The Reverse Ultra Log-Concavity of the Boros-Moll Polynomials

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Abstract. Based on the recurrence relations on the coefficients of the Boros-Moll polynomials $P_m(a) = \sum_i d_i(m)a^i$ derived independently by Kauers and Paule, and Moll, we are led to the discovery of the reverse ultra log-concavity of the sequence $\{d_i(m)\}$. We also show that the sequence $\{i!d_i(m)\}$ is log-concave for $m \geq 1$. Two conjectures are proposed.

Keywords: log-concavity, reverse ultra log-concavity, upper bound, imaginary roots, Boros-Moll polynomials.

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1 Introduction

The main objective of this paper is to prove the reverse ultra log-concavity of the Boros-Moll polynomials. Boros and Moll [1, 2, 3, 7] studied a class of Jacobi polynomials in connection with the following integral:

Theorem 1.1

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a).$$

where $P_m(a)$ can be represented by

$$P_m(a) = 2^{-2m}\sum_k 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k.$$

The polynomials $P_m(a)$ are called Boros-Moll polynomials. Let

$$P_m(a) = \sum_{i=0}^m d_i(m)a^i.$$

The coefficients $d_i(m)$ are positive. They are also log-concave, as conjectured by Moll [7], and recently proved by Kauers and Paule [5].

Theorem 1.2 *For $0 < i < m$, we have*

$$d_i^2(m) \geq d_{i-1}(m)d_{i+1}(m). \tag{1.1}$$

Recall that a sequence $(a_n)_{n\geq 0}$ of real numbers is said to be log-concave if

$$a_n^2 \geq a_{n+1}a_{n-1}, \tag{1.2}$$

for all $n \geq 1$. If the sequence satisfies (1.2) with strict inequality, then it is said to be strictly log-concave. A polynomial is said to be (strictly) log-concave if its sequence of coefficients is (strictly) log-concave. Log-concave sequences and polynomials often arise in combinatorics, algebra and geometry, see, for example, Brenti [4] and Stanley [9].

A sequence $\{a_k\}_{0\leq k\leq n}$ is called ultra log-concave if

$$\left\{ a_k \middle/ \binom{n}{k} \right\}_{0\leq k\leq n} \tag{1.3}$$

is log-concave, see Liggett [6]. Note that this condition can be rewritten as

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \tag{1.4}$$

It is well known that if a polynomial has only real roots, then its coefficients form an ultra log-concave sequence. The above relation (1.4) implies the following inequality

$$ka_k^2 - (k+1)a_{k-1}a_{k+1} \geq 0,$$

from which we can deduce that the sequence $\{k!a_k\}$ is log-concave. This further implies that $\{a_k\}$ is strictly log-concave.

The first result of this paper is to show that $i!\{d_i(m)\}$ is log-concave, as stated below in an equivalent form.

Theorem 1.3 *For $1 \leq i \leq m-1$, we have*

$$i\, d_i^2(m) > (i+1)d_{i-1}(m)d_{i+1}(m). \tag{1.5}$$

Despite the above property of $d_i(m)$, we will show that the reverse ultra log-concavity holds, as described in the following theorem. This is the main result of this paper.

Theorem 1.4 *For $1 \leq i \leq m-1$, we have*

$$\frac{d_{i-1}(m)}{\binom{m}{i-1}}\frac{d_{i+1}(m)}{\binom{m}{i+1}} > \left(\frac{d_i(m)}{\binom{m}{i}}\right)^2, \tag{1.6}$$

or, equivalently,

$$(m-i+1)(i+1)d_{i-1}(m)d_{i+1}(m) - (m-i)id_i(m)^2 > 0. \tag{1.7}$$

We conclude this paper with two conjectures. Roughly speaking, the first conjecture says that in spite of the reverse ultra log-concavity, the ultra log-concavity almost holds in the asymptotic sense. The second conjecture is concerned with the log-concavity of the sequence $d_{i-1}(m)d_{i+1}(m)/d_i^2(m)$ for $m \geq 2$.

