Labeled Partitions and the q-Derangement Numbers

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Abstract. By a re-examination of MacMahon's original proof of his celebrated theorem on the distribution of the major indices over permutations, we give a reformulation of his argument in terms of the structure of labeled partitions. In this framework, we are able to establish a decomposition theorem for labeled partitions that leads to a simple bijective proof of Wachs' formula on the q-derangement numbers.

1 Introduction

We will follow the terminology and notation on permutations and partitions and q-series in Andrews [2] and Stanley [10]. The set of permutations on $\{1, 2, \ldots, n\}$ is denoted by S_n. For any permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, an index i with $1 \le i \le n-1$ is called a *descent* of π if $\pi_i > \pi_{i+1}$. The major index $\mathrm{maj}(\pi)$ of π, introduced by MacMahon [9], is defined as the sum of all descents of π. The following formula is well-known:

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [n]! = 1 \cdot (1+q) \cdot (1+q+q^2) \cdots (1+q+\cdots+q^{n-1}). \qquad (1.1)$$

The underlying idea of MacMahon' proof goes as follows. It is easier to consider sequences and partitions than solely permutations for the purpose of studying the major index. MacMahon established (1.1) by proving an equivalent formula

$$\frac{1}{(q)_n} \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = \frac{1}{(1-q)^n}, \qquad (1.2)$$

where $(q)_n = (1-q)\cdots(1-q^{n-1})$, and $(q)_n^{-1}$ is the generating function for partitions with at most n parts. We will give a reformulation of MacMahon's proof in Section 2 by introducing the notion of standard labeled partitions.

The main objective of this paper is to employ MacMahon's method to deal with the major index of derangements. An integer i with $1 \le i \le n$ is said to be a *fixed point* of $\pi \in S_n$ if $\pi_i = i$, and *derangement point* otherwise. *Derangements* are

permutations with no fixed points. Let D_n be the set of all derangements in S_n. The q-derangement numbers are defined by $d_0(q) = 1$ and for $n \geq 1$

$$d_n(q) = \sum_{\pi \in D_n} q^{\mathrm{maj}(\pi)}.$$

The following elegant formula was first derived by Gessel in his manuscript and was published in [6] as a consequence of the quasi-symmetric generating function encoding the descents and the cycle structure of permutations. A combinatorial proof has been obtained by Wachs [12]:

$$d_n(q) = [n]! \sum_{k=0}^{n} \frac{(-1)^k}{[k]!} q^{\binom{k}{2}}. \tag{1.3}$$

Let us review the combinatorial settings of Wachs for the above formula. Suppose the derangement points of π are $p_1, p_2, \cdots, p_k$. The reduction of π to its *derangement part*, denoted by $dp(\pi)$, is defined as a permutation on $\{1, 2, \cdots, k\}$ induced by the relative order of $\pi_{p_1}, \pi_{p_2}, \cdots, \pi_{p_k}$. For example, the derangement points of $\pi = (1, 5, 3, 7, 6, 2, 9, 8, 4)$ are $2, 4, 5, 6, 7, 9$, and $\pi_2 \pi_4 \pi_5 \pi_6 \pi_7 \pi_9 = (5, 7, 6, 2, 9, 4)$. Then $dp(\pi) = (3, 5, 4, 1, 6, 2)$. Clearly $dp(\pi) \in D_k$ if π has k derangement points. On the other hand, we can insert a fixed point j with $1 \leq j \leq k+1$ into $\pi \in S_k$ to obtain a permutation

$$\bar{\pi} = \pi_1' \pi_2' \cdots \pi_{j-1}' j \, \pi_j' \cdots \pi_k' \in S_{k+1},$$

where $\pi_i' = \pi_i$ if $\pi_i < j$ and $\pi_i' = \pi_i + 1$ if $\pi_i \geq j$. Such an insertion operation produces one extra fixed point.

Wachs [12] has established the following relation.

