The Ratio Monotonicity of the Boros-Moll Polynomials

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Abstract. In their study of a quartic integral, Boros and Moll discovered a special class of Jacobi polynomials, which we call the Boros-Moll polynomials. Kauers and Paule proved the conjecture of Moll that these polynomials are log-concave. In this paper, we show that the Boros-Moll polynomials possess the ratio monotone property which implies the log-concavity and the spiral property. We conclude with a conjecture which is stronger than Moll's conjecture on the ∞-log-concavity.

Keywords: ratio monotone property, spiral property, unimodality, log-concavity, Jacobi polynomials, Boros-Moll polynomials.

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1 Introduction

In this paper, we aim to show that the Boros-Moll polynomials satisfy the ratio monotone property which implies the log-concavity and the spiral property. Boros and Moll [2–6, 9] explored a special class of Jacobi polynomials in their study of a quartic integral. They have shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \tag{1.1}$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(a+1)^j (a-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

Using the Ramanujan's Master Theorem, Boros and Moll [5,9] derived the following formula for $P_m(a)$:

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k, \tag{1.3}$$

which indicates that the coefficients of a^i in $P_m(a)$ is positive for $0 \le i \le m$. Let $d_i(m)$ be given by

$$P_m(a) = \sum_{i=0}^m d_i(m) a^i. \tag{1.4}$$

The polynomials $P_m(a)$ will be called the Boros-Moll polynomials, and the sequence $\{d_i(m)\}_{0\le i\le m}$ of the coefficients will be called a Boros-Moll sequence. From (1.4), it follows that

$$d_i(m) = 2^{-2m}\sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{1.5}$$

Recall that $P_m(a)$ can be expressed as a hypergeometric function:

$$P_m(a) = 2^{-2m}\binom{2m}{m}{}_2F_1(-m, m+1; \frac{1}{2}-m; \frac{a+1}{2}),$$

from which one sees that $P_m(a)$ can be viewed as the Jacobi polynomial $P_m^{(\alpha,\beta)}(a)$ with $\alpha = m+\frac{1}{2}$ and $\beta = -(m+\frac{1}{2})$, where $P_m^{(\alpha,\beta)}(a)$ is given by

$$P_m^{(\alpha,\beta)}(a) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m+\beta}{m-k}\binom{m+k+\alpha+\beta}{k}\left(\frac{1+a}{2}\right)^k.$$

Boros and Moll [3] proved that the sequence $\{d_i(m)\}_{0\le i\le m}$ is unimodal and the maximum element appears in the middle. In other words,

$$d_0(m) < d_1(m) < \cdots < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]-1}(m) > \cdots > d_m(m).$$

They also established the unimodality by taking a different approach [4]. Moll [9] conjectured that the sequence $\{d_i(m)\}_{0\le i\le m}$ is log-concave. Kauers and Paule [8] proved this conjecture based on recurrence relations found by a computer algebra approach.

Recall that a sequence $\{a_i\}_{0\le i\le m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \le \frac{a_1}{a_2} \le \cdots \le \frac{a_{m-1}}{a_m}.$$

A polynomial is said to be log-concave if the sequence of coefficients is log-concave. It is easy to see that if a sequence is log-concave then it is unimodal. A a sequence $\{a_i\}_{0\le i\le m}$ of positive numbers is said to be spiral if

$$a_m \le a_0 \le a_{m-1} \le a_1 \le \cdots \le a_{[\frac{m}{2}]}.$$

Similarly, a polynomial is said to be spiral if its sequence of coefficients is spiral. It is easily seen that a log-concave sequence is not necessarily spiral, and vice versa. Nevertheless, Chen and Xia [7] discovered that the q-derangement numbers are both spiral and log-concave, and introduced the ratio monotone property which implies both log-concavity and the spiral property. The purpose of this paper is to show that the Boros-Moll polynomials possess the ratio monotone property.

