On the Bilateral Series $2\psi_2$

Vincent Y. B. Chen$^1$, William Y. C. Chen$^2$ and Nancy S. S. Gu$^3$

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071
People’s Republic of China

Email: $^1$ybchen@mail.nankai.edu.cn, $^2$chen@nankai.edu.cn, $^3$gu@nankai.edu.cn

Abstract

We obtain a formula which reduces the evaluation of a $2\psi_2$ series to two $2\phi_1$ series. In some sense, this identity may be considered as a companion of Slater’s formulas. We also find that a two-term $2\psi_2$ summation formula due to Slater can be derived from a unilateral summation formula of Andrews by bilateral extension and parameter augmentation.

Keywords: basic hypergeometric series, bilateral series, bilateral extension, parameter augmentation, $q$-Gauss summation.

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1 Introduction

It is well known that many bilateral basic hypergeometric identities can be derived from unilateral identities. Using Cauchy’s method [5, 15, 20, 21] one may obtain bilateral basic hypergeometric identities from terminating unilateral identities. Starting with nonterminating unilateral basic hypergeometric series, Chen and Fu [8] developed a method to construct semi-finite forms by shifting the summation index by $m$. Then the bilateral summations are consequences of the semi-finite forms by letting $m$ tend to infinity. We call this method bilateral extension. In this paper we use bilateral extensions of a $3\phi_2$ series and an identity of Andrews [2] to study the bilateral series $2\psi_2$:

$$2\psi_2 \left[ \begin{array}{c} a, b \cr c, d \end{array} ; q, z \right]. \quad (1.1)$$
The above $2\psi_2$ series is closely related to the question of finding a $q$-extension of Dougall’s bilateral hypergeometric series summation formula [10]:

$$
\sum_{k=-\infty}^{\infty} \frac{(a)_k(b)_k}{(c)_k(d)_k} = \frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)},
$$

where $\text{Re}(c+d-a-b-1) > 0$, $(a)_k = a(a+1)\cdots(a+k-1)$, $k = 1, 2, \ldots$, $(a)_0 = 1$ and $(a)_k = (-1)^k/(1-a)_{-k}$ when $k$ is a negative integer.

Bailey [6] first suggested that there did not exist any $q$-extension of (1.2). Since (1.2) is an extension of the Gauss $2F_1$ summation formula, one naturally expects that a $q$-analogue of (1.2) should be concerned with the following series:

$$
2\psi_2 \left[ \frac{a, b}{c, d} : q, \frac{cd}{abq} \right].
$$

Clearly, when $c$ or $d$ equals $q$, (1.3) reduces to the $q$-Gauss sum [13, Appendix II.8]:

$$
2\phi_1 \left[ \frac{a, b}{c} : q, \frac{c}{ab} \right] = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}, \quad |c/ab| < 1.
$$

Even for the above series (1.3), Gasper [12] pointed out that one could not use analytic continuation to derive an infinite product representation.

On the other hand, many results on the bilateral $2\psi_2$ series (1.1) have been obtained. In [6], Bailey found several transformation formulas for the $2\psi_2$ series (1.1). Later, Slater obtained a general transformation formula for an $r\psi_r$ series in [23] based on Sears’ transformation on the $r+s+1\phi_{r+s}$ series in [22] subject to suitable substitutions and the following relation

$$
\sum_{n=0}^{\infty} f(n) = \sum_{n=-\infty}^{-1} f(-n-1)
$$

to combine two unilateral series to form a bilateral series.

Gasper and Rahman [13] have shown that based on Slater’s transformation formula, one could obtain two expansions of an $r\psi_r$ series in terms of $r\phi_{r-1}$ series [13, Eq. (5.4.4), (5.4.5)]. When $r = 2$, they become

$$
2\psi_2 \left[ \frac{a, b}{c, d} ; q, z \right] = \frac{a(q, qa/b, c/a, d/a, az, q/az, qb, 1/b; q)_{\infty}}{(a/b, qb/a, c, d, q/a, q/b, z, q/az; q)_{\infty}}
$$

$$
\times 2\phi_1 \left[ \frac{qa/c, qa/d}{qa/b} ; q, \frac{cd}{abz} \right] + \text{idem}(a; b)
$$

and

$$
2\psi_2 \left[ \frac{a, b}{c, d} ; q, z \right] = \frac{q \left( q, c/a, c/b, abz/dq, dq^2/abz, q/d; q \right)_{\infty}}{c \left( c, d, q/a, q/b, abz/cd, qcd/abz; q \right)_{\infty}}
$$

$$
\times 2\phi_1 \left[ \frac{qa/c, qb/c}{qd/c} ; q, z \right] + \text{idem}(c; d),
$$

(1.6)
where the symbol “idem(a; b)” after an expression means that the preceding expression is repeated with a and b interchanged.

