Congruences for the Number of Cubic Partitions Derived from Modular Forms

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Abstract

We obtain congruences for the number $a(n)$ of cubic partitions using modular forms. The notion of cubic partitions is introduced by Chan and named by Kim in connection with Ramanujan's cubic continued fractions. Chan has shown that $a(n)$ has several analogous properties to the number $p(n)$ of partitions, including the generating function, the continued fraction, and congruence relations. To be more specific, we show that $a(25n+22) \equiv 0 \pmod 5$, $a(49n+15) \equiv a(49n+29) \equiv a(49n+36) \equiv a(49n+43) \equiv 0 \pmod 7$. Furthermore, we prove that $a(n)$ takes infinitely many even values and infinitely odd values.

Keywords: cubic partition, congruence, modular form, Ramanujan's cubic continued fraction, parity.

AMS Classification: 11F33, 11P83

1 Introduction

The main objective of this paper is to study congruence relations for the number of cubic partitions by constructing suitable modular forms. The number of cubic partitions, denoted by $a(n)$, originated from the work of Chan [6] in connection with Ramanujan's cubic continued fraction which is often denoted by

$$G(q) := \frac{q^{1/3}}{1}{}_{+}\frac{q+q^2}{1}{}_{+}\frac{q^2+q^4}{1}{}_{+}\frac{q^3+q^6}{1}{}_{+\cdots}, \quad |q|<1.$$

On page 366 of his Lost Notebook, Ramanujan claimed that there are many properties of $G(q)$ which are analogous to Rogers-Ramanujan continued fraction $R(q)$ [21]

$$R(q) := \frac{q^{1/5}}{1}{}_{+}\frac{q}{1}{}_{+}\frac{q^2}{1}{}_{+}\frac{q^3}{1}{}_{+\cdots}, \quad |q|<1.$$

Motivated by Ramanujan's observation, many new results on $G(q)$ analogous to those for $R(q)$ have been found, see, e.g., Chan [9]. To give an overview of recent results on $a(n)$, it is informative to recall relevant background on the generating function of $p(n)$ and the Rogers-Ramanujan continued fraction $R(q)$.

Ramanujan obtained many theorems on $R(q)$, see Andrews and Berndt [1]. In particular, he discovered the following beautiful identities on $R(q)$ and $1/R(q)$.

$$\frac{1}{R(q)} - 1 - R(q) = \frac{(q^{1/5};q^{1/5})_\infty}{q^{1/5}(q^5;q^5)_\infty} \tag{1.1}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}. \tag{1.2}$$

Here $(q;q)_\infty$ is the usual notation for $\prod_{n=1}^{\infty}(1-q^n)$.

Berndt [5, p.165] gave a beautiful proof of the following classical identity of Ramanujan by using the continued fraction $R(q)$:

$$\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5}\sum_{n=0}^{\infty} p(5n+4)q^n = 5. \tag{1.3}$$

Dividing (1.2) by (1.1), we get

$$\begin{aligned}&\frac{(q;q)_\infty^6}{q^{4/5}(q^{1/5};q^{1/5})_\infty(q^5;q^5)_\infty^5}\\ &\quad = R^4(q) - R^3(q) + 2R^2(q) - 3R(q) + 5 + \tfrac{3}{R(q)} + \tfrac{2}{R^2(q)} + \tfrac{1}{R^3(q)} + \tfrac{1}{R^4(q)}.\end{aligned} \tag{1.4}$$

Now, (1.3) can be easily deduced from (1.4) by extracting the integer powers of $q^n, n \geq 0$ from both sides of above identity since $R(q)$ has only terms in the form of $q^{n+1/5}$. Ramanujan's congruence on $p(n)$ modulo 5 can be derived directly from (1.3)

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.5}$$

Recently, using two identities of Ramanujan [21] on $G(q)$, see also Berndt [4, p.345, Entry 1], Chan [6] has found the following identities on $G(q)$ and $1/G(q)$ analogous to the above identities (1.1) and (1.2):

$$\frac{1}{G(q)} - 1 - 2G(q) = \frac{(q^{1/3};q^{1/3})_\infty(q^{2/3};q^{2/3})_\infty}{q^{1/3}(q^3;q^3)_\infty(q^6;q^6)_\infty}, \tag{1.6}$$

$$\frac{1}{G^3(q)} - 7 - 8G^3(q) = \frac{(q;q)_\infty^4(q^2;q^2)_\infty^4}{q(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}. \tag{1.7}$$

Motivated by the idea of Berndt, Chan derived the following identity by dividing both sides of (1.7) by (1.6) and then setting $q \to q^3$:

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = q^2\frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}\left(4G^2(q^3) - 2G(q^3) + 3 + \frac{1}{G(q^3)} + \frac{1}{G^2(q^3)}\right). \tag{1.8}$$

Observing that the powers of q in $G(q^3)$ are in the form of $3n+1$, we find

$$\sum_{n=0}^{\infty}\left[q^{3n}\right]\left(4G^2(q^3) - 2G(q^3) + 3 + \frac{1}{G(q^3)} + \frac{1}{G^2(q^3)}\right)q^{3n} = 3.$$

It is now natural to define a function $a(n)$ by the left hand side of (1.8)

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}, \tag{1.9}$$

and it is natural to expect $a(n)$ to have analogous properties to $p(n)$.

