Pairs of Noncrossing Free Dyck Paths and Noncrossing Partitions

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Abstract

Using the bijection between partitions and vacillating tableaux, we establish a correspondence between pairs of noncrossing free Dyck paths of length $2n$ and noncrossing partitions of $[2n+1]$ with $n+1$ blocks. In terms of the number of up steps at odd positions, we find a characterization of Dyck paths constructed from pairs of noncrossing free Dyck paths by using the Labelle merging algorithm.

Keywords: Dyck path, free Dyck path, plane partition, noncrossing partition, vacillating tableau.

AMS Classifications: 05A10, 05A15

1 Introduction

In this paper, we use the bijection between vacillating tableaux and partitions to establish a correspondence between pairs of noncrossing free Dyck paths of length $2n$ and noncrossing partitions of $[2n+1]$ with $n+1$ blocks. Recall that a *Dyck path* is a lattice path from the origin to a point $(2n, 0)$ consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$ that does not go below the x-axis. Moreover, a lattice path from the origin to $(2n, 0)$ using the steps U and D without the restriction for a Dyck path is called a *free Dyck path*. Usually, a (free) Dyck path of length $2n$ is represented as a sequence of n U's and n D's.

A k-tuple $(P_1, P_2, \ldots, P_k)$ of (free) Dyck paths from $(0, 0)$ to $(2n, 0)$ is called *noncrossing* if each P_i never goes below P_{i+1} for $1 \le i \le k-1$. For any positive integer k, a

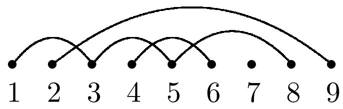

Figure 1.1: The standard representation of $1358-29-46-7$.

k-triangulation of a convex n-gon is a maximal set of diagonals that do not contain $k+1$ mutually intersecting diagonals, where two diagonals are said to be intersecting if they have a common point in the interior. For example, a 1-triangulation is the usual triangulation of a convex n-gon. A 2-triangulation turns out to be a set of $3n-10$ diagonals that do not contain any three mutually intersecting diagonals. Using the determinant formula of Lindström [7], Gessel and Viennot [5], it can be shown that the enumeration of k-tuples of noncrossing Dyck paths can be given by a determinant formula of Catalan numbers [3]. Moreover, this determinant can be explicitly evaluated, e.g., as a specialization of a determinantal formula for the dimensions of irreducible characters of the symplectic group. However, it remains an open problem to find a bijection between k-triangulations and k-tuples of noncrossing Dyck paths for $k \geq 3$. When $k=1$, there are several simple bijections; see, for example, [9, Problem 6.19]. For the case of 2-triangulations, Elizalde recently found a bijection [4].

A *partition* of a finite set S is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of subsets of S such that (i) $B_i \neq \emptyset$ for each i; (ii) $B_i \cap B_j = \emptyset$ if $i \neq j$, and (iii) $B_1 \cup B_2 \cup \cdots \cup B_k = S$. Each B_i is called a *block* of π. A *plane partition* is an array $\delta = (\delta_{ij})_{i,j\geq 1}$ of nonnegative integers such that δ has finitely many nonzero entries and is weakly decreasing in rows and columns. If $\sum \delta_{i,j} = n$, then we say that δ is a plane partition of n and write $|\delta| = n$. A *part* of a plane partition $\delta = (\delta_{ij})$ is a positive entry $\delta_{ij} > 0$.

Let $[n] = \{1, 2, \ldots, n\}$. Given a partition P of $[n]$, the *standard representation* of the partition P is a graph G on $[n]$ such that a block $\{i_1, i_2, \ldots, i_k\}$ of P written in the increasing order $i_1 < i_2 < \cdots < i_k$ corresponds to a path $(i_1, i_2, \ldots, i_k)$. For example, the standard representation of $1358-29-46-7$ is illustrated in Figure 1.1.

Meanwhile, we may view the standard representation of P as a directed graph because each edge can always be considered as an arc (i,j) with $i<j$, and we may say that i is the *left end point* and j is the *right end point*. In this paper, we use Π_n to denote the set of partitions of $[n]$.

Let $k \geq 2$ and $P \in \Pi_n$. Define a *k-crossing* (*k-nesting*) of P as a set of k arcs $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ in the standard representation of P such that $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$ ($i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1$). We use $\mathrm{cr}(P)$ ($\mathrm{ne}(P)$) to denote the maximal k such that P has a k-crossing (k-nesting). In particular, a 2-crossing (2-nesting) is called a *crossing (nesting)* for short.