Setting $d = q$, (1.6) reduces to a three-term transformation formula [13, Appendix III.32] for the $2\phi_1$ series:

$$
2\phi_1 \left[ \begin{array}{ccc} a & b & c \\ d & q, z \end{array} \right] = \frac{(b, c/a, a, q/a; q\infty)}{(c, b/a, z, q/z; q\infty)} 2\phi_1 \left[ \begin{array}{ccc} a & c/a & c \\ q, b/a & q \end{array} \right] + \text{idem}(a; b).
$$

However, it should be noted that when $c$ or $d$ equals $q$, (1.7) does not lead to any nontrivial identity.

The first result of this paper is to give a new formula for the $2\psi_2$ series (1.1) in terms of two $2\phi_1$ series which is different from Slater’s formulas (1.6) and (1.7). It reduces to a different three-term transformation formula (2.4) when $c = q$ compared with the three-term transformation formula (1.8) deduced by Slater’s transformation. Moreover, this identity may be considered as a companion of Slater’s formulas (1.6) and (1.7). Note that Slater’s formulas do not seem to imply the special cases that can be deduced from our formula except for Ramanujan’s $1\psi_1$ summation formula [13, Appendix II.29]. As a consequence, our formula yields a two-term closed product form for the $2\psi_2$ series:

$$
2\psi_2 \left[ \begin{array}{ccc} b, c & q, -aq/bc \\ aq/b, aq/c \end{array} \right] = \frac{(-b, aq/bc, -q/b, b/a, q; q\infty)(aq^2/c^2; q^2\infty)}{(aq/c, -1, q/c, q/b, -aq/bc; q\infty)(b^2/a; q^2\infty)}
$$

$$
+ \frac{(aq/bc, -aq/b, b/a, q; q\infty)(aq^2/c^2; q^2\infty)}{(aq/b, aq/c, -1, -aq/bc, q/c, q; q\infty)(b^2/a; q^2\infty)}.
$$

For comparison, we recall the known formula for the well-poised $2\psi_2$ series [13, Appendix II.30]:

$$
2\psi_2 \left[ \begin{array}{ccc} b, c & q, -aq/bc \\ aq/b, aq/c \end{array} \right] = \frac{(aq/bc; q\infty)(aq^2/b^2, aq^2/c^2, q^2, aq, q/a; q^2\infty)}{(aq/b, aq/c, q/b, q/c, -aq/bc; q\infty)}.
$$

Let us turn our attention back to Dougall’s formula. As pointed out by Askey [4], Bailey seemed to have been partly right concerning his opinion towards the $q$-extension of Dougall’s formula. According to Askey [4], in certain sense the following $q$-extension of Cauchy’s beta integral was similar to a $q$-extension of Dougall’s formula:

$$
\int_{-\infty}^{\infty} \frac{(ct, -dt; q\infty)}{(at, -bt; q\infty)} dt = 2 \frac{(1 - q)(c/a, d/b, -c/b, -d/a, ab, q/ab; q\infty)(q^2; q^2\infty)}{(cd/abq, q; q\infty)(a^2, q^2/a^2, b^2, q^2/b^2; q^2\infty)}.
$$

In fact, this integral can be recast as a two-term summation formula for the $2\psi_2$ series (1.3):

$$
\frac{(c, -d; q\infty)}{(a, -b; q\infty)} 2\psi_2 \left[ \begin{array}{ccc} a & -b \\ c, -d \end{array} ; q, q \right] = \frac{(-c, d; q\infty)}{(-a, b; q\infty)} 2\psi_2 \left[ \begin{array}{ccc} -a & b \\ -c, d \end{array} ; q, q \right]
$$

$$
= 2 \frac{(c/a, d/b, -c/b, -d/a, ab, q/ab; q\infty)(q^2; q^2\infty)}{(cd/abq, q; q\infty)(a^2, q^2/a^2, b^2, q^2/b^2; q^2\infty)}.
$$

(1.12)
As observed by Ismail and Rahman [14], the above two-term summation formula is a special of a transformation formula due to Slater [23]. When \( r = 2 \), by substitutions and the \( q \)-Gauss sum (1.4), Slater’s general transformation on the \( r\psi_r \) series reduces to the following two-term summation formula:

\[
\frac{(c/e, qef/c, q/a, q/b, c/a, c/b; q)_\infty}{(e, f, q/e, q/f, c/ab; q)_\infty} = \frac{q (c/qf, q^2f/c, e/a, e/b, qe/e, q^2/e; q)_\infty}{(e, q/e, q/f, qf/e; q)_\infty} \\
\times \psi_2 \left[ \frac{e/c, e/q}{e/a, e/b} ; q, q \right] + \text{idem}(e; f).
\]  

