CROSSINGS AND NESTINGS IN TANGLED-DIAGRAMS

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Abstract. A tangled-diagram over $[n] = \{1, \dots, n\}$ is a graph of degree less than two whose vertices $1, \dots, n$ are arranged in a horizontal line and whose arcs are drawn in the upper halfplane with a particular notion of crossings and nestings. Generalizing the construction of Chen *et.al.* we prove a bijection between generalized vacillating tableaux with less than k rows and k-noncrossing tangled-diagrams and study their crossings and nestings. We show that the number of k-noncrossing and k-nonnesting tangled-diagrams are equal and enumerate tangled-diagrams.

1. Introduction

The main objective of this paper is to study tangled-diagrams by generalizing the concept of vacillating tableaux introduced by Chen *et.al.* [2]. Tangled-diagrams are motivated from intra-molecular interactions of RNA nucleotides as follows: the primary sequence of an RNA molecule is the sequence of nucleotides **A**, **G**, **U** and **C** together with the Watson-Crick (**A-U**, **G-C**) and (**U-G**) base pairing rules specifying the pairs of nucleotides can potentially form bonds. Single stranded RNA molecules form helical structures whose bonds satisfy the above base pairing rules and which, in many cases, determine their function. One question of central importance is now to predict the 3D-arrangement of the nucleotides, vital for the molecule's functionality. For this purpose it is important to capture the sterical constraints of the base pairings, which then allows to systematically search the configuration space. For a particular class of RNA structures, the pseudoknot RNA structures [6], the notion of diagrams [5] has been used in order to translate the biochemistry of the nucleotide interactions [3, 7] into crossings and nestings of arcs. A diagram is a labeled, partial one-factor graph over $[n]$, represented as follows: all vertices are drawn in a horizontal line and all arcs (representing interactions) are drawn in the upper halfplane. Since *a*

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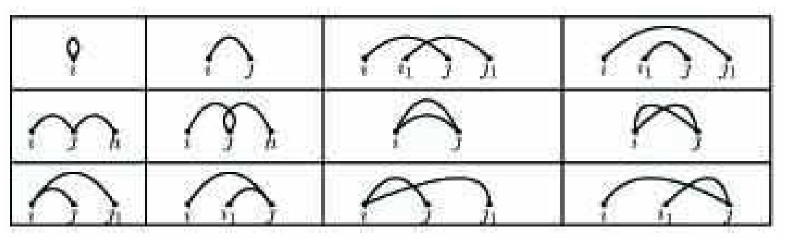
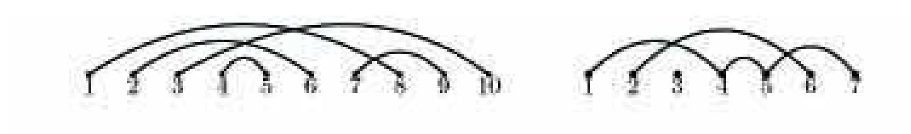
priori restricted to degree ≤ 1 diagrams do not allow to express helix-helix, loop-helix and multiple nucleotide interactions in general [3]. The tangled-diagrams studied in the following are tailored to express these interactions. A tangled-diagram is a labeled graph over the vertices $1, \dots, n$, drawn in a horizontal line and arcs in the upper halfplane. In general, it has isolated points and the following types of arcs



For instance

are two tangled-diagrams and diagrams in which all vertices j of degree two are either incident to loops (j, j) or crossing arcs (i, j) and (j, h), where $i < j < h$ are called braids. In particular matchings and partitions are tangled-diagrams. A matching over the set $[2n] = \{1, 2, \dots, 2n\}$ is just a 1-regular tangled-diagram and a partition corresponds to a tangled-diagram in which any vertex of degree two, j, is incident to the arcs (i, j) and (j, s), where $i < j < s$. For instance



Chen *et al.* observed that there is a bijection between vacillating tableaux and partitions [2] derived from an RSK-insertion idea due to Stanley. In addition they studied enhanced partitions via hesitating tableaux. In the following we integrate both frameworks by generalizing vacillating tableaux as follows: a generalized vacillating tableaux V_λ^{2n} of shape λ and length $2n$ is a sequence $(\lambda^0, \lambda^1, \dots, \lambda^{2n})$ of shapes such that $\lambda^0 = \varnothing$ and $\lambda^{2n} = \lambda$, and $(\lambda^{2i-1}, \lambda^{2i})$ is derived from λ^{2i-2}, for $1 \le i \le n$ by an elementary move, i.e. a step of the form $(\varnothing, \varnothing)$: do nothing twice; $(-\square, \varnothing)$: first remove a square then do nothing; $(\varnothing, +\square)$: first do nothing then adding a square; $(\pm\square, \pm\square)$: add/remove a square at the odd and even steps, respectively. For instance the below sequence is a generalized vacillating tableaux

