Jacobi's Identity and Synchronized Partitions

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Abstract. We obtain a finite form of Jacobi's identity and present a combinatorial proof based on the structure of synchronized partitions.

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1 Introduction

We adopt the common notation on partitions and q-series as used in [1, 8]. The *q-shifted factorial* $(x;q)_n$ is defined by $(x;q)_0 = 1$ and for $n \geq 1$,

$$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x).$$

The *q-binomial coefficient*, or the *Gauss coefficient*, is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \qquad \text{for } 0 \leq k \leq n. \tag{1.1}$$

Note that the parameter q is often omitted in the notation of the Gauss coefficients.

This paper is concerned with Jacobi's identity [12, p.257, Eq.(5)] (see also [18, Theorem 357])

$$(q;q)_\infty^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{\binom{k+1}{2}}. \tag{1.2}$$

Note that the identity (1.2) can be deduced from Jacobi's triple product identity [8, p.15]

$$(z;q)_\infty (q/z;q)_\infty (q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k. \tag{1.3}$$

Rewriting (1.3) in the following form:

$$(z;q)_\infty (z^{-1}q,;q)_\infty (q;q)_\infty = \sum_{k=0}^{\infty} (-1)^k (1-z^{2k+1}) z^{-k} q^{\binom{k+1}{2}}, \tag{1.4}$$

then one obtains (1.2) from (1.4) by dividing both sides by $(1-z)$ and taking the limit $z \to 1$. A combinatorial proof of (1.2) has been found by Joichi and Stanton [13].

Jacobi's identity has many applications. Ramanujan [15, 16] proved the partition congruences modulo 5 and 7 by using this identity. Andrews [2, 3], Ewell [6], and Hirschhorn and Sellers [11] have used this identity in their studies of congruence relations on partition functions. Jacobi's identity (1.2) also plays a role in the study of representing an integer as sum of squares, see, for example, Hirschhorn [10].

The purpose of this paper is to derive the finite form of Jacobi's identity and give a combinatorial proof. We first give a finite form of Jacobi's identity (1.2) by using MacMahon's finite form of Jacobi's triple product identity. Then we give the definitions of synchronized partitions and rooted synchronized partitions and present two simple involutions on synchronized partitions which imply a combinatorial proof of the finite form of Jacobi's identity.

2 A Finite Form of Jacobi's Identity

We obtain the following finite form of Jacobi's identity.

Theorem 2.1 *For $m, n \geq 0$, we have*

$$(q;q)_m (q;q)_n = \sum_{k=-n-1}^{m} (-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix}. \tag{2.5}$$

Proof. We begin with MacMahon's finite form of Jacobi's triple product identity [14, Vol. II, §323] (see also [4, 9])

$$(zq;q)_m (z^{-1};q)_n = \sum_{k=-n}^{m} (-1)^k q^{\binom{k+1}{2}} z^k \begin{bmatrix} m+n \\ n+k \end{bmatrix}. \tag{2.6}$$

Substituting n with $n+1$ in the above identity, we get

$$(zq;q)_m (z^{-1};q)_{n+1} = \sum_{k=-n-1}^{m} (-1)^k q^{\binom{k+1}{2}} z^k \begin{bmatrix} m+n+1 \\ n+1+k \end{bmatrix}. \tag{2.7}$$

Setting

$$f(z) = (zq;q)_m (z^{-1}q;q)_n,$$

then (2.7) becomes

$$(1-z^{-1}) f(z) = \sum_{k=-n-1}^{m} (-1)^k q^{\binom{k+1}{2}} z^k \begin{bmatrix} m+n+1 \\ n+1+k \end{bmatrix}.$$

Differentiating both sides respect to z, we get

$$z^{-2}f(z)+(1-z^{-1})f'(z)=\sum_{k=-n-1}^{m}(-1)^k k q^{\binom{k+1}{2}} z^{k-1}\begin{bmatrix} m+n+1 \\ n+1+k \end{bmatrix}.$$

Setting $z=1$, one obtains (2.6). ∎

Setting $n\to\infty$ and $m\to\infty$ in (2.5), by Tannery's theorem (see[17, p.292]), we get

$$(q;q)_\infty^3=\sum_{k=-\infty}^{\infty}(-1)^k k q^{\binom{k+1}{2}},$$

which is equivalent to Jacobi's identity (1.2).

Replacing m by n in (2.5), we obtain the following identity.