A sequence $\{a_i\}_{0\le i\le m}$ of positive numbers is said to be ratio monotone if

$$\frac{a_0}{a_{m-1}} \le \frac{a_1}{a_{m-2}} \le \cdots \le \frac{a_{i-1}}{a_{m-i}} \le \frac{a_i}{a_{m-1-i}} \le \cdots \le \frac{a_{[\frac{m}{2}]-1}}{a_{m-[\frac{m}{2}]}} \le 1 \tag{1.6}$$

and

$$\frac{a_m}{a_0} \le \frac{a_{m-1}}{a_1} \le \cdots \le \frac{a_{m-i}}{a_i} \le \frac{a_{m-1-i}}{a_{i+1}} \le \cdots \le \frac{a_{m-\left[\frac{m-1}{2}\right]}}{a_{\left[\frac{m-1}{2}\right]}} \le 1. \tag{1.7}$$

If every inequality relation in (1.6) and (1.7) becomes strict, namely, $\le$ can be replaced by $<$, we say that the sequence is strictly ratio monotone. It is easy to see that the ratio monotonicity implies log-concavity. From (1.6) and (1.7), we deduce that

$$\frac{a_i}{a_{i-1}} \ge \frac{a_{m-1-i}}{a_{m-i}} \quad \text{and} \quad \frac{a_{i+1}}{a_i} \le \frac{a_{m-1-i}}{a_{m-i}}.$$

Hence we get

$$\frac{a_i}{a_{i-1}} \ge \frac{a_{i+1}}{a_i}.$$

We will show that the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is strictly ratio monotone. We give an example to demonstrate this property. For $n = 8$, we have

$$\begin{aligned} P_8(a) =& \frac{4023459}{32768} + \frac{3283533}{4096}a + \frac{9804465}{4096}a^2 + \frac{8625375}{2048}a^3 + \frac{9695565}{2048}a^4 \\ &+ \frac{1772199}{512}a^5 + \frac{819819}{512}a^6 + \frac{109395}{256}a^7 + \frac{6435}{128}a^8. \end{aligned}$$

The strictly ratio monotone property is illustrated as follows:

$$\frac{\frac{6435}{128}}{\frac{4023459}{32768}} < \frac{\frac{109395}{256}}{\frac{3283533}{4096}} < \frac{\frac{819819}{512}}{\frac{9804465}{4096}} < \frac{\frac{1772199}{512}}{\frac{8625375}{2048}} < 1,$$

$$\frac{\frac{4023459}{32768}}{\frac{109395}{256}} < \frac{\frac{3283533}{4096}}{\frac{819819}{512}} < \frac{\frac{9804465}{4096}}{\frac{1772199}{512}} < \frac{\frac{8625375}{2048}}{\frac{9695565}{2048}} < 1.$$

The spiral property of $P_8(x)$ is reflected by following order of the coefficients:

$$\begin{aligned} \frac{6435}{128} < \frac{4023459}{32768} < \frac{109395}{256} < \frac{3283533}{4096} < \frac{819819}{512} \\ < \frac{9804465}{4096} < \frac{1772199}{512} < \frac{8625375}{2048} < \frac{9695565}{2048}. \end{aligned}$$

2 Recurrence Relations of Kauers and Paule

We first give a brief review of Kauers and Paule's approach to proving the log-concavity of Boros-Moll sequences [8], because we will use the recurrence relations derived by Kauers and Paule. In fact, they found four recurrence relations for the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ with the RISC package MultiSum [11], which are as follows:

$$d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}d_i(m), \qquad 0\le i\le m+1, \tag{2.1}$$

$$\begin{aligned} d_i(m+1) =& \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)}d_i(m) \\ & - \frac{i(i+1)}{(m+1)(m+1-i)}d_{i+1}(m), \qquad 0\le i\le m, \end{aligned} \tag{2.2}$$

$$\begin{aligned} d_i(m+2) =& \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)}d_i(m+1) \\ & - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)}d_i(m), \qquad 0\le i\le m+1, \end{aligned} \tag{2.3}$$

and for $0\le i\le m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \tag{2.4}$$

Based on the four recurrence relations, Kauers and Paule [8] used a computer algebra approach to derive the following inequality which implies the log-concavity of $d_i(m)$.

Theorem 2.1. *For $0 < i < m$, we have*

$$d_i(m+1) \ge \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}d_i(m). \tag{2.5}$$

The above inequality is also of vital importance for our proof of the ratio monotonicity of the Boros-Moll sequences. In order to establish the strict ratio monotonicity, we need a slightly improved version of the Kauers-Paule inequality, which can be verified by the same techniques.