We prove a bijection between $V_\varnothing^{2n}$, referred to from now on as simply vacillating tableaux, and tangled-diagrams over $[n]$. We show that the notions of k-noncrossing tangled-diagrams and k-nonnesting in tangled-diagrams are in fact dual and enumerate k-noncrossing tangled-diagrams. Restricting the steps of the vacillating tableaux we obtain three wellknown bijections, the bijection between vacillating tableaux with elementary moves $\{(-\square, \varnothing), (\varnothing, +\square)\}$ and matchings [2], the bijection between the vacillating tableaux with elementary moves $\{(-\square, \varnothing), (\varnothing, +\square), (\varnothing, \varnothing), (-\square, +\square)\}$ and partitions and finally the bijection between the vacillating tableaux with elementary moves $\{(-\square, \varnothing), (\varnothing, +\square), (\varnothing, \varnothing), (+\square, -\square)\}$ and enhanced partitions.

2. Tangled-diagrams and vacillating tableaux

2.1. **Tangled-diagrams.** A tangled-diagram is a labeled graph, G_n, over $[n]$ with degree ≤ 2, represented by drawing its vertices in a horizontal line and its arcs (i, j) in the upper halfplane having the following properties: two arcs (i_1, j_1) and (i_2, j_2) such that $i_1 < i_2 < j_1 < j_2$ are crossing and if $i_1 < i_2 < j_2 < j_1$ they are nesting. Two arcs (i, j_1) and (i, j_2) (common lefthand endpoint) and $j_1 < j_2$ can be drawn in two ways: either draw (i, j_1) strictly below (i, j_2) in which case (i, j_1) and (i, j_2) are nesting (at i) or draw (i, j_1) starting above i and intersecting (i, j_2) once, in which case (i, j_1) and (i, j_2) are crossing (at i): The cases of two arcs (i_1, j), (i_2, j), where $i_1 < i_2$

(common righthand endpoint)

and of two arcs $(i,j),(i,j)$, i.e. where i and j are both: right- and lefthand endpoints are completely analogous. Suppose $i<j<h$ and that we are given two arcs (i,j) and (j,h). Then we can draw them intersecting once or not. In the former case (i,j) and (j,h) are crossing: The cases of two arcs (i_1,j), (i_2,j), where $i_1<i_2$ (common righthand endpoint)

The set of all tangled-diagrams is denoted by $\mathcal{G}_n$. A tangled-diagram is k-noncrossing if it contains no k-set of mutually intersecting arcs and k-nonnesting if it contains no k-set of mutually nesting arcs. The set of k-noncrossing and k-nonnesting tangled-diagrams is denoted by $\mathcal{G}_{n,k}$ and $\mathcal{G}_n^k$, respectively.

2.2. **Inflation.** A key observation allowing for the combinatorial interpretation of tangled-diagrams is their "local" inflation. Intuitively, a tangled-diagram with ℓ vertices of degree 2 is resolved into a partial matching over $n+\ell$ vertices. For this purpose we consider the following linear ordering over $\{1,1',\dots,n,n'\}$

$$1<1'<2<2'<\cdots<(n-1)<(n-1)'<n<n' \;. \tag{2.1}$$

Let G_n be a tangled-diagram with exactly ℓ vertices of degree 2. Then the inflation of G_n, $\iota(G_n)$, is a combinatorial graph over $\{1,\dots,n+\ell\}$ vertices with degree ≤ 1 obtained as follows:
$\underline{i<j_1<j_2}$: if (i,j_1), (i,j_2) are crossing, then $((i,j_1),(i,j_2))\mapsto((i,j_1),(i',j_2))$ and if (i,j_1), (i,j_2) are nesting then $((i,j_1),(i,j_2))\mapsto((i,j_2),(i',j_1))$, i.e.: The cases of two arcs (i_1,j), (i_2,j), where $i_1<i_2$ (common righthand endpoint)

$\underline{i_1 < i_2 < j}$: if (i_1, j), (i_2, j) are crossing then $((i_1, j), (i_2, j)) \mapsto ((i_1, j), (i_2, j'))$ and if (i_1, j), (i_2, j) are nesting then $((i_1, j), (i_2, j)) \mapsto ((i_1, j'), (i_2, j))$, i.e.:

$\underline{i < j}$: if $(i, j), (i, j)$ are crossing, then $((i, j), (i, j)) \mapsto ((i, j), (i', j'))$ and if $(i, j), (i, j)$ are nesting, then we set $((i, j), (i, j)) \mapsto ((i, j'), (i', j))$ and if (i, i) is a loop we map $(i, i) \mapsto (i, i')$:

$\underline{i < j < h}$: if (i, j), (j, h) are crossing, then $((i, j), (j, h)) \mapsto ((i, j'), (j', h))$ and $((i, j), (j, h)) \mapsto ((i, j), (j', h))$, otherwise, i.e. we have the following situation