Theorem 2.2 *For $n\geq 0$, we have*

$$(q;q)_n^2=\sum_{k=0}^{n}(-1)^k(2k+1)q^{\binom{k+1}{2}}\begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}. \tag{2.8}$$

3 Synchronized Partitions

In this section, we give a combinatorial proof of the finite form of Jacobi's identity (2.5) by introducing the structures of synchronized partitions and rooted synchronized partitions. Let us recall some common terminology on partitions. *A partition* λ of a positive integer n is a finite weakly decreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^r \lambda_i = n$, denoted by $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_r)$, where the λ_i's are called the parts of λ; the sum of parts is called the *weight* of λ, denoted by $|\lambda|$; the number of parts of λ is called the *length* of λ, denoted by $l(\lambda)$.

A pair of partitions (α,β) of the same length is called a generalized Frobenius partition, see Andrews [2], Corteel and Lovejoy [5]. In a more general setting, a pair of partitions (α,β) that are not necessarily of the same length is also called a generalized Frobenius partition, see Yee [19, 20].

We now give the definitions of *synchronized partitions* and *rooted synchronized partitions*. Assume that $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_r)$ is a partition with distinct parts and $\beta=(\beta_1,\beta_2,\ldots,\beta_s)$ is also a partition with distinct parts under the assumption that the last part β_s may be zero. Then a synchronized partition is a representation of (α,β) as a two-row array such that some $*$ symbols may be added at the end of α or β so that they are of the same length depending on which is of smaller length. We may denote a synchronized partition with underlying partitions α and β by $S(\alpha,\beta)$, or simply (α,β) is no confusion arises. The difference $r-s$ is called the *discrepancy* of the synchronized partition. A synchronized partition with a positive discrepancy k can be represented as follows:

$$S(\alpha,\, \beta) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{s+k} \\ \beta_1 & \beta_2 & \cdots & \beta_s & * & * & * \end{pmatrix}$$

and a synchronized partitions with a negative discrepancy $-k$ ($k > 0$) can be represented as follows:

$$S(\alpha,\, \beta) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r & * & * & * \\ \beta_1 & \beta_2 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_{r+k} \end{pmatrix}.$$

A synchronized partition with zero discrepancy can be simply represented as a two-row array without any star added. A *rooted synchronized partition* is defined as a synchronized partition with a distinguished star symbol, which we denote by $\bar{*}$. Clearly, a rooted synchronized partition has an underlying synchronized partition with nonzero discrepancy.

For example, there are five rooted synchronized partitions of 2 :

$$\begin{pmatrix} 2 \\ \bar{*} \end{pmatrix} \begin{pmatrix} 1 & \bar{*} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{*} \\ 2 \end{pmatrix} \begin{pmatrix} \bar{*} & * \\ 2 & 0 \end{pmatrix} \begin{pmatrix} * & \bar{*} \\ 2 & 0 \end{pmatrix}$$

Let $\mathcal{S}_{m,n}$ denote the set of synchronized partitions $S(\alpha,\beta)$ such that $\alpha_1 \leq m$ and $\beta_1 \leq n$, and let $\mathcal{R}_{m,n}$ be the set of rooted synchronized partitions $S(\alpha,\beta)$ such that $\alpha_1 \leq m$ and $\beta_1 \leq n$. Note that $\mathcal{S}_{m,n}$ are generated by the set of pairs of partitions (α,β) under the same condition by adding some stars to a row if it is of smaller length than the other row. A rooted synchronized partition $S(\alpha,\beta)$ is called *degenerate* if

$$S = \begin{pmatrix} \alpha_1 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_s & \bar{*} & \cdots & * \end{pmatrix}$$

or

$$S = \begin{pmatrix} \alpha_1 & \cdots & \alpha_r & * & \cdots & * & \bar{*} \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_s & 0 \end{pmatrix}$$

where $m \geq \alpha_1 > \alpha_2 > \ldots > \alpha_r \geq 1$ and $n \geq \beta_1 > \beta_2 > \ldots > \beta_s \geq 1$; otherwise $S(\alpha,\beta)$ is called *non-degenerate*.

It is easy to see that the generating function of synchronized partitions in $\mathcal{S}_{m,n}$ equals

$$\sum_{S(\alpha,\beta)\in\mathcal{S}_{m,n}} q^{|\alpha|+|\beta|} = (-q;q)_m(-1;q)_{n+1}, \tag{3.9}$$

and the generating function of synchronized partitions in $\mathcal{S}_{m,n}$ without the zero part equals

$$(-q;q)_m(-q;q)_n. \tag{3.10}$$

On the other hand, the generating function of synchronized partitions in $\mathcal{S}_{m,n}$ with a nonnegative discrepancy k equals

$$q^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix}, \tag{3.11}$$

and the generating function of synchronized partitions in $\mathcal{S}_{m,n}$ with a negative discrepancy $-k$ equals

$$q^{\binom{-k+1}{2}} \begin{bmatrix} m+n+1 \\ n-k+1 \end{bmatrix} = q^{\binom{k}{2}} \begin{bmatrix} m+n+1 \\ n-k+1 \end{bmatrix}. \tag{3.12}$$