Extracting those terms whose powers of q are in the form of $3n+2$ on both sides of (1.8), and then simplifying and setting $q^3 \to q$, Chan established the following elegant identity analogous to (1.3)

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}. \tag{1.10}$$

The above identity immediately leads to the following congruence

$$a(3n+2) \equiv 0 \pmod 3, \tag{1.11}$$

which is analogous to Ramanujan's congruence (1.5) for $p(n)$.

From the point of view of partitions, it is obvious from the generating function (1.9) that $a(n)$ is the number of partition pairs (λ, μ) where $|\lambda| + |\mu| = n$ and μ only has even parts. Chan has called $a(n)$ a certain partition function. Kim [11] called such partitions counted by $a(n)$ cubic partitions owing to the fact that $a(n)$ is close related to Ramanujan's cubic continued fraction.

Based on the cubic partition interpretation of $a(n)$, Chan [8] asked whether there exist a function analogous to Dyson's rank that leads to a combinatorial interpretation of the congruence (1.5). Kim [11] discovered a crank function $N_V^a(m,n)$ for cubic partitions. Let $M'(m,N,n)$ be the number of cubic partitions of n with crank $\equiv m \pmod N$, Kim proved that

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3,$$

which implies (1.11).

Our main results are concerned with congruences for $a(n)$ modulo 5 and 7 which are in the spirit of Ramanujan's classical congruences modulo 7 and 11. Recall that Ramanujan obtained more general congruences modulo 5^k:

$$p(5^k n + r_k) \equiv 0 \pmod{5^k}, \tag{1.12}$$

where $k \geq 1$ and $5^k r_k \equiv 1 \pmod{24}$. In analogy with Ramanujan's congruences, Chan considered the general congruences for $a(n)$ modulo powers of 3. Employing the method of Hirschhorn and Hunt [10] to prove (1.12), Chan [7] derived the following congruence as a consequence of (1.10).

Theorem 1.1. *For $k \geq 1$,*

$$a(3^k n + c_k) \equiv 0 \pmod{3^{k+\delta(k)}}, \tag{1.13}$$

where c_k is the reciprocal modulo 3^k of 8, and $\delta(k) = 1$ if k is even and $\delta(k) = 0$ otherwise.

In the general case, Ramanujan conjectured that there are only three choices for a prime l such that the congruence $p(ln+c) \equiv 0 \pmod{l}$ holds, namely, $l = 5, 7, 11$. This conjecture has been confirmed by Ahlgren and Boylan [3] based on the work of Kiming and Olsson [12]. Chan [7] raised the problem of finding simple congruences for $a(n)$ besides $a(3n+2) \equiv 0$ mod 3. Recently, Sinick [22] has shown that there does not exist other primes l such that $a(ln+c) \equiv 0 \pmod{l}$ except that $l = 3$. In analogy with the results for $p(n)$ due to Ono [17] and Ahlgren [2], Chan [8] obtained the following theorem concerning congruences for $a(n)$ modulo powers of a prime.

Theorem 1.2. *Let $m \geq 5$ be prime and j a positive integer. Then a positive proportion of the primes $Q \equiv -1 \pmod{128m^j}$ have the property that*

$$a\left(\frac{mQn+1}{8}\right) \equiv 0 \pmod{m^j},$$

for every n coprime to Q.

The above theorem implies that for every integer n there exists infinitely many non-nested arithmetic progressions $An + B$ for prime $m \geq 5$ and positive integer j such that

$$a(An+B) \equiv 0 \pmod{m^j}.$$

It should be noted that although the proof of Theorem 1.2 leads to some Ramanujan-type congruences modulo m^j, it does not cover all the congruences in form of $a(An+B) \equiv 0 \pmod{m^j}$. Chan [7] studied the case for $m = 3$, which is not in the scope of Theorem 1.2. This paper is devoted to finding concrete congruences for the cases $m = 5, 7$ and $j = 1$, which are also out of the range of Theorem 1.2 since the Q is larger than 1278 and 2686 for $m = 5$ and 7, respectively. To be precise, we derive the following congruences by constructing suitable modular forms.

Theorem 1.3. *For every nonnegative integer n, we have*

$$a(25n+22) \equiv 0 \pmod{5}.$$

It would be interesting to give a combinatorial interpretation of the above congruence by finding a suitable crank function. In the following theorem, we present some congruences modulo 7.