(1.13)

The second result of this paper is concerned with the above two-term summation formula (1.13) for \( 2\psi_2 \). Andrews [2] established a three-term transformation formula which is the key to proving many of Ramanujan’s identities for partial \( \varphi \)-functions. In view of the symmetry in this formula, he obtained a generalization of Ramanujan’s 1\( \varphi_1 \) summation:

\[
d \sum_{n=0}^{\infty} \frac{(q/bc, acdf; q)_n}{(ad, df; q)_{n+1}} (bd)^n - c \sum_{n=0}^{\infty} \frac{(q/bd, acdf; q)_n}{(ac, cf; q)_{n+1}} (bc)^n \\
= \left( \frac{q, qd/c}{c/d, abcd, acdf, bcdf; q}_\infty \right) \frac{(ac, ad, bc, bd, cf, df; q)_\infty}{(ac, ad, bc, bd, cf, df; q)_\infty}, \quad |bc|, |bd| < 1.
\]  

(1.14)

Using the approach of parameter augmentation developed by Chen and Liu [9], we find that the two-term summation formula (1.13) for \( 2\psi_2 \) series is a consequence of the above identity (1.14) of Andrews by bilateral extension and parameter augmentation.

As is customary, we employ the notation and terminology of basic hypergeometric series in [13]. For \( |q| < 1 \), the \( q \)-shifted factorial is defined by

\[
(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k) \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad \text{for} \ n \in \mathbb{Z}.
\]

For convenience, we shall adopt the following notation for multiple \( q \)-shifted factorials:

\[
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n,
\]

where \( n \) is an integer or infinity. In particular, for a nonnegative integer \( k \), we have

\[
(a; q)_{-k} = \frac{1}{(aq^{-k}; q)_k}.
\]  

(1.15)

The (unilateral) basic hypergeometric series \( r\phi_s \) is defined by

\[
r\phi_s \left[ \begin{array}{cccc}
a_1, & a_2, & \ldots, & a_r \\
b_1, & b_2, & \ldots, & b_s \\
\end{array} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k} \left[ (-1)^k q^{k(s-r)} k! \right]^{1+s-r} z^k,
\]  

(1.16)

while the bilateral basic hypergeometric series \( r\psi_s \) is defined by

\[
r\psi_s \left[ \begin{array}{cccc}
a_1, & a_2, & \ldots, & a_r \\
b_1, & b_2, & \ldots, & b_s \\
\end{array} ; q, z \right] = \sum_{k=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} \left[ (-1)^k q^{k} k! \right]^{s-r} z^k.
\]  

(1.17)
2 An Expansion Formula for the $2\psi_2$ Series

In this section, we derive a representation for the $2\psi_2$ series (1.1) in terms of two $2\phi_1$ series. This formula can be considered as a companion of Slater’s formulas (1.6) and (1.7). We also present some consequences including a two-term infinite product representation for the sum of a well-poised $2\psi_2$ series (1.9).

**Theorem 2.1** We have

$$
2\psi_2 \left[ \begin{array}{cccc}
    a, & b & c, & d \\
    d, & e & q, & z
\end{array} \right] = \frac{(c/b, abz/d, dq/abz, q/d, q; q)_{\infty}}{(c, az/d, q/a, q/b, cd/abz; q)_{\infty}} 2\phi_1 \left[ \begin{array}{cccc}
    cd/abz, & d/a & bq/d \\
    dq/az, & q, & d
\end{array} \right]
$$

$$
- \frac{(cq/d, b/d/a, az/q, q^2/az, q/d, q; q)_{\infty}}{(d/q, c, bq/d, az/d, dq/az, q^2/d, q/a; q)_{\infty}} 2\phi_1 \left[ \begin{array}{cccc}
    aq/d, & bq/d & c/d \\
    cq/d, & q, & z
\end{array} \right],
$$

where $|cd/ab| < |z| < 1$ and $|bq/d| < 1$.