Identifying all vertex-pairs (i, i') allows us to recover the original tangled-diagram and we accordingly have the bijection

$$\iota\colon \mathcal{G}_n \longrightarrow \iota(\mathcal{G}_n)\ . \tag{2.2}$$

ι preserves by definition the maximal number crossing and nesting arcs, respectively. Equivalently, a tangled-diagram G_n is k-noncrossing if and only if its inflation $\iota(G_n)$ is k-noncrossing or k-nonnesting, respectively. For instance the inflation of the tangled-diagram of Section 1 is

2.3. **Vacillating tableaux.** A Young diagram (shape) is a collection of squares arranged in left-justified rows with weakly decreasing number of boxes in each row. A standard Young tableau (SYT) is a filling of the squares by numbers which is strictly decreasing in each row and in each column. We refer to standard Young tableaux as Young tableaux. Elements can be inserted into SYT via the RSK-algorithm [8]. In the following we will refer to SYT simply as tableaux. Our first lemma is instrumental for constructing the bijection between vacillating tableaux and tangled-diagrams in Section 3.

Lemma 1. [2] *Suppose we are given two shapes $\lambda^i \subsetneq \lambda^{i-1}$, which differ by exactly one square and T_{i-1} is a SYT of shape λ^{i-1}. Then there exists a unique j contained in T_{i-1} and a unique tableau T_i such that T_{i-1} is obtained from T_i by inserting j via the* RSK-*algorithm.*

Proof. Suppose we have two shapes $\lambda^i \subsetneq \lambda^{i-1}$, which differ by exactly one square and T_{i-1} is a tableau of shape λ^{i-1}. Let us first assume that λ^{i-1} differs from λ^i by the rightmost square in its first row, containing j. Then j is the unique element of T_{i-1} which, if RSK-inserted into T_i, produces the tableau T_{i-1}. Suppose next the square which is being removed from λ^{i-1} is at the end of row ℓ. Then we remove the square and RSK-insert its element x into the $(\ell-1)$-th row in the square containing y, where y is maximal subject to $y < x$ and such that y, if inserted into row $(\ell-1)$, would push down x in its original position. Since each column is strictly increasing such an y always exists. We can conclude by induction on ℓ that this process results in exactly one element j being removed from T_{i-1} and a filling of λ^i, i.e. a unique tableau T_i. By construction, RSK-insertion of j recovers the tableaux T_{i-1}. □

Definition 1. (Vacillating Tableau) A vacillating tableaux V_λ^{2n} of shape λ and length $2n$ is a sequence $(\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ of shapes such that (i) $\lambda^0 = \varnothing$ and $\lambda^{2n} = \lambda$, and (ii) $(\lambda^{2i-1}, \lambda^{2i})$ is derived from λ^{2i-2}, for $1 \leq i \leq n$ by either $(\varnothing, \varnothing)$: do nothing twice; $(-\square, \varnothing)$: first remove a square then do nothing; $(\varnothing, +\square)$: first do nothing then adding a square; $(\pm\square, \pm\square)$: add/remove a square at the odd and even steps, respectively. Let $\mathcal{V}_\lambda^{2n}$ denote the set of vacillating tableaux.

3. The bijection

Lemma 2. *There exists a mapping from the set of vacillating tableaux of shape $\varnothing$ and length $2n$, $V_\varnothing^{2n}$, into the set of inflations of tangled-diagrams*

$$\phi\colon V_\varnothing^{2n} \longrightarrow \iota(\mathcal{G}_n)\ . \tag{3.1}$$

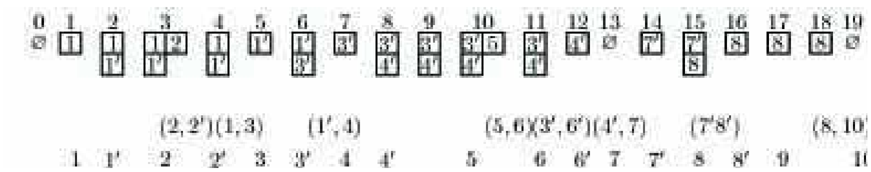
Proof. In order to define ϕ we recursively define a sequence of triples

$$((P_0, T_0, V_0), (P_1, T_1, V_1), \ldots, (P_{2n}, T_{2n}, V_{2n})) \tag{3.2}$$