Theorem 1.4. *For every nonnegative integer n, we have*

$$a(49n+15) \equiv a(49n+29) \equiv a(49n+36) \equiv a(49n+43) \equiv 0 \pmod{7}.$$

The last section of this paper is focused on the parity of $a(n)$. Recall that Kolberg [13] has shown that $p(n)$ takes both even and odd values infinitely often. From numerical evidence, we conjecture that when n tends to infinity the parities of $a(1), a(2), \ldots, a(n)$ are equidistributed. While we have not been able to prove this conjecture, we shall show that there are infinitely many even values of $a(n)$ and there are infinitely many odd values of $a(n)$.

2 Preliminaries

To make this paper self-contained, we give an overview of the background relevant to the proofs of the congruences for $a(n)$ by using modular forms. For more details on the theory of modular forms, see for example, Koblitz [14] and Ono [18].

For a rational integer $N \geq 1$, the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv 0 \pmod{N} \right\}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ act on the complex upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$$

by the linear fractional transformation

$$\gamma z := \frac{az + b}{cz + d}.$$

Suppose that k is a positive integer and χ is a Dirichlet character modulo N.

Definition 2.1. *Let $f(z)$ be a holomorphic function on $\mathbb{H}$ and satisfy the following relation for all $\gamma \in \Gamma_0(N)$ and all $z \in \mathbb{H}$,*

$$f(\gamma z) = \chi(d)(cz + d)^k f(z).$$

In addition, if $f(z)$ is also holomorphic at the cusps of $\Gamma_0(N)$, we call such a function $f(z)$ a modular form of weight k on $\Gamma_0(N)$.

The modular forms of weight k on $\Gamma_0(N)$ with Dirichlet character χ form a finite-dimensional complex vector space denoted by $M_k(\Gamma_0(N), \chi)$. For convenience, we write $M_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi)$ when χ is the trivial Dirichlet character.

Dedekind's eta function is defined by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$. It is well-known that $\eta(z)$ is holomorphic and does not vanish on $\mathbb{H}$.

A function $f(z)$ is called eta-quotient if it can be written in the form of

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z),$$

where $N \geq 1$ and each r_δ is an integer. The following two facts is useful to verify whether an eta-quotient is a modular form, see Ono [18, p.18].

Proposition 2.1. *If $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ is an eta-quotient with*

$$k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z},$$

satisfies the following conditions:

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \tag{2.1}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \tag{2.2}$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z) \tag{2.3}$$

for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, where

$$s := \prod_{\delta|N} \delta^{r_\delta}$$

and $\left(\frac{m}{n}\right)$ is Kronecker symbol.

Based on this proposition, for a given eta-quotient $f(z)$, by checking the conditions (2.1) and (2.2), one can show that $f(z)$ satisfies (2.3). Moreover, if k is a positive integer and $f(z)$ is holomorphic at the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$ because $\eta(z)$ is holomorphic and does not vanish on $\mathbb{H}$. Combined with the following proposition which gives the analytic orders of an eta-quotient at the cusps of $\Gamma_0(N)$, we can deduce that $f(z)$ is a modular form.

Proposition 2.2. *Let c, d and N be positive integers with $d|N$ and $(c,d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions in Proposition 2.1 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta|N} \frac{(d,\delta)^2 r_\delta}{(d, \frac{N}{d}) d\delta}.$$

In the other words, to prove that the above function $f(z)$ is holomorphic at the cusp $\frac{c}{d}$, it suffices to check that

$$\sum_{\delta|N} \frac{(d,\delta)^2 r_\delta}{\delta} \geq 0.$$

Let M be a positive integer and

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

be a function with rational integer coefficients. Define $ord_M(f(z))$ to be the smallest n such that $a(n) \not\equiv 0 \pmod{M}$. Sturm [23] provided the following powerful criterion to determine whether two modular forms are congruent modulo a prime by the verification of a finite number of cases.

Proposition 2.3. *Let p be a prime and $f(z), g(z) \in M_k(\Gamma_0(N))$ with rational integer coefficients. If*

$$ord_p(f(z) - g(z)) > \frac{kN}{12} \prod_d (1 + \frac{1}{d}),$$

where the product is over the prime divisors d of N. Then $f(z) \equiv g(z) \pmod{p}$, i.e., $ord_p(f(z) - g(z)) = \infty$.

We also need the following result due to Lovejoy [15].

Proposition 2.4. *Let*

$$f = \sum_{n=0}^{\infty} u(n) q^n$$

and

$$g = 1 + \sum_{n=1}^{\infty} v(mn) q^{mn}.$$

Define $w(n)$ by

$$fg = \sum_{n=0}^{\infty} w(n) q^n.$$

Let d be a residue class modulo m. Then,

(1) If $u(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for $0 \leq n \leq N$, then $w(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for $0 \leq n \leq N$.

