The Reverse Ultra Log-Concavity of
the Boros-Moll Polynomials

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Abstract. Based on the recurrence relations on the coefficients of the Boros-Moll polynomials
\( P_m(a) = \sum_i d_i(m)a^i \) derived independently by Kauers and Paule, and Moll, we
are led to the discovery of the reverse ultra log-concavity of the sequence \( \{d_i(m)\} \). We
also show that the sequence \( \{ild_i(m)\} \) is log-concave for \( m \geq 1 \). Two conjectures are
proposed.

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Boros-Moll polynomials.

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1 Introduction

The main objective of this paper is to prove the reverse ultra log-concavity of the Boros-
Moll polynomials. Boros and Moll [1, 2, 3, 7] studied a class of Jacobi polynomials in
connection with the following integral:

**Theorem 1.1**

\[
\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),
\]

where \( P_m(a) \) can be represented by

\[
P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k.
\]

The polynomials \( P_m(a) \) are called Boros-Moll polynomials. Let

\[
P_m(a) = \sum_{i=0}^m d_i(m)a^i.
\]

The coefficients \( d_i(m) \) are positive. They are also log-concave, as conjectured by Moll [7],
and recently proved by Kauers and Paule [5].
Theorem 1.2 For $0 < i < m$, we have
\[ d_i^2(m) \geq d_{i-1}(m)d_{i+1}(m). \] (1.1)

Recall that a sequence $(a_n)_{n \geq 0}$ of real numbers is said to be log-concave if
\[ a_n^2 \geq a_{n+1}a_{n-1}, \] (1.2)
for all $n \geq 1$. If the sequence satisfies (1.2) with strict inequality, then it is said to be strictly log-concave. A polynomial is said to be (strictly) log-concave if its sequence of coefficients is (strictly) log-concave. Log-concave sequences and polynomials often arise in combinatorics, algebra and geometry, see, for example, Brenti [4] and Stanley [9].

A sequence $\{a_k\}_{0 \leq k \leq n}$ is called ultra log-concave if
\[ \left\{ \frac{a_k}{\binom{n}{k}} \right\}_{0 \leq k \leq n} \] (1.3)
is log-concave, see Liggett [6]. Note that this condition can be rewritten as
\[ k(n-k)a_k^2 - (n - k + 1)(k + 1)a_{k-1}a_{k+1} \geq 0. \] (1.4)
It is well known that if a polynomial has only real roots, then its coefficients form an ultra log-concave sequence. The above relation (1.4) implies the following inequality
\[ ka_k^2 - (k + 1)a_{k-1}a_{k+1} \geq 0, \]
from which we can deduce that the sequence $\{k!a_k\}$ is log-concave. This further implies that $\{a_k\}$ is strictly log-concave.

The first result of this paper is to show that $i!\{d_i(m)\}$ is log-concave, as stated below in an equivalent form.

Theorem 1.3 For $1 \leq i \leq m - 1$, we have
\[ id_i^2(m) > (i + 1)d_{i-1}(m)d_{i+1}(m). \] (1.5)

Despite the above property of $d_i(m)$, we will show that the reverse ultra log-concavity holds, as described in the following theorem. This is the main result of this paper.

Theorem 1.4 For $1 \leq i \leq m - 1$, we have
\[ \frac{d_{i-1}(m)}{\binom{m}{i-1}} \frac{d_{i+1}(m)}{\binom{m}{i+1}} > \left( \frac{d_i(m)}{\binom{m}{i}} \right)^2, \] (1.6)
or, equivalently,
\[ (m - i + 1)(i + 1)d_{i-1}(m)d_{i+1}(m) - (m - i)d_i(m)^2 > 0. \] (1.7)

We conclude this paper with two conjectures. Roughly speaking, the first conjecture says that in spite of the reverse ultra log-concavity, the ultra log-concavity almost holds in the asymptotic sense. The second conjecture is concerned with the log-concavity of the sequence $d_{i-1}(m)d_{i+1}(m)/d_i^2(m)$ for $m \geq 2$. 