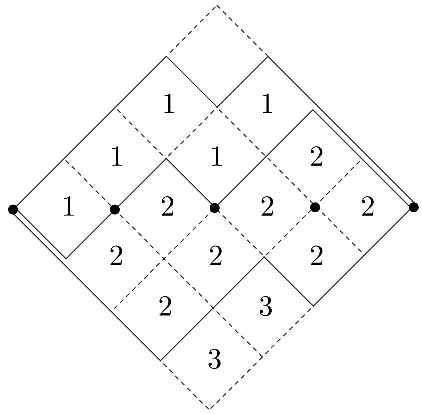

Figure 2.1: A triple of noncrossing free Dyck paths and the corresponding plane partition.

The paper is organized as follows. In Section 2, we give a quick review of the correspondence between k-tuples of noncrossing free Dyck paths and plane partitions. As a consequence, we get the formula for the number of pairs of noncrossing free Dyck paths of length $2n$. Then we give an overview of the bijection between vacillating tableaux and partitions as shown in [1]. We find a correspondence between pairs of noncrossing free Dyck paths and vacillating tableaux such that there is at most one row in each shape. These vacillating tableaux allow us to construct the noncrossing partitions. In Section 3, we give a characterization of Dyck paths obtained from pairs of noncrossing free Dyck paths by applying the Labelle merging algorithm. From this characterization and a theorem of Sulanke, one sees that the Dyck paths with k upsteps at odd positions are counted by the Narayana numbers.

2 Pairs of Noncrossing Free Dyck Paths

We begin with the enumeration of k-tuples of noncrossing free Dyck paths via the correspondence with plane partitions with bounded part size. Given a k-tuple of noncrossing free Dyck paths $(P_1, P_2, \ldots, P_k)$ with each P_i of length $2n$, they must lie in the region bounded by the paths P_0 consisting of n up steps followed by n down steps and the path P_{k+1} consisting of n down steps followed by n up steps. As illustrated in Figure 2.1, we may obtain a plane partition by filling the areas formed by the paths $P_1, P_2, \ldots, P_k$ along with P_0 and P_{k+1}. Suppose that the resulting plane partition π is of shape λ. Then we see that $\lambda_1 \le n$ and $\lambda_1' \le n$ and the largest part of π does not exceed k. Since this correspondence is one-to-one, the enumeration of k-tuples of noncrossing free Dyck paths can be converted into the enumeration of plane partitions with bounded part size.

Let $B(r, c, t)$ be the set of plane partitions with at most r rows and at most c columns, and with the largest part at most t. It is known that

$$\sum_{\pi \in B(n,n,k)} q^{|\pi|} = \frac{[k+1][k+2]^2 \cdots [k+n]^n [k+n+1]^{n-1} \cdots [k+2n-1]}{[1][2]^2 \cdots [n]^n [n+1]^{n-1} \cdots [2n-1]}, \tag{2.1}$$

where $[i] = 1 - q^i$; see, for example, [9, Theorem 7.21.7]. Let $F(n, k)$ denote the number of k-tuples of noncrossing free Dyck paths of length $2n$. Then one can deduce a formula for $F(n, k)$ by setting $q = 1$ in (2.1). In particular,

$$F(n, 2) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}. \tag{2.2}$$

It has been shown by Callan that the above number also equals the number of noncrossing partitions of $[2n + 1]$ with $n + 1$ blocks; see Sloane [8, Sequence A000891]. Hence we are led to find a bijection between the set of pairs of noncrossing free Dyck paths of length $2n$ and the set of noncrossing partitions of $[2n + 1]$ with $n + 1$ blocks.

Our bijection, denoted by ζ, consists of two steps. The first step is to transform a pair of noncrossing free Dyck paths into a vacillating tableau in which each shape has at most one row. Then we use the bijection of Chen-Deng-Du-Stanley-Yan [1] to construct the corresponding noncrossing partition. We now give a brief review of construction in [1]. We assume that the reader is familiar with the RSK algorithm, and we will use row insertion as the basic operation. For notational convenience, we write $\alpha \to \beta$, or equivalently, $\beta \leftarrow \alpha$, to denote that a shape (partition) β is obtained from a shape α by deleting a square, or equivalently, α can be obtained from β by adding a square. Similarly, we use $\alpha \Rightarrow \beta$ to denote that β may be equal to α or may be obtained from α by deleting a square.