2 The Reverse Ultra Log-Concavity

In this section, we give the proofs of and Theorem 1.3 Theorem 1.4. We will need the following three recurrence formulas derived independently by Kauers and Paule [5] and Moll [8]:

$$d_i(m+1) = \frac{m+i}{m+1} d_{i-1}(m) + \frac{4m+2i+3}{2(m+1)} d_i(m), \quad 0 \le i \le m+1, \tag{2.1}$$

$$\begin{aligned} d_i(m+1) =& \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)} d_i(m) \\ & - \frac{i(i+1)}{(m+1)(m+1-i)} d_{i+1}(m), \quad 0 \le i \le m, \end{aligned} \tag{2.2}$$

$$\begin{aligned} d_i(m+2) =& \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) \\ & - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m), \quad 0 \le i \le m+1, \end{aligned} \tag{2.3}$$

Recall that Kauers and Paule [5] obtained the following lower bound on $d_i(m+1)/d_i(m)$ for $0 \le i \le m$,

$$\frac{d_i(m+1)}{d_i(m)} \ge Q(m,i), \tag{2.4}$$

where

$$Q(m,i) = \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}. \tag{2.5}$$

Proof of Theorem 1.3. As the first step, by the recurrence relations (2.1) and (2.2) we may transform Theorem 1.3 to the following equivalent form

$$\begin{aligned} &4(m+1)^2(m-i+1)d_i(m+1)^2 - 4(m+1)(4m^2+7m-2i^2+3)d_i(m)d_i(m+1) \\ &+ (4m+4i+3)(4m^2+7m-i+3)d_i(m)^2 > 0, \end{aligned}$$

which can be recast as

$$\begin{aligned} &4(m+1)^2(m-i+1)\left(\frac{d_i(m+1)}{d_i(m)}\right)^2 - 4(m+1)(4m^2+7m-2i^2+3)\frac{d_i(m+1)}{d_i(m)} \\ &+ (4m+4i+3)(4m^2+7m-i+3) > 0. \end{aligned} \tag{2.6}$$

Notice that the discriminant of the above quadratic form is positive, since

$$\triangle = 16i^2(2i-1)^2(m+1)^2 > 0.$$

Thus the quadratic function on the left hand side of (2.6) has two real roots,

$$x_1 = \frac{4m^2+7m+3-i}{2(m-i+1)(m+1)}, \quad x_2 = \frac{4m+4i+3}{2(m+1)}.$$

Using the lower bound $Q(m,i)$ for $d_i(m+1)/d_i(m)$, we deduce that for $1 \le i \le m-1$,

$$\frac{d_i(m+1)}{d_i(m)} \ge Q(m,i) > x_1 > x_2.$$

Thus we obtain (2.6), and the proof is complete. ∎

In order to prove Theorem 1.4, we need an upper bound for the ratio $d_i(m+1)/d_i(m)$.

Theorem 2.1 *We have for all $m \ge 2$, $1 \le i \le m-1$,*

$$\frac{d_i(m+1)}{d_i(m)} < T(m,i), \tag{2.7}$$

where

$$T(m,i) = \frac{4m^2+7m+3+i\sqrt{4m+4i^2+1}-2i^2}{2(m-i+1)(m+1)}, \tag{2.8}$$

and for $m \ge 1$,

$$\frac{d_0(m+1)}{d_0(m)} = T(m,0), \quad \frac{d_m(m+1)}{d_m(m)} = T(m,m). \tag{2.9}$$

Proof. First, we consider (2.9). Setting $i=0$ in (2.1) gives that for any $m \ge 1$,

$$\frac{d_0(m+1)}{d_0(m)} = \frac{4m+3}{2(m+1)},$$

which agrees with $T(m,0)$. While $i=m$, (2.1) implies

$$\frac{d_m(m+1)}{d_m(m)} = \frac{(2m+3)(2m+1)}{2(m+1)} = T(m,m).$$

Thus (2.9) holds for $m \ge 1$.

We now proceed to conduct induction on m to show that (2.7) is valid for $i \ge 1$. When $m=2$ and $i=1$, we have

$$\frac{d_1(3)}{d_1(2)} = \frac{43}{15} < T(2,1) = \frac{31+\sqrt{13}}{12}.$$

We assume that the theorem holds for m, where $m > 2$. Then we aim to show that it also holds for $m+1$, namely, for $1 \le i \le m$,

$$d_i(m+2) < T(m+1,i)d_i(m+1). \tag{2.10}$$

Using the recurrence (2.3), we may rewrite (2.10) in the following form

$$\begin{aligned}&\frac{-4i^2+8m^2+24m+19}{2(m-i+2)(m+2)}d_i(m+1)\\&-\frac{(m+i+1)(4m+3)(4m+5)}{4(m+1)(m+2)(m-i+2)}d_i(m) < T(m+1,i)d_i(m+1).\end{aligned} \tag{2.11}$$