Theorem 2.2. *Let $m\ge 2$. We have*

$$d_i(m+1) > \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}d_i(m), \qquad 1\le i\le m-1, \tag{2.6}$$

and

$$d_0(m+1) = \frac{4m+3}{2(m+1)}d_0(m), \tag{2.7}$$

$$d_m(m+1) = \frac{(2m+3)(2m+1)}{2(m+1)}d_m(m) = \frac{(2m+3)(2m+1)}{2(m+1)}2^{-m}\binom{2m}{m}. \tag{2.8}$$

To make this paper self-contained, we will present a detailed proof of the above improved version of Theorem 2.1. Before doing so, we remark that (2.3) and (2.4) can be derived from (2.1) and (2.2). Equating the right hand sides of (2.1) and (2.2) and replacing i by $i-1$, we get (2.4). Substituting i with $i+1$ and m with $m+1$ in (2.1) and (2.2), respectively, we obtain two expressions for $d_{i+1}(m+1)$. This yields

$$\begin{aligned}
d_i(m+2) =&\frac{(4m-2i+7)(m+i+2)}{2(m+2)(m+2-i)}d_i(m+1)\\
&-\frac{i(i+1)}{(m+2)(m+2-i)}\left(\frac{m+i+1}{m+1}d_i(m)+\frac{(4m+2i+5)}{2(m+1)}d_{i+1}(m)\right)\\
=&\frac{(4m-2i+7)(m+i+2)}{2(m+2)(m+2-i)}d_i(m+1)-\frac{i(i+1)(m+i+1)}{(m+2)(m+2-i)(m+1)}d_i(m)\\
&-\frac{i(i+1)(4m+2i+5)}{(m+2)(m+2-i)(2m+2)}d_{i+1}(m). \qquad (2.9)
\end{aligned}$$

On the other hand, from (2.2), we have

$$d_{i+1}(m) = -\frac{(m+1)(m+1-i)}{i(i+1)}d_i(m+1)+\frac{(m+i+1)(4m-2i+3)}{2i(i+1)}d_i(m). \qquad (2.10)$$

Substituting (2.10) into (2.9), we obtain (2.3).

We now present a proof of Theorem 2.2.

Proof. Clearly, (2.7) follows from (2.1) by setting $i=0$, and (2.8) can be obtained from (2.2) by setting $i=m$.

We now proceed to prove (2.6) by induction on m. It is easy to verify that (2.6) holds for $m=2$. We assume that (2.6) holds for $n\geq 2$, namely,

$$d_i(n+1) > \frac{4n^2+7n+i+3}{2(n+1-i)(n+1)}d_i(n), \qquad 1\leq i\leq n-1. \qquad (2.11)$$

We aim to show that (2.6) holds for $n+1$, that is,

$$d_i(n+2) > \frac{4(n+1)^2+7(n+1)+i+3}{2(n+2)(n+2-i)}d_i(n+1), \quad 1\leq i\leq n. \qquad (2.12)$$

Observe that for $1\leq i\leq n-1$,

$$\begin{aligned}
&2(n+i+1)(4n+3)(4n+5)(n+1-i)(n+1)-2(4n^2+7n+i+3)\\
&\qquad\times(n+1)(n+1-i)(4n+4i+5) = -4i(1+2i)(n+1)(n+1-i)<0.
\end{aligned}$$

Hence we have for $1\leq i\leq n-1$,

$$\frac{4n^2+7n+i+3}{2(n+1-i)(n+1)} > \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)}. \qquad (2.13)$$

From the inequalities (2.13) and (2.11), we find that for $1 \leq i \leq n-1$,

$$d_i(n+1) > \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)} d_i(n). \tag{2.14}$$

It is easy to check that

$$\frac{\dfrac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - \dfrac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)}} = \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)}.$$

Hence the inequality (2.14) can be rewritten as

$$d_i(n+1) > \frac{\dfrac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - \dfrac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)}} d_i(n).$$

It follows that

$$\begin{aligned} &\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} d_i(n+1) - \frac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)} d_i(n) \\ &\qquad > \frac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)} d_i(n+1). \end{aligned} \tag{2.15}$$

From the recurrence relation (2.3), the left hand side of (2.15) equals $d_i(n+2)$. Thus we have verified the inequality (2.12) for $1 \leq i \leq n-1$. It is still necessary to show that (2.12) is true for $i = n$, that is,

$$d_n(n+2) > \frac{4(n+1)^2+8n+10}{4(n+2)} d_n(n+1). \tag{2.16}$$

Using the formula (1.5), we get

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1},$$

$$d_n(n+2) = \frac{(n+1)(4n^2+18n+21)}{2^{n+4}(2n+3)}\binom{2n+4}{n+2}.$$

It is easily checked that for $n \geq 1$,

$$\frac{d_n(n+2)}{d_n(n+1)} = \frac{(n+1)(4n^2+18n+21)}{2(n+2)(2n+3)} > \frac{4(n+1)^2+8n+10}{4(n+2)}.$$

Hence the proof is complete by induction. ∎

3 The Strictly Ratio Monotone Property

To prove our ratio monotone property, we will establish the some inequalities based on the recurrence relations of Kauers and Paule.