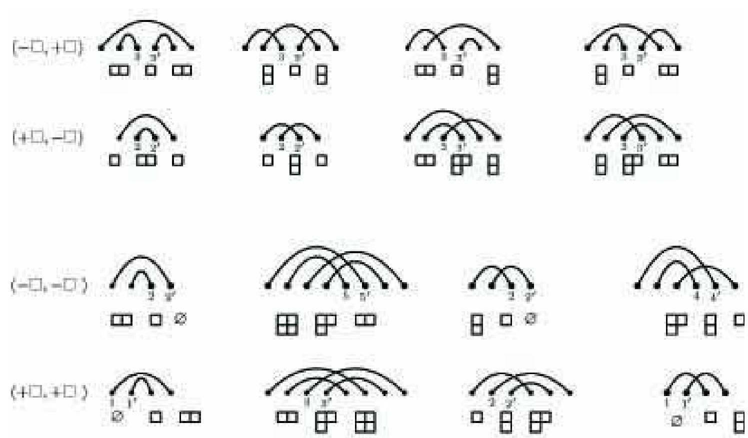
where P_i is a set of arcs, T_i is a tableau of shape λ^i, and $V_i \subset \{1, 1', 2, 2', \ldots, n, n'\}$ is a set of vertices. $P_0 = \varnothing$, $T_0 = \varnothing$ and $V_0 = \varnothing$. We assume that the lefthand and righthand endpoints of all P_i-arcs and the entries of the tableau T_i are contained in $\{1, 1', \ldots, n, n'\}$. Given $(P_{2j-2}, T_{2j-2}, V_{2j-2})$ we derive $(P_{2j-1}, T_{2j-1}, V_{2j-1})$ and (P_{2j}, T_{2j}, V_{2j}) as follows:
$\underline{(\varnothing, \varnothing)}$. If $\lambda^{2j-1} = \lambda^{2j-2}$ and $\lambda^{2j} = \lambda^{2j-1}$, we have $(P_{2j-1}, T_{2j-1}) = (P_{2j-2}, T_{2j-2})$ and $(P_{2j}, T_{2j}) = (P_{2j-1}, T_{2j-1})$ and $V_{2j-1} = V_{2j-2} \cup \{j\}$, $V_{2j} = V_{2j-1}$.
$\underline{(-\square, \varnothing)}$. If $\lambda^{2j-1} \subsetneq \lambda^{2j-2}$ and $\lambda^{2j} = \lambda^{2j-1}$, then T_{2j-1} is the unique tableau of shape λ^{2j-1} such that T_{2j-2} is obtained by RSK-inserting the unique number i via the RSK-algorithm into T_{2j-1} (Lemma 1) and $P_{2j-1} = P_{2j-2} \cup \{(i, j)\}$ and $(P_{2j}, T_{2j}) = (P_{2j-1}, T_{2j-1})$ and $V_{2j-1} = V_{2j-2} \cup \{j\}$, $V_{2j} = V_{2j-1}$.
$\underline{(\varnothing, +\square)}$. If $\lambda^{2j-1} = \lambda^{2j-2}$ and $\lambda^{2j} \supsetneq \lambda^{2j-1}$, then $(P_{2j-1}, T_{2j-1}) = (P_{2j-2}, T_{2j-2})$ and $P_{2j} = P_{2j-1}$ and T_{2j} is obtained from T_{2j-1} by adding the entry j in the square $\lambda^{2j} \setminus \lambda^{2j-1}$ and $V_{2j-1} = V_{2j-2}$, $V_{2j} = V_{2j-1} \cup \{j\}$.
$\underline{(-\square, +\square)}$. If $\lambda^{2j-1} \subsetneq \lambda^{2j-2}$ and $\lambda^{2j} \supsetneq \lambda^{2j-1}$, then T_{2j-1} is the unique tableau of shape λ^{2j-1} such that T_{2j-2} is obtained from T_{2j-1} by RSK-inserting the unique number i, via the RSK-algorithm (Lemma 1). Then we set $P_{2j-1} = P_{2j-2} \cup \{(i, j)\}$ and $P_{2j} = P_{2j-1}$ and T_{2j} is obtained from T_{2j-1} by adding the entry j' in the square $\lambda^{2j} \setminus \lambda^{2j-1}$ and $V_{2j-1} = V_{2j-2} \cup \{j\}$, $V_{2j} = V_{2j-1} \cup \{j'\}$.
$\underline{(+\square, -\square)}$. If $\lambda^{2j-2} \subsetneq \lambda^{2j-1}$ and $\lambda^{2j} \subsetneq \lambda^{2j-1}$ then T_{2j-1} is obtained from T_{2j-2} by adding the entry j in the square $\lambda^{2j-1} \setminus \lambda^{2j-2}$ and the tableau T_{2j} is the unique tableau of shape λ^{2j} such that T_{2j-1} is obtained from T_{2j} by RSK-inserting the unique number i, via the RSK-algorithm (Lemma 1). We then set $P_{2j-1} = P_{2j-2}$ and $P_{2j} = P_{2j-1} \cup \{(i, j')\}$ and $V_{2j-1} = V_{2j-2} \cup \{j\}$, $V_{2j} = V_{2j-1} \cup \{j'\}$.
$\underline{(-\square, -\square)}$. If $\lambda^{2j-1} \subsetneq \lambda^{2j-2}$ and $\lambda^{2j} \subsetneq \lambda^{2j-1}$, let T_{2j-1} be the unique tableau of shape λ^{2j-1} such that T_{2j-2} is obtained from T_{2j-1} by RSK-inserting i_1 (Lemma 1) and T_{2j} be the unique tableau of shape λ^{2j} such that T_{2j-1} is obtained from T_{2j} by RSK-inserting i_2 (Lemma 1) $P_{2j-1} = P_{2j-2} \cup \{(i_1, j)\}$ and $P_{2j} = P_{2j-1} \cup \{(i_2, j')\}$ and $V_{2j-1} = V_{2j-2} \cup \{j\}$, $V_{2j} = V_{2j-1} \cup \{j'\}$.
$\underline{(+\square, +\square)}$. If $\lambda^{2j-1} \supsetneq \lambda^{2j-2}$ and $\lambda^{2j} \supsetneq \lambda^{2j-1}$, we set $P_{2j-1} = P_{2j-2}$, and T_{2j-1} is obtained from T_{2j-2} by adding the entry j in the square $\lambda^{2j-1} \setminus \lambda^{2j-2}$. Furthermore we set $P_{2j} = P_{2j-1}$ and T_{2j} is obtained from T_{2j-1} by adding the entry j' in the square $\lambda^{2j} \setminus \lambda^{2j-1}$ and $V_{2j-1} = V_{2j-2} \cup \{j\}$, $V_{2j} = V_{2j-1} \cup \{j'\}$.
Claim. $\phi(V_{\varnothing}^{2n})$ is the inflation of a tangled-diagram.
First, if $(i, j) \in P_{2n}$, then $i < j$ and secondly any vertex j can occur only as either as lefthand or righthand endpoint of an arc, whence $\phi(V_{\varnothing}^{2n})$ is a 1-diagram. Each step $(+\square, +\square)$ induces a pair

