq, dfq;q)_k}\left(ad^2fq\right)^k q^{2\binom{k}{2}}$$
$$-\frac{d}{(1/ac, 1/cf;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{(bdq^{-k};q)_\infty}{(acq, cfq;q)_k}\left(ac^2fq\right)^k q^{2\binom{k}{2}}$$
$$=\frac{acd^2f(q, qd/c, c/d, abcd, acdf, bcdf, q/acdf;q)_\infty}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df;q)_\infty}. \tag{3.10}$$

Next, applying $E(g\theta)$ to both sides of (3.10) with respect to the parameter b gives

$$\frac{c}{(1/ad, 1/df;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{\left(ad^2fq\right)^k q^{2\binom{k}{2}}}{(adq, dfq;q)_k}E(g\theta)\left\{(bcq^{-k};q)_\infty\right\}$$
$$-\frac{d}{(1/ac, 1/cf;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{\left(ac^2fq\right)^k q^{2\binom{k}{2}}}{(acq, cfq;q)_k}E(g\theta)\left\{(bdq^{-k};q)_\infty\right\}$$
$$=\frac{acd^2f(q, qd/c, c/d, acdf, q/acdf;q)_\infty}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df;q)_\infty}E(g\theta)\left\{abcd, bcdf;q)_\infty\right\}. \tag{3.11}$$

From (3.5) and (3.6), it is evident that

$$E(g\theta)\left\{(bcq^{-k};q)_\infty\right\}=(bcq^{-k}, cgq^{-k};q)_\infty, \tag{3.12}$$

$$E(g\theta)\left\{(bdq^{-k};q)_\infty\right\}=(bdq^{-k}, dgq^{-k};q)_\infty, \tag{3.13}$$

and

$$E(g\theta)\left\{abcd, bcdf; q)_\infty\right\} = \frac{(abcd, acdg, bcdf, cdfg; q)_\infty}{(abc^2d^2fg/q; q)_\infty}. \tag{3.14}$$

Substituting (3.12), (3.13), and (3.14) into (3.11), we see that

$$\begin{aligned}
&\frac{c(bc, cg; q)_\infty}{(1/ad, 1/df; q)_\infty} {}_2\psi_2\left[\begin{matrix} q/bc, & q/cg \\ adq, & dfq \end{matrix}; q, \frac{abc^2d^2fg}{q}\right] \\
&\qquad - \frac{d(bd, dg; q)_\infty}{(1/ac, 1/cf; q)_\infty} {}_2\psi_2\left[\begin{matrix} q/bd, & q/dg \\ acq, & cfq \end{matrix}; q, \frac{abc^2d^2fg}{q}\right] \\
&= \frac{acd^2f(q, qd/c, c/d, abcd, acdf, acdg, bcdf, q/acdf, cdfg; q)_\infty}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df, abc^2d^2fg/q; q)_\infty},
\end{aligned} \tag{3.15}$$

where $|abc^2d^2fg/q| < 1$.

Finally, the proof is completed by replacing a, b, c, d, f, g by c/fq, e, q/ae, f, d/fq, ae/b, respectively, and then setting $aef = \alpha$. ∎

Substitute a, b, c, d, α with qa/e, qb/e, qc/e, q^2/e, fq/e in (3.7), respectively, we may recover the original formula (1.13) due to Slater.

If we set $d = q$ in (3.7), then the second term on the left hand side vanishes, and so we get the q-Gauss summation (1.4) as a special of (3.7).

To conclude this paper, we represent (3.7) in an equivalent form and give the explicit substitutions to reach Askey's q-extension of Cauchy's beta integral (1.11). By the relation

$${}_2\psi_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; q, z\right] = {}_2\psi_2\left[\begin{matrix} q/c, & q/d \\ q/a, & q/b \end{matrix}; q, \frac{cd}{abz}\right], \tag{3.16}$$

we may rewrite (3.7) as

$$\begin{aligned}
&\frac{(q/a, q/b; q)_\infty}{(q/c, q/d; q)_\infty} {}_2\psi_2\left[\begin{matrix} q/c, & q/d \\ q/a, & q/b \end{matrix}; q, q\right] - \frac{\alpha}{q} \frac{(\alpha/a, \alpha/b; q)_\infty}{(\alpha/c, \alpha/d; q)_\infty} {}_2\psi_2\left[\begin{matrix} \alpha/c, & \alpha/d \\ \alpha/a, & \alpha/b \end{matrix}; q, q\right] \\
&\qquad = \frac{(\alpha, q/\alpha, cd/\alpha q, \alpha q^2/cd, q, c/a, c/b, d/a, d/b; q)_\infty}{(c/\alpha, \alpha q/c, d/\alpha, \alpha q/d, c, d, q/c, q/d, cd/abq; q)_\infty},
\end{aligned} \tag{3.17}$$

where $|cd/abq| < 1$. Replacing a, b, c, d, α by q/c, $-q/d$, q/a, $-q/b$, q, respectively, then (3.17) takes the form of Askey's q-extension of Cauchy's beta integral.