In order to derive an upper bound for $d_i(m+1)/d_i(m)$, it is necessary to show that

$$R(m,i) = \frac{-4i^2+8m^2+24m+19}{2(m-i+2)(m+2)} - T(m+1,i) \tag{2.12}$$

is positive. Since $m \geq i$, we have $4m+4i^2+5 < 12m+4m^2+9$. It follows that

$$\begin{aligned}R(m,i) &= \frac{4m^2+9m+5-2i^2-i\sqrt{4m+4i^2+5}}{2(m-i+2)(m+2)}\\&\geq \frac{4m^2+9m+5-2i^2-i(2m+3)}{2(m-i+2)(m+2)}\\&= \frac{(4m^2-2i^2-2mi)+(9m-3i)+5}{2(m-i+2)(m+2)},\end{aligned}$$

which is positive for $1 \leq i \leq m$. Hence (2.11) is equivalent to the following inequality

$$\frac{d_i(m+1)}{d_i(m)} < \frac{(m+i+1)(4m+3)(4m+5)}{4(m+1)(m+2)(m-i+2)R(m,i)}. \tag{2.13}$$

Note that the right hand side of (2.13) can be expressed as

$$F(m,i) = \frac{(m+i+1)(4m+3)(4m+5)}{2(m+1)(4m^2-2i^2+9m+5-i\sqrt{4m+4i^2+5})}. \tag{2.14}$$

By the inductive hypothesis, it suffices to show that for $1 \leq i \leq m-1$,

$$T(m,i) \leq F(m,i). \tag{2.15}$$

Let $A = \sqrt{4m+4i^2+1}$ and $B = \sqrt{4m+4i^2+5}$. It is easy to check that $F(m,i)-T(m,i)$ equals

$$\frac{(i^2-4i^4)-i(5+4m^2+9m-2i^2)A+i(3+4m^2+7m-2i^2)B+i^2AB}{2(m+1)(m-i+1)(4m^2+9m+5-2i^2-iB)}. \tag{2.16}$$

Since

$$(4m^2+9m+5-2i^2)^2-(iB)^2 = (4m+5)^2(m+i+1)(m-i+1) > 0,$$

the denominator of (2.16) is positive. It remains to show that the numerator of (2.16) is also positive. Observe that every term in the numerator contains a factor i. So we may divide this factor. Put

$$\begin{aligned} X &= (i - 4i^3) + iAB \\ Y &= (5 + 4m^2 + 9m - 2i^2)A - (3 + 4m^2 + 7m - 2i^2)B. \end{aligned}$$

We claim that $X > 0$ and $X > Y$. Since $m \geq i$, we have $A \geq 2i + 1$ and $B \geq 2i + 1$. Moreover, since $i \geq 1$, we find

$$X = (i - 4i^3) + iAB \geq i - 4i^3 + i(2i + 1)^2 = 4i^2 + 2i > 0.$$

To show that $X - Y > 0$, we will consider $X^2 - Y^2$. Let us introduce $G(m, i)$ and $H(m, i)$ as given by

$$\begin{aligned} G(m, i) &= (32m^4 - 32m^2i^2 + 128m^3 - 64mi^2 + 190m^2 - 30i^2 + 124m + 30)AB, \\ H(m, i) &= 128m^5 + 608m^4 + 1128m^3 + 1014m^2 + 436m + 128m^4i^2 + 384m^3i^2 \\ &\quad + 408m^2i^2 - 128m^2i^4 + 200mi^2 - 256mi^4 - 120i^4 + 50i^2 + 70. \end{aligned}$$

It can be checked that

$$X^2 - Y^2 = G(m, i) - H(m, i). \tag{2.17}$$

Since $i \leq m - 1$, it is easily seen that $G(m, i) > 0$. To verify $G(m, i) > H(m, i)$, we will show that $G(m, i)^2 - H(m, i)^2 > 0$. In fact, for $1 \leq i \leq m - 1$

$$G(m, i)^2 - H(m, i)^2 = 16(4m + 5)^2(16mi^2 + 12i^2 - 1)(m + i + 1)^2(m - i + 1)^2 > 0.$$

This implies that $X > Y$. Hence the numerator of (2.16) is positive. Consequently, (2.15) holds for $1 \leq i \leq m - 1$.