of arcs of the form (i, j_1), (i', j_2) and each step $(-\square, -\square)$ induces a pair of arcs of the form (i_1, j), (i_2, j'). Each step $(-\square, +\square)$ corresponds to a pair of arcs (h, j), (j', s) where $h < j < j' < s$ and each step $(+\square, -\square)$ induces a pair of arcs of the form (j, s), (h, j'), where $h < j < j' < s$ or a 1-arcs of the form (i, i'). Let ℓ be the number of steps not containing $\varnothing$. By construction each of these adds the 2-set $\{j, j'\}$, whence (V_{2n}, P_{2n}) corresponds to the inflation of a unique tangled-diagram with ℓ vertices of degree 2 and the claim follows. $\square$

Remark 1. The mapping ϕ: if squares are added the corresponding numbers are inserted, if squares are deleted Lemma 1 is used to extract a unique number, which then forms the lefthand endpoint of the derived arcs.



Remark 2. As an illustration of the mapping ϕ: $V_{\varnothing}^{2n} \longrightarrow \iota(\mathcal{G}_n)$ we display systematically all arc-configurations of inflated tangled-diagrams induced by the vacillating tableaux



We proceed by explicitly constructing the inverse of ϕ.

Lemma 3. *There exists a mapping from the set of inflations of tangled-diagrams over* n *vertices,* $\iota(\mathcal{G}_n)$*, into the set of vacillating tableaux of shape* $\varnothing$ *and length* $2n$*,* $\mathcal{V}_{\varnothing}^{2n}$

$$\psi\colon \iota(\mathcal{G}_n) \longrightarrow \mathcal{V}_{\varnothing}^{2n}\ . \tag{3.3}$$

Proof. We define ψ as follows. Let $\iota(G_n)$ be the inflation of the tangled-diagram G_n. We set

$$\iota_i = \begin{cases} (i,i') & \text{iff} \;\; i \text{ has degree 2 in } G_n, \\ i & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $T_{2n} = \varnothing$ be the empty tableau. We will construct inductively a sequence of tableaux T_h of shape $\lambda^h_{\iota(G_n)}$, where $h \in \{0,1,\dots 2n\}$ by considering ι_i for $i = n, n-1, n-2, \dots, 1$. For each ι_j we inductively define the pair of tableaux (T_{2j}, T_{2j-1}):

(I) $\iota_j = j$ is an isolated vertex in $\iota(G_n)$, then we set $T_{2j-1} = T_{2j}$ and $T_{2j-2} = T_{2j-1}$. Accordingly, $\lambda^{2j-1}_{\iota(G_n)} = \lambda^{2j}_{\iota(G_n)}$ and $\lambda^{2j-2}_{\iota(G_n)} = \lambda^{2j-1}_{\iota(G_n)}$ (left to right: $(\varnothing,\varnothing)$).