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2 The Reverse Ultra Log-Concavity

In this section, we give the proofs of Theorem 1.3 and Theorem 1.4. We will need the following three recurrence formulas derived independently by Kauers and Paule [5] and Moll [8]:

\[
d_{i}(m+1) = \frac{m+i}{m+1} d_{i-1}(m) + \frac{4m+2i+3}{2(m+1)} d_{i}(m), \quad 0 \leq i \leq m+1, \quad (2.1)
\]

\[
d_{i}(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)} d_{i}(m)
- \frac{i(i+1)}{(m+1)(m+1-i)} d_{i+1}(m), \quad 0 \leq i \leq m, \quad (2.2)
\]

\[
d_{i}(m+2) = \frac{-4i^2 + 8m^2 + 24m + 19}{2(m+2-i)(m+2)} d_{i}(m+1)
- \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_{i}(m), \quad 0 \leq i \leq m+1, \quad (2.3)
\]

Recall that Kauers and Paule [5] obtained the following lower bound on \(d_{i}(m+1)/d_{i}(m)\) for \(0 \leq i \leq m\),

\[
\frac{d_{i}(m+1)}{d_{i}(m)} \geq Q(m,i), \quad (2.4)
\]

where

\[
Q(m,i) = \frac{4m^2 + 7m + i + 3}{2(m+1-i)(m+1)}. \quad (2.5)
\]

Proof of Theorem 1.3. As the first step, by the recurrence relations (2.1) and (2.2) we may transform Theorem 1.3 to the following equivalent form

\[
4(m+1)^2(m-i+1)d_{i}(m+1)^2 - 4(m+1)(4m^2 + 7m - 2i^2 + 3)d_{i}(m)d_{i}(m+1)
+ (4m+4i+3)(4m^2 + 7m - i + 3)d_{i}(m)^2 > 0,
\]

which can be recast as

\[
4(m+1)^2(m-i+1) \left( \frac{d_{i}(m+1)}{d_{i}(m)} \right)^2 - 4(m+1)(4m^2 + 7m - 2i^2 + 3) \frac{d_{i}(m+1)}{d_{i}(m)}
+ (4m+4i+3)(4m^2 + 7m - i + 3) > 0. \quad (2.6)
\]

Notice that the discriminant of the above quadratic form is positive, since

\[
\Delta = 16i^2(2i-1)^2(m+1)^2 > 0.
\]
Thus the quadratic function on the left hand side of (2.6) has two real roots,

\[ x_1 = \frac{4m^2 + 7m + 3 - i}{2(m - i + 1)(m + 1)}, \quad x_2 = \frac{4m + 4i + 3}{2(m + 1)}. \]

Using the lower bound \( Q(m, i) \) for \( d_i(m+1)/d_i(m) \), we deduce that for \( 1 \leq i \leq m - 1 \),

\[ \frac{d_i(m+1)}{d_i(m)} \geq Q(m, i) > x_1 > x_2. \]

Thus we obtain (2.6), and the proof is complete. \[\square\]

In order to prove Theorem 1.4, we need an upper bound for the ratio \( d_i(m+1)/d_i(m) \).

**Theorem 2.1** We have for all \( m \geq 2, 1 \leq i \leq m - 1 \),

\[ \frac{d_i(m+1)}{d_i(m)} < T(m, i), \quad (2.7) \]

where

\[ T(m, i) = \frac{4m^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1} - 2i^2}{2(m - i + 1)(m + 1)}, \quad (2.8) \]

and for \( m \geq 1 \),

\[ \frac{d_0(m+1)}{d_0(m)} = T(m, 0), \quad \frac{d_m(m+1)}{d_m(m)} = T(m, m). \quad (2.9) \]

**Proof.** First, we consider (2.9). Setting \( i = 0 \) in (2.1) gives that for any \( m \geq 1 \),

\[ \frac{d_0(m+1)}{d_0(m)} = \frac{4m + 3}{2(m + 1)}, \]

which agrees with \( T(m, 0) \). While \( i = m \), (2.1) implies

\[ \frac{d_m(m+1)}{d_m(m)} = \frac{(2m + 3)(2m + 1)}{2(m + 1)} = T(m, m). \]

Thus (2.9) holds for \( m \geq 1 \).

We now proceed to conduct induction on \( m \) to show that (2.7) is valid for \( i \geq 1 \). When \( m = 2 \) and \( i = 1 \), we have

\[ \frac{d_1(3)}{d_1(2)} = \frac{43}{15} < T(2, 1) = \frac{31 + \sqrt{13}}{12}. \]

We assume that the theorem holds for \( m \), where \( m > 2 \). Then we aim to show that it also holds for \( m + 1 \), namely, for \( 1 \leq i \leq m \),

\[ d_i(m+2) < T(m+1, i)d_i(m+1). \quad (2.10) \]
Using the recurrence (2.3), we may rewrite (2.10) in the following form
\[
\frac{-4i^2 + 8m^2 + 24m + 19}{2(m - i + 2)(m + 2)}d_i(m + 1)
- \frac{(m + i + 1)(4m + 3)(4m + 5)}{4(m + 1)(m + 2)(m - i + 2)}d_i(m) < T(m + 1, i)d_i(m + 1). \tag{2.11}
\]