Theorem 1.1. *Let* $\sigma \in D_k$ *and* $k \leq n$. *Then we have*

$$\sum_{\substack{dp(\pi)=\sigma \\ \pi \in S_n}} q^{\mathrm{maj}(\pi)} = q^{\mathrm{maj}(\sigma)} \begin{bmatrix} n \\ k \end{bmatrix}, \tag{1.4}$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ *is the* q-*binomial coefficient.*

By summing over all derangements $\sigma \in D_k$ and then summing over all k for the above relation (1.4), and applying (1.1) gives

$$[n]! = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} d_k(q).$$

Thus (1.3) follows from the q-binomial inversion [1, Corollary 3.38],

In order to justify the relation (1.4), Wachs found a bijection on S_n by rearranging a permutation π according to *excedant* (where $\pi_i > i$), fixed point, and *subcedant* (where $\pi_i < i$). She showed that this bijection preserves the major index

by considering 9 cases. Then a result of Garsia-Gessel [4, Theorem 3.1] on shuffles of permutations is applied to establish Theorem 1.1.

Inspired by MacMahon's proof of (1.1), we find it much easier to deal with an equivalent form of (1.4):

$$\frac{1}{(q)_n} \sum_{\substack{dp(\pi)=\sigma \\ \pi \in S_n}} q^{\mathrm{maj}(\pi)} = \frac{1}{(q)_k(q)_{n-k}} q^{\mathrm{maj}(\sigma)}. \tag{1.5}$$

We will use the terminology of labeled partitions and will introduce the notion of standard labeled partitions. In such terms, MacMahon's proof can be easily stated. Moreover, a combinatorial reasoning of (1.5) becomes quite natural, which is analogous to the decomposition of a permutation by separating the derangements from the fixed points.

2   Labeled Partitions

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We say that λ is a partition with at most n parts. We write $|\lambda| = \lambda_1 + \cdots + \lambda_n$. A *labeled partition* is defined as a pair (λ, π) of a partition λ and a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$. A labeled partition is also represented in the following two row form as in Andrews [2, p. 43]:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}.$$

A labeled partition (λ, π) is said to be *standard* if $\pi_i > \pi_{i+1}$ implies $\lambda_i > \lambda_{i+1}$. For example, the labeled partition in (2.1) is standard.

A labeled partition (λ, π) is standard if $\lambda_i = \lambda_{i+1}$ implies $\lambda_i < \lambda_{i+1}$.

The following Lemma 2.1 is straightforward to verify, which is MacMahon's approach to study the major index with the aid of partitions, see MacMahon [9], Andrews [2, Theorem 3.7], Knuth [8, p. 18] or [7]. This method was further extended by Stanley [11]. For other applications, see [4].

Lemma 2.1. *Given $\pi \in S_n$, there is a bijection $\psi_\pi \colon \lambda \mapsto \mu$ from partitions λ with at most n parts to standard labeled partitions (μ, π) such that $|\lambda| + \mathrm{maj}(\pi) = |\mu|$.*

The bijection ψ_π (or simply ψ when π is understood from the context), is given as follows:

$$\mu = \psi_\pi(\lambda) = (\lambda_1 + \phi_1, \lambda_2 + \phi_2, \cdots, \lambda_n + \phi_n),$$

where ϕ_i is the number of descents in $\pi_i\pi_{i+1} \cdots \pi_n$. One may also view ψ as the operation of adding 1 to $\lambda_1, \ldots, \lambda_i$ whenever i is a descent of π.

We now give a restatement of MacMahon's proof of (1.2) in the above terminology of labeled partitions.

Proof of (1.2). Given a sequence $a_1 a_2 \cdots a_n$ of nonnegative integers, we associate it with a weight $q^{a_1 + a_2 + \cdots + a_n}$. Let us construct a two row array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

By permuting the columns of the above array, one can get a unique standard labeled partition (μ, π) with $|\mu| = a_1 + a_2 + \cdots + a_n$. Applying Lemma 2.1, we obtain a partition λ with $|\lambda| + \mathrm{maj}(\pi) = \mu$. Clearly, the above steps are reversible. This completes the proof. $\square$

An Example. Let $a_1 a_2 \ldots a_9$ be the sequence with a two line array

$$\begin{pmatrix} 3 & 6 & 8 & 3 & 1 & 3 & 6 & 4 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}.$$

Permuting the columns we get the a standard labeled partition:

$$\begin{pmatrix} \mu \\ \pi \end{pmatrix} = \begin{pmatrix} 8 & 8 & 6 & 6 & 4 & 3 & 3 & 3 & 1 \\ 3 & \underline{9} & 2 & 7 & \underline{8} & 1 & 4 & \underline{6} & 5 \end{pmatrix}, \tag{2.1}$$

where we have underlined the descents of π.