**Proof.** We start with a three-term transformation of $3\phi_2$ series [13, Appendix III.33]:

$$
3\phi_2 \left[ \begin{array}{cccc}
    a, & b, & c, & de \\
    d, & e & q, & abc
\end{array} \right] = \frac{(c/b, e/c, eq/a, q/d; q)_{\infty}}{(e/cd/q, a/e, bcq/e; q)_{\infty}} 3\phi_2 \left[ \begin{array}{cccc}
    c, & d/a, & ce/q & bq \\
    cq/a, & bcq/e & q & d
\end{array} \right]
$$

$$
- \frac{(eq/d, bcq/de, bcq^2/de; q)_{\infty}}{(d/q, c, bq/de, cdq/a, e, bcq/eq; q)_{\infty}} 3\phi_2 \left[ \begin{array}{cccc}
    aq/d, & bq/d, & c/d & de \\
    c/dq/eq & de & q & abc
\end{array} \right],
$$

where $|bq/d|, |de/abc| < 1$.

Shifting the index of summation on the left hand side of the above identity by $m$ such that the new sum runs from $-m$ to infinity, and then replacing $a, b, d, e$ by $aq^{-m}, bq^{-m}, dq^{-m}, eq^{-m}$, respectively, we get

$$
\sum_{k=-m}^{\infty} \frac{(a, b, cq^m; q)_k}{(q^{m+1}, d, e; q)_k} \left( \frac{de}{abc} \right)^k = \frac{(eq/c, eq/d, q; q)_m (e/b, e/c, eq^{1+m}/a, q^{1+m}/d; q)_{\infty}}{(c/q, a/q, bcq/e; q)_{\infty}} 3\phi_2 \left[ \begin{array}{cccc}
    c, & d/a, & ce/q & bq \\
    cq/a, & bcq/e & q & d
\end{array} \right]
$$

$$
- \frac{(bcq^2/de, eq/d, q; q)_m (q^{1+m}/d, eq/d, b; q)_{\infty}}{(d/q, e, bq/d; q)_{\infty}} 3\phi_2 \left[ \begin{array}{cccc}
    aq/d, & bq/d, & c/q & de \\
    c/dq/eq & de & q & abc
\end{array} \right],
$$

where $|bq/d|, |de/abc| < 1$.

Setting $m \to \infty$ in (2.2) and assuming $|c| < 1$, Tannery’s theorem [7] enables us to interchange the limit and the summation. This gives

$$
2\psi_2 \left[ \begin{array}{cccc}
    a, & b & d, & e \\
    d, & e & q, & abc
\end{array} \right] = \frac{(eq/c, eq/d, q, e/b, e/c; q)_{\infty}}{(c/q, a/q, bcq/e; q)_{\infty}} 2\phi_1 \left[ \begin{array}{cccc}
    c, & d/a, & bq \\
    bcq/e & q & d
\end{array} \right]
$$

$$
- \frac{(bcq^2/de, eq/d, q; q)_m (q^{1+m}/d, eq/d; q)_{\infty}}{(d/q, e, bq/d; q)_{\infty}} 2\phi_1 \left[ \begin{array}{cccc}
    aq/d, & bq/d, & c/dq/eq, & de \\
    eq/d & de & q & abc
\end{array} \right],
$$

where $|bq/d|, |de/abc| < 1$. 

5
where $|bq/d|, |c|, |de/abc| < 1$.

By the substitutions $c \to de/azb$ and $e \to c$ in (2.3), we get the desired formula. ■

Note that Theorem 2.1 may be considered as a bilateral extension of the following three-term transformation formula [13, Appendix III.31]

$$2\phi_1 \left[ \begin{array}{c} a, & b \cr c, & d \end{array} ; q, z \right] = \frac{(abz/d, q/d; q)_\infty}{(az/d, q/a; q)_\infty} 2\phi_1 \left[ \begin{array}{c} d/a, & dq/az \cr bq/d \end{array} ; q, \frac{bq}{d} \right] - \frac{(b/a, az/q, q^2/az, q/d; q)_\infty}{(d/q, bq/d, az/d, dq/az, q/a; q)_\infty} 2\phi_1 \left[ \begin{array}{c} aq/d, & bq/d \cr q^2/d \end{array} ; q, z \right], \quad (2.4)$$

where $|bq/d|, |z| < 1$. It is clear that (2.4) is a special case of (2.1) for $c = q$.