In order to derive an upper bound for \(d_i(m + 1)/d_i(m)\), it is necessary to show that
\[
R(m, i) = \frac{-4i^2 + 8m^2 + 24m + 19}{2(m - i + 2)(m + 2)} - T(m + 1, i)
\tag{2.12}
\]
is positive. Since \(m \geq i\), we have \(4m + 4i^2 + 5 < 12m + 4m^2 + 9\). It follows that
\[
R(m, i) = \frac{4m^2 + 9m + 5 - 2i^2 - i\sqrt{4m + 4i^2 + 5}}{2(m - i + 2)(m + 2)}
\geq \frac{4m^2 + 9m + 5 - 2i^2 - i(2m + 3)}{2(m - i + 2)(m + 2)}
= \frac{(4m^2 - 2i^2 - 2mi) + (9m - 3i) + 5}{2(m - i + 2)(m + 2)},
\]
which is positive for \(1 \leq i \leq m\). Hence (2.11) is equivalent to the following inequality
\[
\frac{d_i(m + 1)}{d_i(m)} < \frac{(m + i + 1)(4m + 3)(4m + 5)}{4(m + 1)(m + 2)(m - i + 2)R(m, i)}. \tag{2.13}
\]

Note that the right hand side of (2.13) can be expressed as
\[
F(m, i) = \frac{(m + i + 1)(4m + 3)(4m + 5)}{2(m + 1)(4m^2 - 2i^2 + 9m + 5 - i\sqrt{4m + 4i^2 + 5})}. \tag{2.14}
\]

By the inductive hypothesis, it suffices to show that for \(1 \leq i \leq m - 1\),
\[
T(m, i) \leq F(m, i). \tag{2.15}
\]

Let \(A = \sqrt{4m + 4i^2 + 1}\) and \(B = \sqrt{4m + 4i^2 + 5}\). It is easy to check that \(F(m, i) - T(m, i)\) equals
\[
\frac{(i^2 - 4i^4) - i(5 + 4m^2 + 9m - 2i^2)A + i(3 + 4m^2 + 7m - 2i^2)B + i^2AB}{2(m + 1)(m - i + 1)(4m^2 + 9m + 5 - 2i^2 - iB)}. \tag{2.16}
\]

Since
\[
(4m^2 + 9m + 5 - 2i^2)^2 - (iB)^2 = (4m + 5)^2(m + i + 1)(m - i + 1) > 0,
\]

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the denominator of (2.16) is positive. It remains to show that the numerator of (2.16) is also positive. Observe that every term in the numerator contains a factor $i$. So we may divide this factor. Put

$$X = (i - 4i^3) + iAB$$
$$Y = (5 + 4m^2 + 9m - 2i^2)A - (3 + 4m^2 + 7m - 2i^2)B.$$  

We claim that $X > 0$ and $X > Y$. Since $m \geq i$, we have $A \geq 2i + 1$ and $B \geq 2i + 1$. Moreover, since $i \geq 1$, we find

$$X = (i - 4i^3) + iAB \geq i - 4i^3 + i(2i + 1)^2 = 4i^2 + 2i > 0.$$  

To show that $X - Y > 0$, we will consider $X^2 - Y^2$. Let us introduce $G(m, i)$ and $H(m, i)$ as given by

$$G(m, i) = (32m^4 - 32m^2i^2 + 128m^3 - 64mi^2 + 190m^2 - 30i^2 + 124m + 30)AB,$$
$$H(m, i) = 128m^5 + 608m^4 + 1128m^3 + 1014m^2 + 436m + 128m^4i^2 + 384m^3i^2 + 408m^2i^2 - 128m^2i - 256mi^2 - 120i^4 + 50i^2 + 70.$$  

It can be checked that

$$X^2 - Y^2 = G(m, i) - H(m, i). \tag{2.17}$$  

Since $i \leq m - 1$, it is easily seen that $G(m, i) > 0$. To verify $G(m, i) > H(m, i)$, we will show that $G(m, i)^2 - H(m, i)^2 > 0$. In fact, for $1 \leq i \leq m - 1$

$$G(m, i)^2 - H(m, i)^2 = 16(4m + 5)^2(16mi^2 + 12i^2 - 1)(m + i + 1)^2(m - i + 1)^2 > 0.$$  

This implies that $X > Y$. Hence the numerator of (2.16) is positive. Consequently, (2.15) holds for $1 \leq i \leq m - 1$.