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References

[1] G. E. Andrews, On the foundations of combinatorial theory V. Eulerian differential operators, Studies in Appl. Math., **50** (1971), 345-375.

[2] G. E. Andrews, Ramanujan's "lost" notebook. I. Partial θ-functions, Adv. in Math., **41** (1981), 137-172.

[3] G. Andrews and R. Askey, Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials. Higher Combinatorics, M. Aigner, ed., Reidell Publ. Co., Dordrecht, Holland 1977, 3-26.

[4] R. Askey, A q-extension of Cauchy's form of the beta integral, Quart. J. Math. (Oxford) (2), **32** (1981), 255-266.

[5] W. N. Bailey, Series of hypergeometric type which are infinite in both directions, Quart. J. Math. (Oxford), **7** (1936), 105-115.

[6] W. N. Bailey, On the basic bilateral hypergeometric series ${}_2\psi_2$, Quart. J. Math. (Oxford) (2), **1** (1950), 194-198.

[7] R. P. Boas, Tannery's theorem, Math. Mag., **38** (1965), pp. 66.

[8] W. Y. C. Chen and A. M. Fu, Semi-finite forms of bilateral basic hypergeometric series, Proc. Amer. Math. Soc., **134** (2006), 1719-1725.

[9] W. Y. C. Chen and Z. G. Liu, Parameter augmentation for basic hypergeometric series I, in: B. E. Sagan, R. P. Stanley (eds.), Mathematical Essays in honor of Gian-Carlo Rota, Birkhäuser, Basel, 1998, 111-129.

[10] J. Dougall, On Vandermonde's theorem and some more general expansions, Proc. Edin. Math. Soc., **25** (1907), 114-132.

[11] H. Exton, q-Hypergeometric Functions and Applications, Ellis Horwood, Chichester, 1983, pp. 347.

[12] G. Gasper, Elementary derivations of summation and transformation formulas for q-series, Fields Inst. Commun., **14** (1997), 55-70.

[13] G. Gasper and M. Rahman, Basic Hypergeometric Series, Second Ed., Cambridge University Press, Cambridge, 2004.

[14] M. Ismail and M. Rahman, Some basic bilateral sums and integrals, Pacific J. Math., **170** (2) (1995), 497-515.

[15] F. Jouhet and M. Schosser, Another proof of Bailey's ${}_6\psi_6$ summation theorem, Aequationes Math., **70** (2005), 43-50.

[16] L. J. Rogers, On the expansion of some infinite products, Proc. London Math. Soc., **24** (1893), 337-352.

[17] L. J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc., **25** (1894), 318-343.

[18] L. J. Rogers, Third memoir on the expansion of certain infinite products, Proc. London Math. Soc., **26** (1895), 15-32.

[19] S. Roman, More on the umbral calculus, with emphasis on the q-umbral calculus, J. Math. Anal. Appl., **107** (1985), 222-254.

[20] M. Schlosser, A simple proof of Bailey's very well-poised ${}_6\psi_6$ summation, Proc. Amer. Math. Soc., **130** (2002), 1113-1123.

[21] M. Schlosser, Abel-Rothe type generalizations of Jacobi's triple product identity, Theory and Applications of Special Functions. A Volume Dedicated to Mizan Rahman (M. E. H. Ismail and E. Koelink, eds.), Dev. Math., **13** (2005), 383-400.

[22] D. B. Sears, Transformation theory of basic hypergeometric functions of any order, Proc. London Math. Soc., **53** (2) (1951), 181-191.

[23] L. J. Slater, General transformations of bilateral series, Quart. J. Math. (Oxford) (2), **3** (1952), 73-80.

[24] L. J. Slater, A note on equivalent product theorems, Math. Gazette, **38** (1954), 127-128.

[25] L. J. Slater, Some new results on equivalent products, Proc. Camb. Phil. Soc., **50** (1954), 394-403.

[26] L. J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.