Since Slater’s formula (1.7) and our formula (2.1) deal with the same series, we are naturally led to an identity on $2\phi_1$ series. The right hand sides of (1.7) and (2.1) give rise to the following identity by replacing $a, b, c, z$ by $d=b, dz=q, adz/c, bq/c$, respectively,

$$2\phi_1 \left[ \begin{array}{c} a, & b \cr c, & q, z \end{array} \right] = \frac{(abz/c, q/c; q)_\infty}{(az/c, q/a; q)_\infty} 2\phi_1 \left[ \begin{array}{c} cq/abz, & c/a \cr cq/az \end{array} ; q, \frac{bq}{c} \right] + \left( \frac{q(1-a)(b, q/z, d/aq, aq^2/d, cq/adz, adz/c, q/c; q)_\infty}{d(d/c/az, 1/a, aq/c, dz/c, cq/dz, q/d; q)_\infty} + \frac{(azq/c, dz/b, d/c, cq/d, q^2/dz, a; q)_\infty}{(d/q, z, c, d^2, aq/c, dz/c, cq/dz; q)_\infty} \right) 2\phi_1 \left[ \begin{array}{c} q/b, & z \cr azq/c \end{array} ; q, \frac{bq}{c} \right]. \quad (2.5)$$

It is worth noting that the parameter $d$ occurs only in the factors of the second term on the right hand side of (2.5). Hence the sum of the two products in the parentheses does not depend on $d$. This fact does not seem to be obvious by direct verification.

Setting $d = aq$, it follows that

$$2\phi_1 \left[ \begin{array}{c} a, & b \cr c, & q, z \end{array} \right] = \frac{(abz/c, q/c; q)_\infty}{(az/c, q/a; q)_\infty} 2\phi_1 \left[ \begin{array}{c} cq/abz, & c/a \cr cq/az \end{array} ; q, \frac{bq}{c} \right] + \frac{(az, b, c/a, q/az; q)_\infty}{(z, c, q/a, c/az; q)_\infty} 2\phi_1 \left[ \begin{array}{c} q/b, & z \cr azq/c \end{array} ; q, \frac{bq}{c} \right]. \quad (2.6)$$

From Heine’s transformation [13, Appendix III.1]

$$2\phi_1 \left[ \begin{array}{c} a, & b \cr c, & q, z \end{array} \right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} 2\phi_1 \left[ \begin{array}{c} c/b, & z \cr az \end{array} ; q, b \right], \quad (2.7)$$

it is easily seen that (2.6) is equivalent to (2.4) by the substitution $c \to d$.

**Corollary 2.2** We have

$$2\psi_2 \left[ \begin{array}{c} a, & b \cr c, & d \end{array} ; q, z \right] = \frac{(abz/d, c/b, dq/abz, q/d; q)_\infty}{(c, az/d, q/a, q/b, cd/abz; q)_\infty} 2\phi_1 \left[ \begin{array}{c} cd/abz, & d/a \cr dq/az \end{array} ; q, \frac{bq}{d} \right] + \frac{(d/a, b, az, q/az; q; q)_\infty}{(d, c, d/az, q/a; q)_\infty} 2\phi_1 \left[ \begin{array}{c} c/b, & z \cr azq/d \end{array} ; q, \frac{bq}{d} \right], \quad (2.8)$$

6
where $|cd/ab| < |z| < 1$ and $|bq/d| < 1$.

**Proof.** By Heine’s transformation (2.7), the second term on the right hand side of (2.1) equals

$$-rac{(cq/d, b, d/a, az/q, q^2/az, q/d, q; q)_\infty}{(d/q, c, bq/d, az/d, dq/az, q^2/d, q/a; q)_\infty} 2\phi_1\left[ \frac{aq/d, bq/d}{cq/d}; q, z \right]$$

$$= -\frac{(b, d/a, az/q, q^2/az, q/d, q; azq/d; q)_\infty}{(d/q, c, az/d, dq/az, q^2/d, q/a, z; q)_\infty} 2\phi_1\left[ \frac{c/b, z}{azq/d}; q, \frac{bq}{d} \right]$$

$$= -\frac{(d/a, b, az/q; az, q; q)_\infty}{(d, c, d/az, z, q/a, q)_\infty} \frac{(1 - az/q)(1 - q/d)(1 - d/az)}{(1 - d/q)(1 - az/d)(1 - q/az)}$$

$$\times 2\phi_1\left[ \frac{c/b, z}{azq/d}; q, \frac{bq}{d} \right]$$

$$= \frac{(d/a, b, az/q; az, q; q)_\infty}{(d, c, d/az, z, q/a, q)_\infty} 2\phi_1\left[ \frac{c/b, z}{azq/d}; q, \frac{bq}{d} \right]. \tag{2.9}$$

**Remark 2.3** Corollary 2.2 can also be obtained from the following three-term transformation formula [13, Appendix III.34]

$$3\phi_2\left[ \frac{a, b, c}{d, e}; q, \frac{de}{abc} \right] = \frac{(e/b, e/c; q)_\infty}{(e, e/bc; q)_\infty} 3\phi_2\left[ \frac{d/a, b, c}{d, bcq/e}; q, q \right]$$

$$+ \frac{(d/a, b, c, de/bc; q)_\infty}{(d, e, bc/e, de/abc; q)_\infty} 3\phi_2\left[ \frac{e/b, e/c, de/abc}{de/bc, eq/bc}; q, q \right].$$

Shifting the summation index by $m$ on the left hand side and replacing $a, c, d, e$ by $aq^{-m}, eq^{-m}, dq^{-m}, eq^{-m}$, respectively, we are led to (2.8) by taking the limit $m \to \infty$ and making suitable substitutions.

