Referee's report for "Labeled Partitions and the q-derangement numbers" by Chen and Xu

In this paper, the authors present a decomposition theorem on partitions, which is used to give a new proof of a formula of Wachs on the major index enumerator of permutations with a given derangement part. I have looked at Wachs' proof and do not agree with the authors view that their proof is simpler. However I do find that the new proof provides an interesting alternative approach, which could make the paper acceptable for publication. In particular, the decomposition theorem (Theorem 2.2) is quite appealing.

The following issues need to be addressed before I can recommend acceptance:

(1) Page 3, line 1: The mention of "9 cases" might give the impression that Wachs' proof is long and complicated, when in fact, the 9 cases comprise a short and simple proof of a lemma on descent sets, which could have been left to the reader. The reference to "9 cases" does not appropriately describe the difference in the proofs, nor does it provide a rationale for publication, and should be omitted. The essence of the Wachs proof is a descent set preserving bijection between the set of permutations with a fixed derangement part and the set of shuffles of two fixed permutations. An old formula of Garsia and Gessel analogous to (1.4) with shuffles playing the role of permutations with a fixed derangement part is then applied. On the other hand, the main contribution of the Chen-Xu proof is the direct usage of MacMahon's partition technique, which was also used by Garsia and Gessel in proving their shuffle result.

(2) More care needs to be given to the descriptions of the critical bijections of Lemma 2.1 and Theorem 2.2. In Lemma 2.1, is ψ_π taking the partition λ to the partition μ or is it taking the partition λ to the standard labeled partition (μ, π)? In the first part of the statement of Lemma 2.1 it's the former and in the second part of the statement it's the latter. In the first line of page 5, ψ has yet another interpretation; it takes labeled partitions to standard labeled partitions. In Theorem 2.2 the domain and codomain should be explicitly given. With clear and consistent descriptions of these bijections, the proof of (1.5) would be easier to follow. In fact, the claim on line 2 of page 5 is not quite correct. The image of the composition should be the pair (β, γ) such that (β, σ) is a *standard* labeled partition and γ is a partition with at most $n - k$ parts.

(3) Proof of Theorem 2.2: More care and details are needed here; specifically:
   (a) I believe there is a problem with the definition of s given in paragraph 3. It seems to me that s is not uniquely defined. For example if $(\beta, \sigma) = (855331, 315264)$ and $\gamma = 8$ then for $i = 1$ we have $r = 1$ and $t = 2$. Two values of s satisfy $\pi^0_{s-1} < s \leq \pi^0_s$; namely $s = 1$ and $s = 5$.
   (b) In the last sentence of paragraph 3, the primes haven't been defined.
   (c) Line -2 of the proof: I don't see why $\sigma_f \geq \pi_{f+1} - 1$. Suppose $f + 1$ is not a fixed point of π. Then $\sigma_f = \pi_{f+1} - m$ where m is the number of fixed points of π that are less than π_{f+1}. How do we know that there can't be two such fixed points, in which case, the inequality $\sigma_f \geq \pi_{f+1} - 1$ would be false? Am I missing something easy here?

Typos:
Page 3, line -14: Sentence should be deleted.
Page 4, line 6: μ should be $|\mu|$
Page 5, line 4: λ should be $|\lambda|$
Page 5, line 10: π should be π_i