Up to now, we still need to consider the case $i = m$. It remains to show that

$$\frac{d_m(m + 2)}{d_m(m + 1)} < T(m + 1, m).$$

By direct computation, we find that

$$d_m(m + 2) = \frac{(m + 1)(4m^2 + 18m + 21)}{2^{m+4}(2m + 3)} \binom{2m + 4}{m + 2}$$

and

$$d_m(m + 1) = \frac{2m + 3}{2^{m+2}} \binom{2m + 2}{m + 1}.$$

We get

$$\frac{d_m(m + 2)}{d_m(m + 1)} = \frac{(m + 1)(4m^2 + 18m + 21)}{2(2m + 3)(m + 2)}.$$

On the other hand,

$$T(m+1,m) = \frac{2m^2+15m+14+m\sqrt{4m^2+4m+5}}{4(m+2)}.$$

We see that for $m \geq 2$,

$$\begin{aligned}
&T(m+1,m) > \frac{d_m(m+2)}{d_m(m+1)} \\
\Longleftrightarrow &(2m^2+15m+14+m\sqrt{4m^2+4m+5})(2m+3) - 2(m+1)(4m^2+18m+21) > 0 \\
\Longleftrightarrow &(2m^2+3m)\sqrt{4m^2+4m+5} > 4m^3+8m^2+5m \\
\Longleftrightarrow &\left((2m^2+3m)\sqrt{4m^2+4m+5}\right)^2 - \left(4m^3+8m^2+5m\right)^2 > 0 \\
\Longleftrightarrow &4m^2(4m+5) > 0,
\end{aligned}$$

which is evident. This completes the proof of the theorem. ∎

We are now ready to prove Theorem 1.4. Like the first step in the proof of Theorem 1.3, we use the recurrences (2.1) and (2.2) to restate (1.7) as follows

$$\begin{aligned}
&4(m-i+1)^2(m+1)^2\left(\frac{d_i(m+1)}{d_i(m)}\right)^2 \\
&-4(m-i+1)(m+1)(4m^2-2i^2+7m+3)\frac{d_i(m+1)}{d_i(m)} \\
&-(32mi^2-56m^3-73m^2-42m+13i^2-9-16m^4+16i^2m^2) < 0. \qquad (2.18)
\end{aligned}$$

Observe that the discriminant of the above quadratic form is positive for $i \geq 1$, since

$$\triangle = 16i^2(m+1)^2(4i^2+4m+1)(m-i+1)^2 > 0.$$

It follows that the quadratic function on the left hand side of (2.18) has two real roots

$$x_1 = \frac{4m^2-2i^2+7m+3-i\sqrt{4m+4i^2+1}}{2(m-i+1)(m+1)},$$

$$x_2 = \frac{4m^2-2i^2+7m+3+i\sqrt{4m+4i^2+1}}{2(m-i+1)(m+1)}.$$

By the definition of $Q(m,i)$ in (2.5), we see that $x_1 < Q(m,i)$. Note that $x_2 = T(m,i)$ as given in Theorem 2.1. In view of Theorem 2.1, we deduce that

$$x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2,$$

for $1 \leq i \leq m-1$. Hence we conclude that (2.18) holds. This completes the proof of Theorem 1.4. ∎

We conclude this paper with two conjectures. Let

$$c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)}, \quad 1 \leq i \leq m-1.$$

Then Theorem 1.3 and Theorem 1.4 lead to the following bounds on $c_i(m)$ for $1 \leq i \leq m-1$,

$$1 + \frac{1}{i} \leq c_i(m) \leq \left(1 + \frac{1}{i}\right)\left(1 + \frac{1}{m-i}\right). \tag{2.19}$$

Numerical evidence indicates that the upper bound in (2.19) is very close to $c_i(m)$ even when m is small. Let $u_i(m) = \left(1 + \frac{1}{i}\right)\left(1 + \frac{1}{m-i}\right)$. For example, when $m = 6$, the values of $c_i(m)/u_i(m)$ are given below

0.9462708849, 0.9642110408, 0.9752109510, 0.9821688283, 0.9867303609.

Conjecture 2.2 *For $1 \leq i \leq m-1$, we have*

$$\lim_{m\to\infty} \frac{c_i(m)}{u_i(m)} = 1. \tag{2.20}$$

Conjecture 2.3 *For $m \geq 2$, the sequence $\{1/c_i(m)\}_{i=2}^{m-2}$ is log-concave.*

Conjecture 2.2 implies that the Boros-Moll polynomials are almost ultra log-concave. Further conjectures can be made based on Conjecture 2.3 in the spirit of Moll's conjectures on the k-log-concavity [7].

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