Lemma 3.1. *Let $m \geq 2$ be an integer. Then we have for $1 \leq j \leq m-1$,*

$$\frac{m-j}{j+1} > \frac{d_{j+1}(m)}{d_j(m)}. \tag{3.1}$$

Proof. From (2.2) and Theorem 2.2, we find that for $1 \leq j \leq m-1$,

$$\begin{aligned}(4m-2j+3)(m+j+1)d_j(m)-2j(j+1)d_{j+1}(m) &= 2(m+1-j)(m+1)d_j(m+1)\\ &> (4m^2+7m+j+3)d_j(m),\end{aligned}$$

which implies (3.1). ∎

The following lemma gives an upper bound on the ratio $d_i(m+1)/d_i(m)$, which is crucial for the proof of the main result of this paper.

Lemma 3.2. *Let $m \geq 2$ be a positive integer. We have for $0 \leq i \leq m$,*

$$d_i(m+1) \leq B(m,i)d_i(m), \tag{3.2}$$

where $B(m,i)$ is defined by

$$B(m,i) = \frac{A(m,i)}{2(i+2)(4m+2i+5)(m+1)(m-i+1)} \tag{3.3}$$

with

$$\begin{aligned}A(m,i) =&30+96m^2+94m+37i+72m^2i+8m^2i^2-i^3\\ &+99mi+5i^2+13mi^2+16m^3i+32m^3.\end{aligned} \tag{3.4}$$

Proof. We proceed by induction on m. It is easily seen that the lemma holds for $m=2$. We assume that the lemma is true for $n \geq 2$, i.e.,

$$d_i(n+1) \leq B(n,i)d_i(n), \qquad 0 \leq i \leq n, \tag{3.5}$$

where $B(n,i)$ is defined by (3.3). It will be shown that the lemma holds for $n+1$, that is,

$$d_i(n+2) \leq B(n+1,i)d_i(n+1), \quad 0 \leq i \leq n+1. \tag{3.6}$$

For $0 \leq i \leq n$, let

$$F(n,i) =(4n+2i+9)(i+2)(4n+5)(4n+3)(n+i+1),$$

$$
\begin{aligned}
G(n,i) =& -2(-90-23i-202n+51i^3+60i^2-144n^2-32n^3 \\
&-80n^2i-8n^2i^2-97ni+13ni^2-16n^3i+16ni^3+8i^4)(n+1).
\end{aligned}
$$

We claim that

$$
\frac{F(n,i)}{G(n,i)} \geq B(n,i), \qquad 0 \leq i \leq n. \tag{3.7}
$$

Keeping in mind that $A(n,i)$ is defined by (3.4), it is easy to check that

$$
\begin{aligned}
&2(i+2)(4n+2i+5)(n+1)(n-i+1)F(n,i) - A(n,i)G(n,i) \\
&= (128n^4i^4 - 32n^3i^5 - 80n^2i^6 - 16ni^7) + (618n^3i^4 - 222ni^6 - 16i^7 - 284n^2i^5) \\
&\quad + (844ni^3 - 170i^4) + (1502n^2i^3 - 338i^5) + (984n^2i^4 - 142i^6) \\
&\quad + (844n^3i^3 - 590ni^5) + 256n^5i^2 + 720i + 10i^3 + 788i^2 + 3984n^2i \\
&\quad + 2656ni + 3568ni^2 + 3136n^3i + 4600n^3i^2 + 256n^5i \\
&\quad + 1344n^4i + 324ni^4 + 176n^4i^3 + 5908n^2i^2 + 1728n^4i^2.
\end{aligned}
$$

We are now in a position to see that the above expression is always nonnegative since the expression in every parenthesis is nonnegative for $0 \leq i \leq n$. For example,

$$
128n^4i^4 - 32n^3i^5 - 80n^2i^6 - 16ni^7 \geq 128n^4i^4 - 32n^4i^4 - 80n^4i^4 - 16n^4i^4 = 0.
$$

Thus we have

$$
2(i+2)(4n+2i+5)(n+1)(n-i+1)F(n,i) - A(n,i)G(n,i) \geq 0. \tag{3.8}
$$

It is easy to see that $G(n,i)$ is positive for $0 \leq i \leq n$, and hence (3.7) can be deduced from (3.8). From the inductive hypothesis (3.5) and (3.8), it follows that for $0 \leq i \leq n$,

$$
\frac{F(n,i)}{G(n,i)} d_i(n) \geq B(n,i) d_i(n) \geq d_i(n+1). \tag{3.9}
$$

It is a routine to verify that

$$
\frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)\left(\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - B(n+1,i)\right)} = \frac{F(n,i)}{G(n,i)}.
$$