Applying ψ^{-1} gives

$$\begin{pmatrix} \lambda \\ \pi \end{pmatrix} = \begin{pmatrix} 5 & 5 & 4 & 4 & 2 & 2 & 2 & 2 & 1 \\ 3 & \underline{9} & 2 & 7 & \underline{8} & 1 & 4 & \underline{6} & 5 \end{pmatrix}.$$

We remark that the idea of standard labeled partitions appeared in [4, p. 292], though it was not used to prove (1.2).

We now come to the main result of this note, which is a decomposition theorem on standard labeled partitions in terms of the fixed points. Let $\begin{pmatrix} \mu \\ \pi \end{pmatrix}$ be a standard labeled partition with $\pi \in S_n$. Assume that π has $n - k$ fixed points. Let $i_1 < i_2 < \cdots < i_{n-k}$ be the fixed points, let $j_1 < j_2 < \cdots < j_k$ be the derangement points of π, and let $dp(\pi) = \sigma \in D_k$. We now define the following decomposition of a standard labeled partition:

$$\varphi \colon \begin{pmatrix} \mu \\ \pi \end{pmatrix} \mapsto (\beta, \gamma), \tag{2.2}$$

where $\beta = \mu_{j_1} \mu_{j_2} \cdots \mu_{j_k}$ and $\gamma = \mu_{i_1} \mu_{i_2} \cdots \mu_{i_{n-k}}$ are the partitions corresponding to derangement points and fixed points, respectively. Evidently $|\mu| = |\beta| + |\gamma|$.

The following is the main theorem of this paper.

Theorem 2.2. *For given $\sigma \in D_k$, the decomposition φ of $\begin{pmatrix} \mu \\ \pi \end{pmatrix}$ with $dp(\pi) = \sigma$ is a bijection from standard labeled partitions to pairs of partitions such that (β, σ) is a standard labeled partition.*

We note that the above theorem and Lemma 2.1 lead to a combinatorial interpretation of the relation (1.5). Since (β, σ) is a standard labeled partition, we may

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find a partition α such that $\psi\binom{\alpha}{\sigma} = \binom{\beta}{\sigma}$. Consequently, the bijection $\varphi \circ \psi$ maps a labeled partition $\binom{\lambda}{\pi}$ to a pair (α, γ) of partitions, where α has at most k parts and γ has at most $n - k$ parts. Moreover, the following relation holds:

$$\lambda + \mathrm{maj}(\pi) = |\alpha| + |\gamma| + \mathrm{maj}(\sigma), \tag{2.3}$$

which implies (1.5).

Proof of Theorem 2.2. We first show that (β, σ) is standard. It suffices to show that if $\pi_i > \pi_j$ with $\pi_{i+1}, \ldots, \pi_{j-1}$ being fixed points, then $\mu_i > \mu_j$. If $j = i+1$, since $\binom{\mu}{\pi}$ is standard, we have $\mu_i > \mu_j$. We now consider the case $i < j - 1$, and we claim that either $\pi_i > \pi_{i+1} = i+1$ or $\pi_{j-1} = j - 1 > \pi_j$ holds; Otherwise, it follows that $\pi < i + 1 \leq j - 1 < \pi_j$, a contradiction. Therefore, we have either $\mu_i > \mu_{i+1}$ or $\mu_{j-1} > \mu_j$. It is deduced that $\mu_i > \mu_j$.

We now proceed to construct the map φ' which is guided by the procedure of inserting the fixed points of π to the derangement σ on $\{1, 2, \ldots, k\}$. We will show that φ' and φ are inverse to each other, which implies that φ is a bijection.