Definition 2.1. *A vacillating tableau V_λ^{2n} of shape λ and length $2n$ is a sequence $(\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ of partitions such that*

$$\emptyset = \lambda^0 \Rightarrow \lambda^1 \Leftarrow \lambda^2 \Rightarrow \lambda^3 \Leftarrow \lambda^4 \Rightarrow \cdots \Rightarrow \lambda^{2n-1} \Leftarrow \lambda^{2n} = \lambda.$$

Given a partition P, let $E(P)$ denote the set of arcs in the standard representation of P. To construct a vacillating tableau, we will derive a sequence $(T_0, T_1, T_2, \ldots, T_{2n})$ of tableaux. Then the vacillating tableau is just the sequence of the shapes of these tableaux. We work our way backwards from $T_{2n} = \emptyset$ by determining the tableau T_{i-1} from T_i. We will make use of the notation $T' = T \leftarrow i$ to denote that a Young tableau T' is obtained from T by row inserting an element i into T. If a tableau T contains a corner element i, then we use $T' = T - i$ to denote the tableau obtained from T by deleting the square occupied by the element i. In the above notation, we can construct the tableaux T_{2k-1}, T_{2k-2} from T_{2k} $(k \leq n)$ by the following rules.

1. Let $T_{2k-1} = T_{2k}$ if the integer k does not appear in T_{2k}. Otherwise, $T_{2k-1} = T_{2k} - k$.
2. $T_{2k-2} = T_{2k-1}$ if $E(P)$ does not have any arc of the form (i, k). Otherwise, there is a unique integer $i < k$ such that $(i, k) \in E(P)$. Then we set $T_{2k-2} = T_{2k-1} \leftarrow i$.

Let λ^i be the shape of T_i for $0 \le i \le 2n$. Then $(\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^{2n})$ is the required vacillating tableau.

The inverse procedure can be described as follows. Given a vacillating tableau $V = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$, we will recursively generate a sequence $(T_0, T_1, \ldots, T_{2n})$, where T_i is a SYT of shape λ^i. Let T_0 be the empty SYT (on the empty alphabet). Below are the rules to construct T_i from T_{i-1}:

1. If $\lambda^i = \lambda^{i-1}$, then $T_i = T_{i-1}$.
2. If $\lambda^i \rightarrow \lambda^{i-1}$, then $i = 2k$ for some integer $k \in [n]$. Determine the tableau T_i such that $T_i - k = T_{i-1}$, equivalently, T_i is obtained from T_{i-1} by adding the integer k in the position of $\lambda^i \setminus \lambda^{i-1}$.
3. If $\lambda^i \leftarrow \lambda^{i-1}$, then $i = 2k - 1$ for some integer $k \in [n]$. Set T_i to be the unique SYT (on a suitable alphabet) of shape λ^i such that $T_{i-1} = T_i \leftarrow j$. Moreover, we record the arc $A_i = (j, k)$.

After the completion of the above procedure we are led to a set of arcs generated in Step 3. These arcs form a standard representation of a partition P of $[n]$.

Example 2.2. *Given the vacillating tableau*

$$(\emptyset, \emptyset, 1, 1, 11, 11, 111, 11, 11, 1, 2, 1, 1, \emptyset, \emptyset),$$

the sequence (A_i, T_i) is as follows:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
T_i	$\emptyset$	$\emptyset$	1	1	1	1	1	2	2	3	35	3	3	$\emptyset$	$\emptyset$
					2	2	2	3	3						
							3								
A_i								$(1,4)$		$(2,5)$		$(5,6)$		$(3,7)$	

The corresponding partition is $P =$ 14-256-37.

The following result can be derived from [1, Theorem 6].