(II) $\iota_j = j$ is the righthand endpoint of exactly one arc (i,j) but not a lefthand endpoint, then we set $T_{2j-1} = T_{2j}$ and obtain T_{2j-2} by adding i via the RSK-algorithm to T_{2j-1}. Consequently we have $\lambda^{2j-1}_{\iota(G_n)} = \lambda^{2j}_{\iota(G_n)}$ and $\lambda^{2j-2}_{\iota(G_n)} \supsetneq \lambda^{2j-1}_{\iota(G_n)}$. (left to right: $(-\square,\varnothing)$).

(III) j is the lefthand endpoint of exactly one arc (j,k) but not a righthand endpoint, then first set T_{2j-1} to be the tableau obtained by removing the square with entry j from T_{2j} and let $T_{2j-2} = T_{2j-1}$. Therefore $\lambda^{2j-1}_{\iota(G_n)} \subsetneq \lambda^{2j}_{\iota(G_n)}$ and $\lambda^{2j-2}_{\iota(G_n)} = \lambda^{2j-1}_{\iota(G_n)}$. (left to right: $(\varnothing,+\square)$).

(IV) j is a lefthand and righthand endpoint, then we have the two $\iota(G_n)$-arcs (i,j) and (j',h), where $i < j < j' < h$. T_{2j-1} is obtained by removing the square with entry j' in T_{2j} first and T_{2j-2} via inserting i in T_{2j-1} via the RSK-algorithm. Accordingly we derive the shapes $\lambda^{2j-1}_{\iota(G_n)} \subsetneq \lambda^{2j}_{\iota(G_n)}$ and $\lambda^{2j-2}_{\iota(G_n)} \supsetneq \lambda^{2j-1}_{\iota(G_n)}$. (left to right: $(-\square,+\square)$).

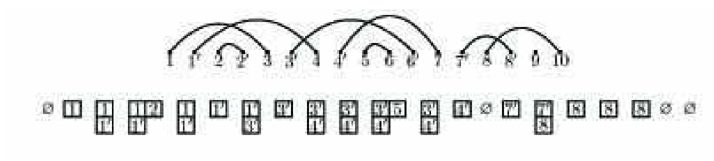
(V) j is a righthand endpoint of degree 2, then we have the two $\iota(G_n)$-arcs (i,j) and (h,j'). T_{2j-1} is obtained by inserting h via the RSK-algorithm into T_{2j} and T_{2j-2} is obtained by RSK-inserting i into T_{2j-1} via the RSK-algorithm. We derive $\lambda^{2j-1}_{\iota(G_n)} \supsetneq \lambda^{2j}_{\iota(G_n)}$ and $\lambda^{2j-2}_{\iota(G_n)} \supsetneq \lambda^{2j-1}_{\iota(G_n)}$ (left to right:$(-\square,-\square)$).

(VI) j is a lefthand endpoint of degree 2, then we have the two $\iota(G_n)$-arcs (j,r) and (j',h). T_{2j-1} is obtained by removing the square with entry j' from the tableau T_{2j} and T_{2j-2} is obtained by removing the square with entry j from the T_{2j-1}. Then we have $\lambda^{2j-1}_{\iota(G_n)} \subsetneq \lambda^{2j}_{\iota(G_n)}$ and $\lambda^{2j-2}_{\iota(G_n)} \subsetneq \lambda^{2j-1}_{\iota(G_n)}$ (left to right: $(+\square,+\square)$).

(VII) j is a lefthand and righthand endpoint of crossing arcs or a loop, then we have the two $\iota(G_n)$-arcs (j,s) and (h,j'), $h < j < j' < s$ or an arc of the form (j,j'). T_{2j-1} is obtained by RSK-inserting h (j) into the tableau T_{2j} and T_{2j-2} is obtained by removing the square with entry

j (j) from the T_{2j-1} (left to right: $(+\square, -\square)$).
Therefore ψ maps the inflation of a tangled-diagram into a vacillating tableau and the lemma follows. $\square$

Remark 3. From inflations of tangled-diagrams to vacillating tableaux: starting from right to left the vacillating tableaux is obtained via the RSK-algorithm as follows: if j is a righthand endpoint it gives rise to RSK-insertion of its (unique) lefthand endpoint and if j is a lefthand endpoint the square containing j is removed.



Theorem 1. *There exists a bijection between the set of vacillating tableaux of shape $\varnothing$ and length $2n$, $\mathcal{V}_\varnothing^{2n}$ and the set of tangled-diagrams over n vertices, $\mathcal{G}_n$*

$$\beta\colon \mathcal{V}_\varnothing^{2n} \longrightarrow \mathcal{G}_n \ . \tag{3.5}$$