Let $(\mu^0, \pi^0) = (\beta, \sigma)$. We assume that (μ^i, π^i) is obtained from (μ^{i-1}, π^{i-1}) by *inserting* γ_i. We find the first position r so that the insertion of γ_i at the proper position produces a partition. This partition is denoted μ^i. Clearly, $\mu^i_{r-1} > \mu^i_r = \gamma_i$. Assume that $\mu^i_r = \cdots = \mu^i_t > \mu^i_{t+1}$ for some $t \geq r$. If $r = t$ then we set $s = r$. Otherwise we find the position s such that $\pi^{i-1}_{s-1} < s \leq \pi^{i-1}_s$, (here we have taken π^{i-1}_{r-1} as $-\infty$ and π^{i-1}_t as ∞). Now insert s as a fixed point into π^{i-1} to generate π^i. Note that the position s is judiciously chosen so that the subsequence $\pi^i_r, \pi^i_{r+1}, \cdots, \pi^i_t$, which is the same as $\pi^{i-1'}_r, \cdots, \pi^{i-1'}_{s-1}, s, \pi^{i-1'}_s, \cdots, \pi^{i-1'}_{t-1}$, is increasing, and hence π^i is a standard labeled partition.

Since μ^{n-k} is the partition obtained from β by inserting $\gamma_1, \ldots, \gamma_{n-k}$, we must have $\mu^{n-k} = \mu$. From the above procedure, one sees that π^{n-k} is constructed from $\pi_0 = \sigma$ by inserting fixed points, therefore we have $dp(\pi^{n-k}) = \sigma$. It follows that for a given $\sigma \in D_k$, we have $\varphi\varphi'(\beta, \gamma) = (\beta, \gamma)$.

Now it is only necessary to show that $\pi^{n-k} = \pi$. For simplicity, we write π^{n-k} as $\bar{\pi}$. We prove by contradiction. By removing same fixed points, we may assume that the first fixed point f of π is different from the first fixed point $\bar{f}$ of $\bar{\pi}$. Furthermore, we may assume that $f < \bar{f}$. Clearly, $\mu_f = \mu_{\bar{f}}$. Since (μ, π) and $(\mu, \bar{\pi})$ are standard labeled partitions, we have

$$\pi_f < \pi_{f+1} < \cdots < \pi_{\bar{f}}, \text{ and } \bar{\pi}_f < \bar{\pi}_{f+1} < \cdots < \bar{\pi}_{\bar{f}}.$$

So we get $\bar{\pi}_f = \sigma_f \geq \pi_{f+1} - 1 \geq \pi_f = f$. By assumption, f is not a fixed point of $\bar{\pi}$. It follows that $\bar{\pi}_f > f$. Hence $\bar{\pi}_{\bar{f}} > \bar{f}$, a contradiction. $\square$

An Example.

Let $\binom{\lambda}{\pi} = \begin{pmatrix} 5 & 4 & 4 & 4 & 4 & 3 & 2 \\ \underline{5} & \underline{2} & 1 & 4 & \underline{7} & 3 & 6 \end{pmatrix}$. Applying ψ, we get $\binom{\mu}{\pi} = \begin{pmatrix} 8 & 6 & 5 & 5 & 5 & 3 & 2 \\ 5 & 2 & 1 & 4 & 7 & 3 & 6 \end{pmatrix}$.

The fixed points of π are $2, 4$. Hence $\sigma = dp(\pi) = (3\ 1\ 5\ 2\ 4)$. Applying φ on (μ, π) gives $(\beta, \gamma) = ((8\ 5\ 5\ 3\ 2), (6\ 5))$. Finally, applying ψ^{-1} to (β, σ), we

obtain $\binom{\alpha}{\sigma} = \left(\begin{smallmatrix} 6 & 4 & 4 & 3 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{smallmatrix}\right)$. Based on $\sigma = (3\ 1\ 5\ 2\ 4)$, we conclude that $\varphi\binom{\lambda}{\pi} = ((6\ 4\ 4\ 3\ 2),\ (6\ 5))$.

Conversely, given $\sigma = (3\ 1\ 5\ 2\ 4)$ and $(\beta, \gamma) = ((8\ 5\ 5\ 3\ 2),\ (6\ 5))$, we have

$$\binom{\beta}{\sigma} = \begin{pmatrix} 8 & 5 & 5 & 3 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix} \overset{\gamma_1=6}{\longmapsto} \begin{pmatrix} 8 & \underline{6} & 5 & 5 & 3 & 2 \\ 4 & \underline{2} & 1 & 6 & 3 & 5 \end{pmatrix} \overset{\gamma_2=5}{\longmapsto} \begin{pmatrix} 8 & 6 & 5 & \underline{5} & 5 & 3 & 2 \\ 5 & 2 & 1 & \underline{4} & 7 & 3 & 6 \end{pmatrix} = \binom{\mu}{\pi}.$$

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