Theorem 2.3. *Let $P \in \Pi_n$ and $(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$ be the corresponding vacillating tableau. Then cr(P) is the largest number of rows among λ^i, and ne(P) is the largest number of columns among λ^i.*

We are now ready to describe the bijection ζ. Let (P, Q) be a pair of noncrossing free Dyck paths, and let $P = p_1p_2\cdots p_{2n}$ and $Q = q_1q_2\cdots q_{2n}$. Based on P and Q, we form the sequence (p_i, q_i), where $i = 1, 2, \ldots, 2n$. The bijection ζ consists of two phases. First, we transform (P, Q) into a vacillating tableau

$$\mathcal{V}_{\emptyset}^{4n+2} = (\lambda^0, \lambda^1, \ldots, \lambda^{4n+2})$$

of empty shape such that there is at most one row in each λ^i and there are a total number of n operations of adding a square in the process to obtain λ^{4n+2} from λ^0. Once a vacillating tableau is constructed, we may turn the vacillating tableau into a partition by the bijection in [1].

For $1 \leq i \leq 2n$, we have the following procedure to determine λ^k for $0 \leq k \leq 4n+2$. Keep in mind that all the involved tableaux have at most one row. Roughly speaking, we will map the four possibilities of (p_i, q_i) to the four possibilities of the relation

$$\lambda^{2i-1} \Leftarrow \lambda^{2i} \Rightarrow \lambda^{2i+1}. \tag{2.3}$$

Specifically, we have the rules:

1. $\lambda^0 = \lambda^1 = \lambda^{4n+2} = \emptyset$.
2. If $(p_i, q_i) = (U, U)$, then set $\lambda^{2i-1} \leftarrow \lambda^{2i} \rightarrow \lambda^{2i+1}$.
3. If $(p_i, q_i) = (U, D)$, then set $\lambda^{2i-1} \leftarrow \lambda^{2i} = \lambda^{2i+1}$.
4. If $(p_i, q_i) = (D, U)$, then set $\lambda^{2i-1} = \lambda^{2i} \rightarrow \lambda^{2i+1}$.
5. If $(p_i, q_i) = (D, D)$, then set $\lambda^{2i-1} = \lambda^{2i} = \lambda^{2i+1}$.

Let $P_i = p_1p_2\cdots p_i$ and $Q_i = q_1q_2\cdots q_i$, and let $|(P_i, Q_i)|_{(U,U)}$ denote the number of the pairs $(p_j, q_j) = (U, U)$ in (P_i, Q_i). Similarly, we can define $|(P_i, Q_i)|_{(U,D)}$, $|(P_i, Q_i)|_{(D,U)}$ and $|(P_i, Q_i)|_{(D,D)}$. Evidently, the number of U's in P_i equals $|(P_i, Q_i)|_{(U,U)} + |(P_i, Q_i)|_{(U,D)}$, and the number of U's in Q_i equals $|(P_i, Q_i)|_{(U,U)} + |(P_i, Q_i)|_{(D,U)}$. Since P and Q are noncrossing, the number of U's in P_i is not less than that in Q_i. It follows that

$$|(P_i, Q_i)|_{(U,U)} + |(P_i, Q_i)|_{(U,D)} \geq |(P_i, Q_i)|_{(U,U)} + |(P_i, Q_i)|_{(D,U)}.$$

Hence we find that

$$|(P_i, Q_i)|_{(U,D)} \geq |(P_i, Q_i)|_{(D,U)}. \tag{2.4}$$

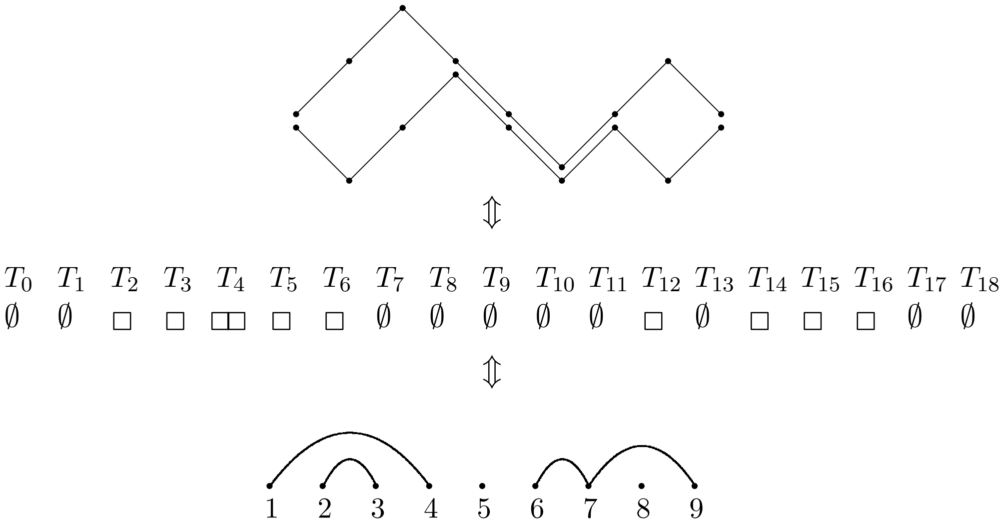

Figure 2.2: A pair of noncrossing free Dyck paths and its corresponding partition.