From the above identity and (3.9), it follows that for $0 \leq i \leq n$,

$$
\frac{(n+1+i)(4n+3)(4n+5)d_i(n)}{4(n+1)(n+2)(n+2-i)\left(\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - B(n+1,i)\right)}
$$

$$= \frac{F(n,i)}{G(n,i)} d_i(n) \geq d_i(n+1). \tag{3.10}$$

Since

$$\frac{-4i^2 + 8n^2 + 24n + 19}{2(n+2-i)(n+2)} - B(n+1,i)$$

is positive for $0 \leq i \leq n$, (3.10) can be rewritten as

$$\begin{aligned} &\frac{-4i^2 + 8n^2 + 24n + 19}{2(n+2-i)(n+2)} d_i(n+1) \\ &\qquad - \frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)} d_i(n) \leq B(n+1,i) d_i(n+1). \end{aligned} \tag{3.11}$$

From the recurrence relation (2.3), we see that

$$\frac{-4i^2 + 8n^2 + 24n + 19}{2(n+2-i)(n+2)} d_i(n+1) - \frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)} d_i(n) = d_i(n+2). \tag{3.12}$$

In view of (3.11) and (3.12), we find that the inequality (3.6) holds for $0 \leq i \leq n$. It remains to verify that (3.6) holds for $i = n+1$, that is,

$$d_{n+1}(n+2) \leq B(n+1, n+1) d_{n+1}(n+1). \tag{3.13}$$

By the definition (3.3) of $B(n,i)$, we have

$$B(n+1, n+1) = \frac{501 + 212n^3 + 692n^2 + 975n + 24n^4}{2(n+3)(6n+11)(n+2)}.$$

From the formula (1.5) for $d_i(m)$, we get

$$d_{n+1}(n+1) = 2^{-n-1} \binom{2n+2}{n+1}$$

and

$$d_{n+1}(n+2) = 2^{-n-2} \binom{2n+3}{n+1} + 2^{-n-2}(n+2) \binom{2n+4}{n+2}.$$

Therefore, for $n \geq 0$, we have

$$\frac{d_{n+1}(n+2)}{d_{n+1}(n+1)} = \frac{(2n+3)(2n+5)}{2(n+2)} \leq \frac{501 + 212n^3 + 692n^2 + 975n + 24n^4}{2(n+3)(6n+11)(n+2)}.$$

This completes the proof of the lemma. ∎

Lemma 3.3. *Let $B(m,j)$ be defined by (3.3) and $m \geq 2$ be an integer. Then we have for $1 \leq j \leq m$,*

$$d_{j-1}(m) \leq \frac{2(m+1)B(m,j) - (4m+2j+3)}{2(m+j)} d_j(m). \tag{3.14}$$

Proof. From the recurrence relation (2.1) and Lemma 3.2, we find that for $0 \le j \le m$,

$$\begin{aligned} 2(m+1)d_j(m+1) &= 2(m+j)d_{j-1}(m) + (4m+2j+3)d_j(m) \\ &\le 2(m+1)B(m,j)d_j(m), \end{aligned} \tag{3.15}$$

where $B(m,j)$ is defined by (3.3). Then (3.15) implies (3.14). ∎

Lemma 3.4. *Let m be a positive integer. For $0 \le i \le \frac{m}{2}$, we have*

$$\frac{2(2m-i)}{2(m+1)B(m,m-i)-(6m-2i+3)} > \frac{2(m+1)B(m,i)-(4m+2i+3)}{2(m+i)}, \tag{3.16}$$

where $B(m,i)$ is defined by (3.3).

Proof. For $0 \le i \le m$, let

$$N(m,i) = 2(2m-i)(m-i+2)(6m-2i+5)(i+1), \tag{3.17}$$

$$\begin{aligned} M(m,i) &= 4(3m-i)(2m-i)(m-i)^2 + (80m^3 - 155m^2 i) + (80m^2 - 108mi) \\ &\quad + (20m - 20i) + (94mi^2 - 19i^3) + 28i^2, \end{aligned} \tag{3.18}$$

$$C(m,i) = i(24m^2 + 52m + 8m^2 i + 37mi + 4i^3 + 12mi^2 + 20 + 19i^2 + 28i), \tag{3.19}$$