As a consequence of Corollary 2.2, we may deduce the following expansion of a $2\psi_2$ series in terms of a $2\phi_1$ series [11, Eq. (3.13.1.7)]. Setting $z = q/a$ in (2.8), the second summation on the right hand side vanishes. It follows from (2.7) that

$$2\psi_2\left[ \frac{a, b}{c, d}; q, \frac{q}{a} \right] = \frac{(c/b, d/b, bq/a, q; q)_\infty}{(c, d, q/a, q/b, q)_\infty} 2\phi_1\left[ \frac{bq/c, bq/d}{bq/a}; q, \frac{cd}{bq} \right], \tag{2.10}$$

which was originally derived from a double sum transformation formula of Slater, see [11, Section 3.13].

**Corollary 2.4** We have

$$2\psi_2\left[ \frac{b, c}{aq/b, aq/c}; q, \frac{-aq}{bc} \right] = \frac{(-b, aq/bc, -q/b, b/a, q; q)_\infty(aq^2/c^2; q^2)_\infty}{(aq/c, q/c, q/b, -aq/bc; q)_\infty(b^2/a; q^2)_\infty}$$

$$+ \frac{(aq/bc, b, -aq/b, -b/a, q; q)_\infty(aq^2/c^2; q^2)_\infty}{(aq/b, aq/c, q/b, -aq/bc, q/c; q)_\infty(b^2/a; q^2)_\infty}, \tag{2.11}$$

where $|aq/bc| < 1$. 

7
Proof. Setting \( c = cq/a, d = cq/b, \) and \( z = -cq/ab \) in (2.8), we find that the summations on the right hand side of the identity are both equal to
\[
\sum_{k=0}^{\infty} \frac{(c^2q^2/a^2b^2;q^2)_k}{(q^2;q^2)_k} \left( \frac{b^2}{c} \right)^k,
\]
which can be summed by the Cauchy \( q \)-binomial theorem \([13, \text{Appendix II.3}]\)
\[
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1.
\]
Thus the following relation holds
\[
2^{\psi_2} \left[ \frac{a}{cq/a}, \frac{b}{cq/b}, q, -\frac{cq}{ab} \right] = \frac{(-b, cq/ab, -q/b, b/c, q; q)\infty(cq^2/a^2; q^2)\infty}{(cq/a, -1, q/a, q/b, -cq/ab; q)\infty(b^2/c; q^2)\infty}
\frac{(cq/ab, b, -cq/b, -b/c, q; q)\infty(cq^2/a^2; q^2)\infty}{(cq/b, cq/a, -1, -cq/ab, q/a; q)\infty(b^2/c; q^2)\infty}.
\]
The proof is thus completed by interchanging \( a \) and \( c \). \( \Box \)

Combining (2.11) and (1.10), we are led to the following identity
\[
(-b, -q/b, b/a, aq/b; q)\infty + (b, q/b, -b/a, -aq/b; q)\infty = \frac{2(aq, q/a, b^2/a, aq^2/b^2; q^2)\infty}{(q; q^2)\infty}.
\]
To restate the above identity in a symmetric form, we replace \( a \) by \( b/a \) in (2.14).

**Theorem 2.5** We have
\[
(a, -b, q/a, -q/b; q)\infty + (-a, b, -q/a, q/b; q)\infty = \frac{2(ab, q^2/ab, aq/b, bq/a; q^2)\infty}{(q; q^2)\infty}. \tag{2.15}
\]

More identities on sums of infinite products have been found by Bailey \([5]\) and Slater \([24–26]\).