To show that $(\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^{4n+2})$ is a valid vacillating tableau, we should justify that the above constructions are feasible. Clearly, the items 1, 2, 3 and 5 are well defined. So we may restrict our attention to item 4, in which case $(p_i, q_i) = (D, U)$. We proceed to show that $\lambda^{2i} = \lambda^{2i-1}$ is not an empty shape. Combining (2.4) and the relations $|(P_i, Q_i)|_{(D,U)} = |(P_{i-1}, Q_{i-1})|_{(D,U)} + 1$ and $|(P_i, Q_i)|_{(U,D)} = |(P_{i-1}, Q_{i-1})|_{(U,D)}$, we see that

$$|(P_{i-1}, Q_{i-1})|_{(U,D)} > |(P_{i-1}, Q_{i-1})|_{(D,U)}.$$

From the construction of $(\lambda^0, \lambda^1, \ldots, \lambda^{2i-1})$ it can be seen that $\lambda^{2i-1} \neq \emptyset$. Thus we have reached the conclusion that $(\lambda^0, \lambda^1, \ldots, \lambda^{4n+2})$ is a vacillating tableau.

It is easily seen from Theorem 2.3 that the partition corresponding to the above vacillating tableau, denoted by R, is noncrossing because λ^i contains at most one row for any i. It remains to show that the resulting partition R contains exactly $n+1$ blocks. Since the free Dyck path P is of length $2n$, there are n left end points in the standard representation of R. This implies that there are n arcs in the standard representation of R. On the other hand, R contains $2n+1$ elements, since the vacillating tableau is of length $4n+2$. So we may deduce that there are $n+1$ blocks in R. It is not difficult to see that the above procedure is reversible. Therefore, we have established the following result.

Theorem 2.4. *The above map ζ is a bijection between the set of pairs of noncrossing free Dyck paths of length $2n$ and the set of noncrossing partitions of $[2n+1]$ with $n+1$ blocks.*

Figure 2.2 is an illustration of the bijection ζ.

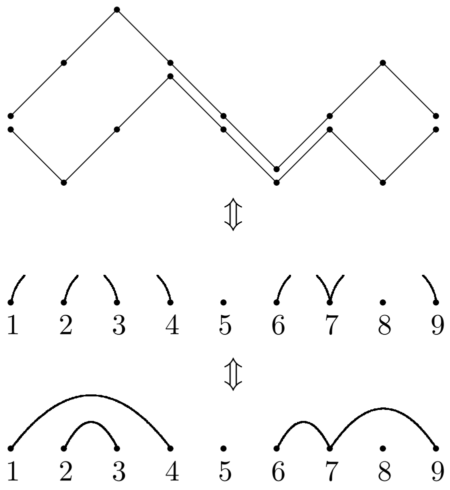

Figure 2.3: A pair of noncrossing free Dyck paths and the corresponding partition.

We remark that the bijection ζ can be described in a simpler manner. For a given pair (P, Q) of noncrossing free Dyck paths, let $P = p_1p_2\cdots p_{2n}$ and $Q = q_1q_2\cdots q_{2n}$, and let l_i (resp. r_i) denote the left-degree (resp. the right-degree) of vertex i in the standard representation of the partition corresponding to (P, Q), i.e., the number of vertices j with $j < i$ (resp. $j > i$) connected to i. First, set $l_1 = r_{2n+1} = 0$. Then the pair (r_i, l_{i+1}) is determined by the following rules for $1 \le i \le 2n$.

1. If $(p_i, q_i) = (U, U)$, then set $(r_i, l_{i+1}) = (1, 1)$.
2. If $(p_i, q_i) = (U, D)$, then set $(r_i, l_{i+1}) = (1, 0)$.
3. If $(p_i, q_i) = (D, U)$, then set $(r_i, l_{i+1}) = (0, 1)$.
4. If $(p_i, q_i) = (D, D)$, then set $(r_i, l_{i+1}) = (0, 0)$.