$$D(m,i) = 2(i+2)(4m+2i+5)(m-i+1)(i+m). \tag{3.20}$$

Note that $N(m,i), M(m,i), C(m,i)$ and $D(m,i)$ are all nonnegative for $0 \le i \le \frac{m}{2}$, since the sum in every parenthesis in (3.17), (3.18), (3.19) and (3.20) is nonnegative for $0 \le i \le \frac{m}{2}$. It is easy to check that

$$\begin{aligned} & N(m,i)D(m,i) - C(m,i)M(m,i) \\ =& (312m^5 i^2 + 36m^2 i^5 + 276m^3 i^4 - 612m^4 i^3 - 12mi^6) + (2040m^4 i^2 - 2533m^3 i^3) \\ & + (129mi^5 - 43i^6) + (384m^6 - 752m^5 i) + (3568m^4 - 3328m^3 i) \\ & + (1952m^5 - 2792m^4 i) + (4280m^3 i^2 - 2976m^2 i^3) + (2800m^3 - 1240m^2 i) \\ & + (3868m^2 i^2 - 1080mi^3) + 1240mi^2 + 1488mi^4 + 540i^4 + 800m^2 + 1159m^2 i^4. \end{aligned}$$

Observe that the expression in every parenthesis in the above sum is nonnegative for $0 \le i \le \frac{m}{2}$. Moreover, one sees the term $800m^2$ is certainly positive. It follows that

$$N(m,i)D(m,i) - C(m,i)M(m,i) > 0, \qquad 0 \le i \le \frac{m}{2}. \tag{3.21}$$

Recall that $B(n,i)$ is defined by (3.3). It is easy to check that

$$\frac{2(m+1)B(m,i)-(4m+2i+3)}{2(m+i)} = \frac{C(m,i)}{D(m,i)},$$

$$\frac{2(2m-i)}{2(m+1)B(m,m-i)-(6m-2i+3)}=\frac{N(m,i)}{M(m,i)}.$$

Thus the inequality (3.21) is equivalent to (3.16). This completes the proof of the lemma. ∎

The main result of this paper is stated as follows.

Theorem 3.5. *Let $m\geq 2$ be an integer. Then the Boros-Moll sequence $\{d_i(m)\}_{0\leq i\leq m}$ satisfies the strictly ratio monotone property. To be precise, we have*

$$\frac{d_m(m)}{d_0(m)}<\frac{d_{m-1}(m)}{d_1(m)}<\cdots<\frac{d_{m-i}(m)}{d_i(m)}<\frac{d_{m-i-1}(m)}{d_{i+1}(m)}<\cdots<\frac{d_{m-[\frac{m-1}{2}]}(m)}{d_{[\frac{m-1}{2}]}(m)}<1 \quad (3.22)$$

and

$$\frac{d_0(m)}{d_{m-1}(m)}<\frac{d_1(m)}{d_{m-2}(m)}<\cdots<\frac{d_{i-1}(m)}{d_{m-i}(m)}<\frac{d_i(m)}{d_{m-i-1}(m)}<\cdots<\frac{d_{[\frac{m}{2}]-1}(m)}{d_{m-[\frac{m}{2}]}(m)}<1. \quad (3.23)$$

Proof. It is clear that Theorem 3.5 holds for $m=2,3,4$. We now assume that $m\geq 5$. First we consider (3.22). In order to verify

$$\frac{d_m(m)}{d_0(m)}<\frac{d_{m-1}(m)}{d_1(m)}, \quad (3.24)$$

we invoke the formula (1.5) to get

$$\frac{d_1(m)}{d_0(m)}=\frac{2^{-2m}\sum_{k=1}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{m}k}{2^{-2m}\sum_{k=0}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{m}}<\frac{\sum_{k=1}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{m}m}{\sum_{k=1}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{m}}=m, \quad (3.25)$$

and

$$\frac{d_{m-1}(m)}{d_m(m)}=\frac{2^{-m}\binom{2m-1}{m}+2^{-m}\binom{2m}{m}m}{2^{-m}\binom{2m}{m}}>m. \quad (3.26)$$

Combining (3.25) and (3.26), we obtain

$$\frac{d_1(m)}{d_0(m)}<\frac{d_{m-1}(m)}{d_m(m)},$$

which yields (3.24).