Proof. According to Lemma 2 and Lemma 3 we have the following mappings $\phi\colon \mathcal{V}_\varnothing^{2n} \longrightarrow \iota(\mathcal{G}_n)$ and $\psi\colon \iota(\mathcal{G}_n) \longrightarrow \mathcal{V}_\varnothing^{2n}$. We next show that ϕ and ψ are indeed inverses with respect to each other. By definition ϕ extracts arcs such that their respective lefthand-endpoints if RSK-inserted (Lemma 1) recover the tableaux of the preceding step. We observe that by definition, ψ reverses this extraction: it explicitly RSK-inserts the lefthand-endpoints of arcs. Therefore we have the following situation

$$\phi\circ\psi(\iota(G_n)) = \phi((\lambda_{\iota(G_n)})_0^{2n}) = \iota(G_n) \qquad \text{and} \qquad \psi\circ\phi(\mathcal{V}_\varnothing^{2n}) = \mathcal{V}_\varnothing^{2n} \ , \tag{3.6}$$

from which we conclude that ϕ and ψ are bijective. Since G_n is in one to one correspondence with $\iota(G_n)$ the proof of the theorem is complete. $\square$

By construction the bijection $\iota\colon \mathcal{G}_n \longrightarrow \iota(\mathcal{G}_n)$ preserves the maximal number crossing and nesting arcs, respectively. Equivalently, a tangled-diagram G_n is k-noncrossing if and only if its inflation $\iota(G_n)$ is k-noncrossing or k-nonnesting [2]. Indeed, this follows immediately from the definition of the inflation.

Theorem 2. *A tangled-diagram G_n is k-noncrossing if and only if all shapes λ^i in its corresponding vacillating tableau have less than k rows, i.e. $\phi\colon \mathcal{V}_\varnothing^{2n} \longrightarrow \mathcal{G}_n$ maps vacillating tableaux having less than k rows into k-noncrossing tangled-diagrams. Furthermore there is a bijection between the set of k-noncrossing and k-nonnesting tangled-diagrams.*

Theorem 2 is the generalization of the corresponding result in [2] to tangled-diagrams. Since the inflation map allows to interpret a tangled-diagram with ℓ vertices of degree 2 over n vertices as a partial matching over $n+\ell$ vertices its proof is analogous.

We next observe that restricting the steps for vacillating tableaux produces the bijections of Chen *et.al.* [2]. Let $\mathcal{M}_k(n)$, $\mathcal{P}_k(n)$ and $\mathcal{B}_k(n)$ denote the set of k-noncrossing matchings, partitions and braids, respectively. If a tableaux-sequence $V_\varnothing^{2n}$ is obtained via certain steps $s \in S$ we write $V_\varnothing^{2n} \models S$.

Corollary 1. *Let β_i denote the restriction of the bijection $\beta\colon \mathcal{V}_\varnothing^{2n} \longrightarrow \mathcal{G}_n$ in Theorem 1. Then β induces the bijections*

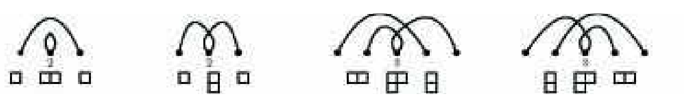
$$\beta_1\colon \{V_\varnothing^{2n} \mid V_\varnothing^{2n} \models (-\square,\varnothing),(\varnothing,+\square) \text{ and has} \leq k \text{ rows}\} \rightarrow \mathcal{M}_k(n)\ . \tag{3.7}$$

$$\beta_2\colon \{V_\varnothing^{2n} \mid V_\varnothing^{2n} \models (-\square,\varnothing),(\varnothing,+\square),(\varnothing,\varnothing),(-\square,+\square) \text{ and has} \leq k \text{ rows}\} \rightarrow \mathcal{P}_k(n)\ . \tag{3.8}$$

$$\beta_3\colon \{V_\varnothing^{2n} \mid V_\varnothing^{2n} \models (-\square,\varnothing),(\varnothing,+\square),(\varnothing,\varnothing),(+\square,-\square) \text{ and has} \leq k \text{ rows}\} \rightarrow \mathcal{B}_k(n)\ . \tag{3.9}$$

Remark 4. For partitions we can illustrate the correspondences between the elementary steps and associated tangled-diagram arc-configurations as follows:

Remark 5. For braids we derive the following correspondences. They illustrate one key difference between partitions and braids: for fixed crossing number braids are more restricted due to the fact that they already have *a priori* "local" crossings at their non-loop-vertices of degree 2.



Let $D_{2,k}(n)$ and $\tilde{D}_{2,k}(n)$ be the numbers of k-noncrossing tangled-diagrams and tangled-diagrams without isolated points over $[n]$, respectively. Furthermore let $f_k(2n-\ell)$ be the number of k-noncrossing matchings over $2n-\ell$ vertices. We show that the enumeration of tangled-diagrams can be reduced to the enumeration of matchings via the inflation map. W.l.o.g. we can restrict our analysis to the case of tangled-diagrams without isolated points since the number of tangled-diagrams over $[n]$ is given by $D_{2,k}(n) = \sum_{i=0}^{n} \binom{n}{i} \tilde{D}_{2,k}(n-i)$.