While no attempt will be made to derive a closed product formula for the series (1.3), we obtain a formula involving a product and a summation which has the advantage that it reduces to the \( q \)-Gauss summation (1.4) when \( c = q \) or \( d = q \). Combining Corollary 2.2 and Cauchy’s \( q \)-binomial theorem (2.13), we deduce

**Corollary 2.6**
\[
2^{\psi_2} \left[ \frac{a}{c}, \frac{b}{d}, q, \frac{cd}{abq} \right] = \frac{(c/b, c/q, q^2/c, q/d; q)\infty}{(c, c/bq, a/q, b/d; q)\infty} \sum_{k=0}^{\infty} \frac{(d/a; q)_k}{(bq^2/c; q)_k} \left( \frac{bq}{d} \right)^k
+ \frac{(c/a, d/a, b, cd/bq, bq^2/cd, q; q)\infty}{(c, d, bq/c, bq/d, q/a, cd/abq; q)\infty}, \tag{2.16}
\]
where \(|bq/d|, |cd/abq| < 1|\).
3 A Two-term Summation Formula for $2\psi_2$

In this section, we show that a two-term summation formula for the $2\psi_2$ series (1.13) due to Slater can be derived from an identity of Andrews (1.14) by bilateral extension and parameter augmentation.

We recall that the $q$-difference operator, or Euler derivative, is defined as

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$  \hfill (3.1)

The $q$-shift operator $\eta$ in the literature [1,19] is defined as follows:

$$\eta\{f(a)\} = f(aq) \quad \text{and} \quad \eta^{-1}\{f(a)\} = f(aq^{-1}),$$  \hfill (3.2)

which was introduced by Rogers in [16–18].

In [19], Roman combined $q$-differential operator and the $q$-shift operator to build an operator which was denoted by $\theta$ in [9]:

$$\theta = \eta^{-1}D_q.$$  \hfill (3.3)

In [9], Chen and Liu introduced the operator:

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^n}{(q;q)_n},$$  \hfill (3.4)

and proved the following basic relations:

$$E(b\theta) \{(at; q)_\infty\} = (at, bt; q)_\infty,$$  \hfill (3.5)

$$E(b\theta) \{(as, at; q)_\infty\} = \frac{(as, at, bs, bt; q)_\infty}{(abst/q; q)_\infty}, \quad |abst/q| < 1.$$  \hfill (3.6)

The procedure to apply the operator $E(b\theta)$ in order to derive a new identity is called parameter augmentation.

The following theorem is equivalent to Slater’s formula (1.13), as pointed out by Ismail and Rahman [14]. We proceed to demonstrate how to derive it from the identity (1.14) of Andrews by bilateral extension and parameter augmentation.

**Theorem 3.1** We have

$$2\psi_2 \left[ a, \frac{b}{c}, \frac{cd}{d}; q, \frac{aq}{abq} \right] \frac{\alpha (q/c, q/d, \alpha/a, \alpha/b; q)_\infty}{q (q/a, q/b, \alpha/c, \alpha/d; q)_\infty} 2\psi_2 \left[ \frac{aq/\alpha}{c q/\alpha}, \frac{b q/\alpha}{d q/\alpha}; q, \frac{cd}{abq} \right]$$

$$= \left( \alpha, \frac{q/\alpha}{c}, \frac{cd/\alpha q}{d/\alpha}, \frac{\alpha/\alpha q^2}{cd}, \frac{q}{c/\alpha}, \frac{c/a, c/b, d/a, d/b; q)_\infty}{c/\alpha, q^2/c, d/\alpha, q^2/d, c, d, q/a, q/b, cd/abq; q)_\infty}, \right.$$  \hfill (3.7)

where $|cd/abq| < 1$. 

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Proof. Shifting the index of summation by \( m \) and then replacing \( a, b, f \) by \( aq^{-m}, bq^m, f q^{-m} \) in (1.14), respectively, we obtain

\[
\frac{d(1 - adq^{-m})(1 - dfq^{-m})(adq^{-1,m}, dfq^{-1,m}; q)_m}{(1 - acq^{-m})(1 - cfq^{-m})(acq^{-1,m}, cfq^{-1,m}; q)_m} \sum_{k=-m}^{\infty} \frac{(q/bc, acdf q^{-m}; q)_k}{(adq, dfq; q)_k} (bdq^m)_k
\]

\[
= \frac{c(q^{1-m}/bc, acdf q^{-2m}; q)_m (bdq^m)_m}{(1 - adq^{-m})(1 - dfq^{-m})(adq^{-1,m}, dfq^{-1,m}; q)_m} \sum_{k=-m}^{\infty} \frac{(q/bd, acdf q^{-m}; q)_k}{(acq, cfq; q)_k} (bcq^m)_k
\]

(3.8)

Letting \( m \to \infty \) in (3.8) and employing Tannery’s theorem, we get

\[
\frac{c(bc; q)_\infty}{(1/ad, 1/df; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(q/bc; q)_k}{(adq, dfq; q)_k} (-abcd^2 f)^k q^{(k)}
\]

\[
- \frac{d(bd; q)_\infty}{(1/ac, 1/cf; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(q/bd; q)_k}{(acq, cfq; q)_k} (-abc^2 df)^k q^{(k)}
\]