The next step is to show that

$$\frac{d_{m-i}(m)}{d_i(m)}<\frac{d_{m-i-1}(m)}{d_{i+1}(m)}, \qquad 1\leq i\leq\left[\frac{m-1}{2}\right]-1. \quad (3.27)$$

By the assumption $m \geq 5$, we have $\left[\frac{m-1}{2}\right] - 1 \geq 1$. Substituting j with i in (3.1), we have for $1 \leq i \leq \left[\frac{m-1}{2}\right] - 1$,

$$\frac{d_{i+1}(m)}{d_i(m)} < \frac{m-i}{i+1}. \tag{3.28}$$

On the other hand, since $1 \leq i \leq \left[\frac{m-1}{2}\right] - 1$, we have $m - \left[\frac{m-1}{2}\right] \leq m - i - 1 \leq m - 2$. Hence we may substitute j with $m - i - 1$ in (3.1) to deduce that

$$\frac{d_{m-i-1}(m)}{d_{m-i}(m)} > \frac{m-i}{i+1}. \tag{3.29}$$

From (3.28) and (3.29), it follows that for $1 \leq i \leq \left[\frac{m-1}{2}\right] - 1$,

$$\frac{d_{i+1}(m)}{d_i(m)} < \frac{m-i}{i+1} < \frac{d_{m-i-1}(m)}{d_{m-i}(m)}.$$

Hence we have verified (3.27).

It remains to show that the last ratio in (3.22) is smaller than 1. Since $\left[\frac{m-1}{2}\right] < m - \left[\frac{m-1}{2}\right]$, it is easily seen that for $m - \left[\frac{m-1}{2}\right] \leq k \leq m$, we have

$$\binom{k}{\left[\frac{m-1}{2}\right]} \geq \binom{k}{m - \left[\frac{m-1}{2}\right]}.$$

Based on the formula (1.5) and the above relation, we obtain that

$$\begin{aligned}
d_{\left[\frac{m-1}{2}\right]}(m) &= 2^{-2m} \sum_{k=\left[\frac{m-1}{2}\right]}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{\left[\frac{m-1}{2}\right]} \\
&> 2^{-2m} \sum_{k=m-\left[\frac{m-1}{2}\right]}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{\left[\frac{m-1}{2}\right]} \\
&\geq 2^{-2m} \sum_{k=m-\left[\frac{m-1}{2}\right]}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{m-\left[\frac{m-1}{2}\right]} \\
&= d_{m-\left[\frac{m-1}{2}\right]}(m),
\end{aligned}$$

leading to the relation

$$\frac{d_{m-\left[\frac{m-1}{2}\right]}(m)}{d_{\left[\frac{m-1}{2}\right]}(m)} < 1.$$

This completes the proof of (3.22).

We now turn our attention to the proof (3.23), which will rely on the bound $B(n,i)$ and Lemmas 3.3 and 3.4. First, rewrite (3.14) as

$$\frac{d_{i-1}(m)}{d_i(m)} \leq \frac{2(m+1)B(m,i)-(4m+2i+3)}{2(m+i)}, \quad 1 \leq i \leq m. \tag{3.30}$$

For $1 \leq i \leq \left[\frac{m}{2}\right]$, we have $m - \left[\frac{m}{2}\right] \leq m-i \leq m-1$. It follows that

$$\begin{aligned} & 2(m+1)B(m,j)-(4m+2j+3) \\ =& \frac{j(24m^2+8m^2j+52m+37mj+19j^2+28j+20+12mj^2+4j^3)}{(j+2)(4m+2j+5)(m-j+1)}, \end{aligned} \tag{3.31}$$

which is positive for $1 \leq j \leq m$. Substituting j with $m-i$ in (3.31), we obtain that

$$2(m+1)B(m,m-i)-(6m-2i+3) > 0, \qquad 1 \leq i \leq \left[\frac{m}{2}\right].$$

Hence we can substitute j with $m-i$ in (3.14) to deduce that for $1 \leq i \leq \left[\frac{m}{2}\right]$,

$$\frac{d_{m-i}(m)}{d_{m-i-1}(m)} \geq \frac{2(2m-i)}{2(m+1)B(m,m-i)-(6m-2i+3)}. \tag{3.32}$$

Combining (3.30), (3.32) and Lemma 3.4, we obtain that for $1 \leq i \leq \left[\frac{m}{2}\right]$,

$$\frac{d_{i-1}(m)}{d_i(m)} < \frac{d_{m-i}(m)}{d_{m-i-1}(m)},$$

which can be restated as

$$\frac{d_{i-1}(m)}{d_{m-i}(m)} < \frac{d_i(m)}{d_{m-i-1}(m)}, \qquad 1 \leq i \leq \left[\frac{m}{2}\right]. \tag{3.33}$$