From (3.11) and (3.12) it follows that the generating function for rooted synchronized partitions in $\mathcal{R}_{m,n}$ equals

$$\sum_{k=0}^{m} kq^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix} + \sum_{k=1}^{n+1} kq^{\binom{k}{2}} \begin{bmatrix} m+n+1 \\ n-k+1 \end{bmatrix}.$$

Let us define the sign of a rooted synchronized partition $S(\alpha, \beta)$ as $(-1)^{\delta(S)}$, where $\delta(S)$ is the number of stars in $S(\alpha, \beta)$ under the assumption that a star with the bar in the top row is not counted. The sign of a synchronized partition equals $(-1)^k$, where k is the discrepancy. We now give a sign reversing involution on the set of non-degenerate rooted synchronized partitions.

Theorem 3.1 *There is a sign reversing involution τ on the set of non-degenerate rooted synchronized partitions of p in $\mathcal{R}_{m,n}$.*

Proof. For a non-degenerate rooted synchronized partition $S(\alpha, \beta) \in \mathcal{R}_{m,n}$, we proceed to construct a non-degenerate rooted synchronized partition $S(\alpha', \beta')$. We consider the following two cases.

Case 1: The partition β has a zero part.

- If $l(\alpha) > l(\beta)$, then replace the zero part by a star $*$.
- If $l(\alpha) < l(\beta)$, then delete the whole column of the zero part.

Case 2: The partition β has no zero part.

- If $l(\alpha) > l(\beta)$, then replace the first '$*$' on the bottom row by a zero part.
- If $l(\alpha) < l(\beta)$, then add a zero part along with a star on the top as a column.

The above bijection can be illustrated as follows:

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_s & a_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & 0 & * & \bar{*} & * \end{pmatrix} \xleftrightarrow{l(\alpha)>l(\beta)} \begin{pmatrix} \alpha_1 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & * & * & \bar{*} & * \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_r & * & \bar{*} & * & * \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_{s-1} & 0 \end{pmatrix} \xleftrightarrow{l(\alpha)<l(\beta)} \begin{pmatrix} \alpha_1 & \cdots & \alpha_r & * & \bar{*} & * \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_{s-1} \end{pmatrix}.$$

It is easy to check that the above construction is a sign reversing involution. ∎

Next, we give a bijection between degenerate rooted synchronized partitions and synchronized partitions without the zero part.

Theorem 3.2 *There is a sign preserving bijection between the set of degenerate rooted synchronized partitions of* p *in* $\mathcal{R}_{m,n}$ *and the set of synchronized partitions of* p *in* $\mathcal{S}_{m,n}$ *that do not contain the zero part.*

Proof. For a degenerated rooted synchronized partition $S(\alpha,\beta)$ in $\mathcal{R}_{m,n}$, we can construct a synchronized partition $S(\alpha',\beta')$ in $\mathcal{S}_{m,n}$ that do not contain the zero part.

Case 1 If $l(\alpha) > l(\beta)$, then delete the bar to the first '$*$' on the bottom row.

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_s & a_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_s & \bar{*} & \cdots & * \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha_1 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_s & * & \cdots & * \end{pmatrix}$$

Case 2 If $l(\alpha) < l(\beta)$, then delete a zero part on the bottom row along with a barred star on the top row.

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_r & * & \cdots & * & \bar{*} \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_s & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha_1 & \cdots & \alpha_r & * & \cdots & * \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_s \end{pmatrix}$$

Clearly, the procedure is reversible and it preserves the signs. ∎

We are now ready to give a combinatorial interpretation of finite form of Jacobi's identity (2.5). It is easy to see that $(q;q)_n(q;q)_m$ is the generating function of signed synchronized partitions $\mathcal{S}_{m,n}$ without the zero part. Note that

$$\sum_{k=-n-1}^{m}(-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix} = \sum_{k=0}^{m}(-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix} + \sum_{k=1}^{n+1}(-1)^{k-1} k q^{\binom{k}{2}} \begin{bmatrix} m+n+1 \\ n-k+1 \end{bmatrix}$$

is the generating function of signed rooted synchronized partitions in $\mathcal{R}_{m,n}$. Combining Theorem 3.1 and Theorem 3.2, we are led to a combinatorial interpretation of the finite form (2.5).

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