Theorem 3. *The number of k-noncrossing tangled-diagrams over $[n]$ is given by*

$$\tilde{D}_{2,k}(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} f_k(2n-\ell) \ . \tag{3.10}$$

and in particular for $k=3$ we have

$$\tilde{D}_{2,3}(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} \left(C_{\frac{2n-\ell}{2}}\, C_{\frac{2n-\ell}{2}+2} - C^2_{\frac{2n-\ell}{2}+1} \right) \ . \tag{3.11}$$

Proof. Let $\tilde{\mathcal{D}}_{2,k}(n,V)$ denote the set of tangled-diagrams in which $V = \{i_1,\dots,i_h\}$ is the set of vertices of degree 1 (where $h \equiv 0 \mod 2$ by definition of $\tilde{\mathcal{D}}_{2,k}(n,V)$) and let $\mathcal{M}_k(\{1,1',\dots,n,n'\} \setminus V')$, where $V' = \{i_1',\dots,i_h'\}$ denote the set of matchings over $\{1,1',\dots,n,n'\} \setminus V'$. By construction, (eq. (**??**), eq. (**??**) and eq. (**??**)) the inflation is a well defined mapping

$$\iota\colon \tilde{\mathcal{D}}_{2,k}(n,V) \longrightarrow \mathcal{M}_k(\{1,1',\dots,n,n'\} \setminus V') \tag{3.12}$$

with inverse κ defined by identifying all pairs (x,x'), where $x,x' \in \{1,1',\dots,n,n'\} \setminus V'$. Obviously, we have $|\mathcal{M}_k(\{1,1',\dots,n,n'\} \setminus V')| = f_k(2n-\ell)$ and we obtain

$$\tilde{D}_{2,k}(n) = \sum_{V \subset [n]} \tilde{D}_{2,k}(n,V) = \sum_{\ell=0}^{n} \binom{n}{\ell} f_k(2n-\ell) \ . \tag{3.13}$$

Suppose $n \equiv 0 \mod 2$ and let C_m denote the m-th Catalan number, then we have [4]

$$f_3(n) = C_{\frac{n}{2}}\, C_{\frac{n}{2}+2} - C^2_{\frac{n}{2}+1} \ . \tag{3.14}$$

and the theorem follows. □

Remark 6. The first 10 numbers of 3-noncrossing tangled-diagrams are given by

n	1	2	3	4	5	6	7	8	9	10
$D_{2,3}(n)$	2	7	39	292	2635	27019	304162	3677313	47036624	629772754

The enumeration of 3-noncrossing partitions and 3-noncrossing enhanced partitions, which are in bijection with braids without isolated points has been derived in [1].

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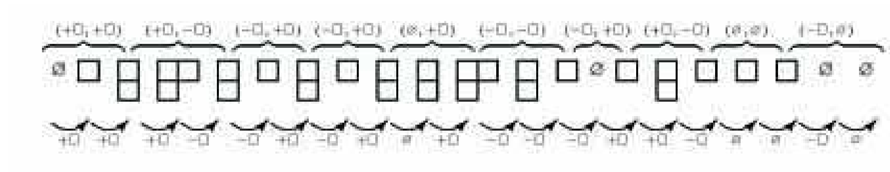

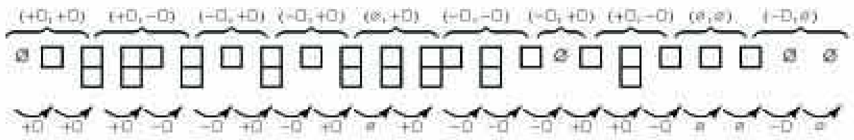

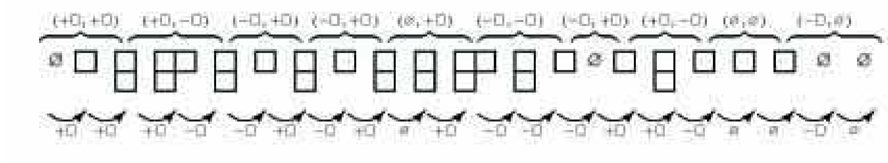

$$
\begin{array}{lcl}
(+\square, -\square\ ) & \longleftrightarrow & \{(+e_1, -e_1), (+e_2, -e_2), (+e_1, -e_2), (+e_2, -e_1)\} \\
(-\square, +\square\ ) & \longleftrightarrow & \{(-e_1, +e_1), (-e_2, +e_2), (-e_1, +e_2), (-e_2, +e_1)\} \\
(\ \varnothing, +\square\ ) & \longleftrightarrow & \{(0, +e_1), (0, +e_2)\} \\
(-\square,\ \ \varnothing\ ) & \longleftrightarrow & \{(-e_1, 0), (-e_2, 0)\} \\
(\ \varnothing,\ \ \varnothing\ ) & \longleftrightarrow & \{(0, 0)\}
\end{array}
$$