At this point, it is necessary to show that

$$\frac{d_{\left[\frac{m}{2}\right]-1}(m)}{d_{m-\left[\frac{m}{2}\right]}(m)} < 1. \tag{3.34}$$

For $i = \left[\frac{m}{2}\right]$, (3.33) becomes

$$\frac{d_{\left[\frac{m}{2}\right]-1}(m)}{d_{m-\left[\frac{m}{2}\right]}(m)} < \frac{d_{\left[\frac{m}{2}\right]}(m)}{d_{m-\left[\frac{m}{2}\right]-1}(m)}. \tag{3.35}$$

When m is even, we have $\left[\frac{m}{2}\right] = m - \left[\frac{m}{2}\right]$. From (3.35) it follows that

$$\frac{d_{\left[\frac{m}{2}\right]-1}(m)}{d_{m-\left[\frac{m}{2}\right]}(m)} < \frac{d_{m-\left[\frac{m}{2}\right]}(m)}{d_{\left[\frac{m}{2}\right]-1}(m)},$$

which implies (3.34). When m is odd, we have $\left[\frac{m}{2}\right] = m - \left[\frac{m}{2}\right] - 1$. Then (3.34) immediately follows from (3.35). This completes the proof of Theorem 3.5. ∎

As a corollary of Theorem 3.5, we obtain the spiral property of the Boros-Moll sequences.

Corollary 3.6. *Let m be a positive integer, then the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is spiral.*

We remark that the proof of Theorem 3.5 does not explain how the expression (3.3) for $B(m,i)$ is derived. In fact, it has been found by a heuristic approach by considering an approximate equation and by adjusting the coefficients. It would be interesting to find a proof without guessing a formula for $B(m,i)$.

4 A Conjecture

Moll made a conjecture on a property of the Boros-Moll sequences which is stronger than the log-concavity. Given a sequence $A=\{a_i\}_{0\le i\le n}$, define the operator $\mathcal{L}$ by $\mathcal{L}(A)=S=\{b_i\}_{0\le i\le n}$, where

$$b_i=a_i^2-a_{i-1}a_{i+1},\qquad 0\le i\le n,$$

with the convention that $a_{-1}=a_{n+1}=0$. We say that $\{a_i\}_{0\le i\le n}$ is k-log-concave if $\mathcal{L}^j\left(\{a_i\}_{0\le i\le n}\right)$ is log-concave for every $0\le j\le k-1$, and that $\{a_i\}_{0\le i\le n}$ is ∞-log-concave if $\mathcal{L}^k\left(\{a_i\}_{0\le i\le n}\right)$ is log-concave for every $k\ge 0$. Similarly, we say that $\{a_i\}_{0\le i\le n}$ is j-ratio-monotone (resp. j-strictly-ratio-monotone) if $\mathcal{L}^k\left(\{a_i\}_{0\le i\le n}\right)$ is ratio monotone (resp. strictly ratio monotone) for every $0\le k\le j-1$, and that $\{a_i\}_{0\le i\le n}$ is ∞-ratio-monotone (resp. ∞-strictly-ratio-monotone) if $\mathcal{L}^k\left(\{a_i\}_{0\le i\le n}\right)$ is ratio monotone (resp. strictly ratio monotone) for every $k\ge 0$.

Moll [9] has conjectured that the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is ∞-log-concave. We propose a stronger conjecture.

Conjecture 4.1. *Suppose that $m\ge 2$ is a positive integer, then the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is ∞-strictly-ratio-monotone.*

We have verified that the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is 2-strictly-ratio-monotone for $2\le m\le 100$. For example, $\mathcal{L}\left(\{d_i(8)\}_{0\le i\le 8}\right)$ is given by

$$b_0=\frac{16188222324681}{1073741824},\qquad b_1=\frac{46804848752277}{134217728},\qquad b_2=\frac{39484127036475}{16777216},$$
$$b_3=\frac{53734360083525}{8388608},\qquad b_4=\frac{32860456870725}{4194304},\qquad b_5=\frac{4614148779669}{1048576},$$
$$b_6=\frac{284363773551}{262144},\qquad b_7=\frac{836466345}{8192},\qquad b_8=\frac{41409225}{16384}.$$

It is easy to verify that

$$\frac{b_8}{b_0}<\frac{b_7}{b_1}<\frac{b_6}{b_2}<\frac{b_5}{b_3}<1,\qquad \frac{b_0}{b_7}<\frac{b_1}{b_6}<\frac{b_2}{b_5}<\frac{b_3}{b_4}<1.$$

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