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Chen, William Y. C. (1-LANL)
The theory of compositionals. (English summary)
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Whereas Joyal's theory of species gives a set-theoretic interpretation of operations with univariate formal power series, the author's theory of compositionals gives a set-theoretic interpretation of operations with formal power series in an infinite number of variables. A composition is a vector with an infinite number of entries each of which is a finite set. Pólya's plethystic composition and its set-theoretic interpretation are obtained as a particular instance of the operation of substitution of a summable family of power series into a power series. In another paper [J. Combin. Theory Ser. A **64** (1993), no. 2, 149–188; MR 95g:05014], also arising from his Ph.D. thesis ["On the combinatorics of plethysm", Ph.D. Thesis, Massachusetts Inst. Tech., Cambridge, MA, 1991], the author developed a plethystic umbral calculus in this combinatorial setting.

Chen's theory is rigorous, but perhaps difficult to follow. For example, the text begins with a series of 17 numbered definitions without examples, "species" diagrams, or much motivation. Similarly, the reviewer's theory of genera ["The iterated logarithmic algebra", Ph.D. Thesis, Massachusetts Inst. Tech., Cambridge, MA, 1989] is difficult to understand. The key to understanding compositionals and genera can be found in the theory of colored species independently discovered by M. A. Méndez and O. A. Nava Z. [J. Combin. Theory Ser. A **64** (1993), no. 1, 102–129; MR 95f:05008]. As the author notes on page 82, a composition $C\colon E_1 + E_2 + \cdots$ should be viewed as a colored set by assuming the elements of E_i have color i. Méndez and Nava Z. allow an arbitrary set of colors; however, modelling plethysm requires a colored monoid structure such as $\mathbf{N}$ in the author's theory. The author's definition of general composition (Definition 1.27) involving a de-composition followed by a partition of each block corresponds simply to colored partitions.

F. Bergeron's theory of **S**-species [J. Combin. Theory Ser. A **46** (1987), no. 2, 291–305; MR 89b:05017] (also cast by Méndez and Nava Z. in terms of colored species) is unfairly termed "unnatural" by the author. However, Bergeron's work is quite "natural" in both the technical and usual sense of the word: **S**-species are natural in that they enjoy a universal property of factoring the cycle index operator from species to symmetric functions. Moreover, **S**-species can be simply defined. Instead of considering functors from sets to sets, Bergeron considers functors from permutations to sets. The operations on species generalize to **S**-species in a fairly obvious way.

The author ends the paper by generalizing to plethystic Schröder trees and plethystic rooted forests, the bijection he found [Proc. Nat. Acad. Sci. U.S.A. **87** (1990), no. 24, 9635–9639; MR 91i:05011] between Schröder trees and forests of trees of depth one (small trees). This bijection was used to obtain many classical results of tree enumeration, and notably gives a combinatorial explanation of the cancellation that occurs in the Lagrange inversion formula. In this context, the author gives a combinatorial expansion of the plethystic inverse of a series.

There are a number of minor points to note for the reader: In Definition 2.10, the notation $n|C_n$ means C_n is a stepped composition (cf. Definition 2.7). The definition of the cycle index Z_G not given in Proposition 2.32 may be found in [A. Joyal, Adv. in Math. **42** (1981), no. 1, 1–82; MR 84d:05025]. The remark "see the next corollary" on page 72 is no longer followed by its reference [Lemma 4.26 in the author's thesis]. The order in the plethystic lattice should be the transitive closure of the relation defined on page 73. The construction of maximal [resp. minimal] reduced compounds denoted by the author by both $\widehat{1}$ and $\tilde{C}$ [resp. $\widehat{0}$ and $\widehat{C}$] on page 74 is actually required for the proof of Theorem 4.1. $X[C]$ should be $U[C]$ on page 76. Although there are no diagrams in the paper, for understanding the covering relationship mentioned in Definition 5.4, the reader should draw his plethystic trees with the root on top.

Reviewed by *Daniel Elliott Loeb*