(3.9)

Now, (3.9) can be written as

\[
\frac{c}{(1/ad, 1/df; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(bcq^{-k}; q)_\infty (ad^2 f q)^k q^{2(k)}(k)}{(adq, dfq; q)_k}
\]

\[
- \frac{d}{(1/ac, 1/cf; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(bdq^{-k}; q)_\infty (ac^2 f q)^k q^{2(k)}(k)}{(acq, cfq; q)_k}
\]

(3.10)

Next, applying \( E(g\theta) \) to both sides of (3.10) with respect to the parameter \( b \) gives

\[
\frac{c}{(1/ad, 1/df; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(ad^2 f q)^k q^{2(k)}(k)}{(adq, dfq; q)_k} E(g\theta) \{ (bcq^{-k}; q)_\infty \}
\]

\[
- \frac{d}{(1/ac, 1/cf; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(ac^2 f q)^k q^{2(k)}(k)}{(acq, cfq; q)_k} E(g\theta) \{ (bdq^{-k}; q)_\infty \}
\]

(3.11)

From (3.5) and (3.6), it is evident that

\[
E(g\theta) \{ (bcq^{-k}; q)_\infty \} = (bcq^{-k}, cgq^{-k}; q)_\infty,
\]

(3.12)

\[
E(g\theta) \{ (bdq^{-k}; q)_\infty \} = (bdq^{-k}, dgq^{-k}; q)_\infty,
\]

(3.13)
and
\[ E(g\theta) \{abcd, bcdf; q\}_\infty = \frac{(abcd, acdg, bcdf, cdfg; q)_\infty}{(abc^2d^2fg/g; q)_\infty}. \] (3.14)

Substituting (3.12), (3.13), and (3.14) into (3.11), we see that
\[ c(b, cg; q)_\infty \geq \frac{q}{(ad, df; q)_\infty} q_{\psi_2} \left[ \frac{q}{bc}, \frac{q}{cg}, \frac{abc^2d^2fg}{q} \right] \]
\[ - \frac{d(b, dg; q)_\infty}{(ac, cf; q)_\infty} \geq \frac{q}{ac}, \frac{q}{cf}, \frac{abc^2d^2fg}{q} \]
\[ = \frac{acd^2f(q, qd/c, c/d, abcd, acdf, acdg, bcdf, q/acdf, cdfg; q)_\infty}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df, abc^2d^2fg/g; q)_\infty}, \] (3.15)

where \(|abc^2d^2fg/g| < 1\).

Finally, the proof is completed by replacing \(a, b, c, d, g\) by \(c = fq, e, q/ae, f, d/eq, ae/b\), respectively, and then setting \(ae = \alpha\).

Substitute \(a, b, c, d, \alpha\) with \(qa = e, qb = e, qc = e, q^2 = e, fq = e\) in (3.7), respectively, we may recover the original formula (1.13) due to Slater.

If we set \(d = q\) in (3.7), then the second term on the left hand side vanishes, and so we get the \(q\)-Gauss summation (1.4) as a special of (3.7).

To conclude this paper, we represent (3.7) in an equivalent form and give the explicit substitutions to reach Askey’s \(q\)-extension of Cauchy’s beta integral (1.11). By the relation
\[ 2\psi_2 \left[ \frac{a}{c}, \frac{b}{d} ; q, z \right] = 2\psi_2 \left[ \frac{q}{c}, \frac{q}{d} ; q, \frac{cd}{a} \right], \] (3.16)
we may rewrite (3.7) as
\[ \frac{(q/a, q/b; q)_\infty}{(q/c, q/d; q)_\infty} 2\psi_2 \left[ \frac{q}{c}, \frac{q}{d} ; q, q \right] = \frac{(\alpha/a, \alpha/b, \alpha/c; q)_\infty}{(\alpha/c, \alpha/d; q)_\infty} 2\psi_2 \left[ \frac{\alpha}{c}, \frac{\alpha}{d} ; q, q \right] \]
\[ = \frac{(\alpha, q/\alpha, cd/\alphaq, \alpha^2q/cd, q/c, c/a, c/b, d/a, d/b; q)_\infty}{(c/\alpha, \alphaq/c, d/\alpha, \alphaq/d, c, d, q/c, q/d, cd/\alpha; q)_\infty}, \] (3.17)
where \(|cd/\alpha| < 1\). Replacing \(a, b, c, d, \alpha\) by \(q/c, -q/d, q/a, -q/b, q\), respectively, then (3.17) takes the form of Askey’s \(q\)-extension of Cauchy’s beta integral.

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References