Observe that the degree sequences $(l_1, l_2, \ldots, l_{2n+1})$ and $(r_1, r_2, \ldots, r_{2n+1})$ consisting of only zero and one entries. As shown in Figure 2.3, we may use half arcs (intuitively called left half arcs and right half arcs) to represent the left and right degrees. We have the following unique way to pair up half arcs in order to form a noncrossing partition. At each step we always try to find the leftmost left half arc and pair it up with the nearest right half arc on its left. Iterate this procedure until all the half arcs are paired. Finally, we obtain the standard representation of the desired partition, say R.

It is necessary to show that the resulting partition R contains $2n + 1$ elements and $n+1$ blocks and is noncrossing. First, from the sequence $(p_1, q_1), (p_2, q_2), \ldots, (p_{2n}, q_{2n})$ we determine that degrees $(r_1, l_2), (r_2, l_3), \ldots, (r_{2n}, l_{2n+1})$. Clearly, the underlying set of the

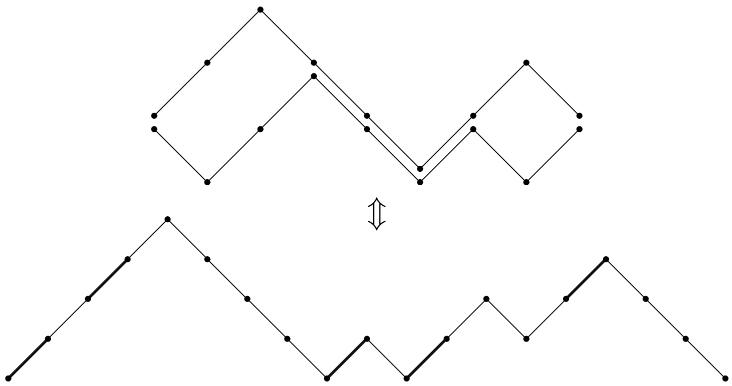

Figure 3.1: The Labelle merging algorithm.

partition R is $[2n+1]$. From the definition of (r_i, l_{i+1}), one sees that the total number of half arcs equals the number of U's in P and Q. Since P and Q are free Dyck paths of length $2n$, the total number of U's in P and Q equals $2n$. It follows that there are n arcs in the standard representation of R. Moreover, R is noncrossing since at each step the leftmost left half arc is paired up with the nearest right half arc on its left, and this operation does not cause any crossing. It is not hard to check that the above procedure is reversible. Figure 2.3 gives an illustration of this procedure.

3 The Labelle Merging Algorithm

To conclude, we note that one can merge a pair of noncrossing free Dyck paths of length $2n$ into a single Dyck path of length $4n+2$ with $n+1$ up steps at odd positions. Labelle [6] gives an algorithm to merge a pair of noncrossing free Dyck paths into a 2-Motzkin path. It is noticed that one can further transform a 2-Motzkin path into a Dyck path by using a bijection due to Delest and Viennot [2]. We will present an equivalent algorithm that directly transforms a pair of noncrossing free Dyck paths into a single Dyck path, and will call it the Labelle merging algorithm.

Let $P = p_1p_2\cdots p_{2n}$ and $Q = q_1q_2\cdots q_{2n}$ be a pair of noncrossing free Dyck paths of length $2n$. Let $Q' = q_1'q_2'\cdots q_{2n}'$, where we define $D = U'$ and $U = D'$. Then we merge P and Q into a Dyck path

$$Up_1q_1'p_2q_2'\cdots p_{2n}q_{2n}'D.$$

The following theorem gives a characterization of the Dyck paths corresponding to pairs of noncrossing free Dyck paths.

Theorem 3.1. *The Labelle merging algorithm is a bijection between noncrossing free Dyck paths of length $2n$ and Dyck paths of length $4n+2$ with $n+1$ up steps at odd positions.*

The verification of the above statement is omitted. According to a theorem of Sulanke [10], the number of Dyck paths of length $2n$ with k up steps at odd positions equals the Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}$. Hence it equals the number of noncrossing partitions on $[2n+1]$ with $n+1$ blocks, which is also given by the Narayana number. Figure 3.1 is an illustration of the Labelle merging algorithm, where thick lines represent up steps at odd positions.

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