Up to now, we still need to consider the case $i = m$. It remains to show that

$$\frac{d_m(m + 2)}{d_m(m + 1)} < T(m + 1, m).$$

By direct computation, we find that

$$d_m(m + 2) = \frac{(m + 1)(4m^2 + 18m + 21)}{2m + 4} \cdot \frac{(2m + 4)}{m + 2}$$

and

$$d_m(m + 1) = \frac{2m + 3}{2} \cdot \frac{(2m + 2)}{m + 1}.$$  

We get

$$\frac{d_m(m + 2)}{d_m(m + 1)} = \frac{(m + 1)(4m^2 + 18m + 21)}{2(2m + 3)(m + 2)}.$$  

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On the other hand,

\[ T(m+1,m) = \frac{2m^2 + 15m + 14 + m\sqrt{4m^2 + 4m + 5}}{4(m+2)}. \]

We see that for \( m \geq 2 \),

\[ T(m+1,m) > \frac{d_m(m+2)}{d_m(m+1)} \]

\[ \iff (2m^2 + 15m + 14 + m\sqrt{4m^2 + 4m + 5})(2m+3) > 2(m+1)(4m^2 + 18m + 21) > 0 \]

\[ \iff (2m^2 + 3m)\sqrt{4m^2 + 4m + 5} > 4m^3 + 8m^2 + 5m \]

\[ \iff \left( (2m^2 + 3m)\sqrt{4m^2 + 4m + 5} \right)^2 - (4m^3 + 8m^2 + 5m)^2 > 0 \]

\[ \iff 4m^2(4m+5) > 0, \]

which is evident. This completes the proof of the theorem.

We are now ready to prove Theorem 1.4. Like the first step in the proof of Theorem 1.3, we use the recurrences (2.1) and (2.2) to restate (1.7) as follows

\[ 4(m - i + 1)^2(m+1)^2 \left( \frac{d_i(m+1)}{d_i(m)} \right)^2 \]

\[ - 4(m - i + 1)(m+1)(4m^2 - 2i^2 + 7m + 3) \frac{d_i(m+1)}{d_i(m)} \]

\[ - (32m^2 - 56m^3 - 73m^2 - 42m + 13i^2 - 9 - 16m^4 + 16i^2m^2) < 0. \quad (2.18) \]

Observe that the discriminant of the above quadratic form is positive for \( i \geq 1 \), since

\[ \Delta = 16i^2(m+1)^2(4i^2 + 4m + 1)(m - i + 1)^2 > 0. \]

It follows that the quadratic function on the left hand side of (2.18) has two real roots

\[ x_1 = \frac{4m^2 - 2i^2 + 7m + 3 - i\sqrt{4m^2 + 4i^2 + 1}}{2(m - i + 1)(m+1)}, \]

\[ x_2 = \frac{4m^2 - 2i^2 + 7m + 3 + i\sqrt{4m^2 + 4i^2 + 1}}{2(m - i + 1)(m+1)}. \]

By the definition of \( Q(m, i) \) in (2.5), we see that \( x_1 < Q(m, i) \). Note that \( x_2 = T(m, i) \) as given in Theorem 2.1. In view of Theorem 2.1, we deduce that

\[ x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2, \]

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for $1 \leq i \leq m - 1$. Hence we conclude that (2.18) holds. This completes the proof of Theorem 1.4.

We conclude this paper with two conjectures. Let

$$c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)}, \quad 1 \leq i \leq m - 1.$$ 

Then Theorem 1.3 and Theorem 1.4 lead to the following bounds on $c_i(m)$ for $1 \leq i \leq m - 1$,

$$1 + \frac{1}{i} \leq c_i(m) \leq \left(1 + \frac{1}{i}\right) \left(1 + \frac{1}{m-i}\right). \quad (2.19)$$

Numerical evidence indicates that the upper bound in (2.19) is very close to $c_i(m)$ even when $m$ is small. Let $u_i(m) = (1 + \frac{1}{i}) (1 + \frac{1}{m-i})$. For example, when $m = 6$, the values of $c_i(m)/u_i(m)$ are given below

0.9462708849, 0.9642110408, 0.9752109510, 0.9821688283, 0.9867303609.

**Conjecture 2.2** For $1 \leq i \leq m - 1$, we have

$$\lim_{m \to \infty} \frac{c_i(m)}{u_i(m)} = 1. \quad (2.20)$$

**Conjecture 2.3** For $m \geq 2$, the sequence $\{1/c_i(m)\}_{i=2}^{m-2}$ is log-concave.

Conjecture 2.2 implies that the Boros-Moll polynomials are almost ultra log-concave. Further conjectures can be made based on Conjecture 2.3 in the spirit of Moll’s conjectures on the $k$-log-concavity [7].

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**References**