(2) If $w(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for all n, then $u(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for all n.

The following two propositions will also be used to construct modular forms, see Koblitz [14].

Proposition 2.5. *Suppose $f(z) \in M_k(\Gamma_0(N))$ with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} u(n) q^n.$$

Then for any positive integer $m|N$,

$$f(z)|U(m) := \sum_{n=0}^{\infty} u(mn) q^n$$

is the Fourier expansion of a modular form in $M_k(\Gamma_0(N))$.

Proposition 2.6. *Let χ_1 be a Dirichlet character modulo M, and let χ_2 be a primitive Dirichlet character modulo N. Let*

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(M, \chi_1)$$

and

$$g(z) = \sum_{n=0}^{\infty} a(n) \chi_2(n) q^n.$$

Then $g(z) \in M_k(MN^2, \chi_1 \chi_2^2)$. In particular, if $f(z) \in M_k(\Gamma_0(M))$ and χ_2 is quadratic, then $g(z) \in M_k(\Gamma_0(MN^2))$.

3 Congruences for the Number of Cubic Partitions

In this section, we give the proofs of Theorem 1.3 and Theorem 1.4 using the technique of modular forms due to Ono [16]. The following congruence relation is well-known, see, for example, Ono [16]. We include a proof for the sake of completeness.

Lemma 3.1. *If $p \geq 3$ is a prime, then*

$$\frac{(q;q)_\infty^p}{(q^p;q^p)_\infty} \equiv 1 \pmod{p}. \tag{3.1}$$

Proof. Using the well-known binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

it is easily seen that

$$\frac{(1-X)^p}{1-X^p} = \frac{\sum_{k=0}^{p} \binom{p}{k} (-1)^k X^k}{1-X^p} \equiv \frac{1-X^p}{1-X^p} \equiv 1 \pmod{p},$$

since $p \,|\binom{p}{k}$ for $0 < k < p$. It follows that

$$\frac{(q;q)_\infty^p}{(q^p;q^p)_\infty} = \prod_{k=1}^{\infty} \frac{(1-q^k)^p}{(1-q^{kp})} \equiv 1 \pmod{p},$$

as desired. ∎

We first consider Theorem 1.3, that is, for $n \geq 0$,

$$a(25n+22) \equiv 0 \pmod{5}.$$

Proof. To establish the claimed congruence relation, we shall construct an eta-quotient with the following expansion in $q = e^{2\pi i z}$,

$$g(z) = \sum_{n \geq 0} b(n) q^n.$$

We assume that $g(z)$ satisfies the following conditions

(1) $g(z)$ is a modular form;

(2) If for all $n \geq 0$, $b(25n+25) \equiv 0 \pmod{5}$ then $a(25n+22) \equiv 0 \pmod{5}$;

(3) The function

$$g(z)|U(25) = \sum_{n \geq 0} b(25n) q^n \equiv 0 \pmod{5}$$

is also a modular form.

In order to satisfy the second condition, we consider the function $h(q)$ of the following form

$$h(q)=\prod_i (q^{25r_i};q^{25r_i})_\infty^{s_i}\prod_j \Big(\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}\Big)^{t_j},$$

where r_i, s_i, t_j are integers. By the above Lemma 3.1, it is easily seen that for any integers r_i, s_i and t_j the expansion of $h(q)$ has the following form modulo 5,

$$h(q)\equiv 1+\sum_{m\geq 1} c(m)q^{25m} \pmod 5,$$

where $c(m)$ are integers. Now, we set

$$g(z)=h(q)\sum_{n=0}^{\infty} a(n)q^{n+3}=h(q)\sum_{n\geq 3}^{\infty} a(n-3)q^n. \tag{3.2}$$

Since $h(q)$ is a series in q^{25} modulo 5 with constant term 1, we can make use of Proposition 2.4 (2) to deduce that if for any n, $b(25n+25)\equiv 0 \pmod 5$, then we have $a(25n+22)\equiv 0 \pmod 5$.

We now proceed to determine the parameters r_i, s_i and t_j in $g(z)$ to make it a modular form. Consider the case $r_1=1, s_1=1, r_2=2, s_2=1$ and $t_1=2$, namely,

$$g(z)=(q^{25};q^{25})_\infty(q^{50};q^{50})_\infty\Big(\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}\Big)^2\sum_{n=0}^{\infty} a(n)q^{n+3}. \tag{3.3}$$

We are going to show that $g(z)$ satisfies the conditions (2.1) and (2.2) in Proposition 2.1. Recalling the definition of $\eta(z)$, we can rewrite $g(z)$ as an eta-quotient

$$\begin{aligned} g(z) &= \frac{\eta(25z)\eta(50z)}{\eta(z)\eta(2z)}\left(\frac{\eta^5(z)}{\eta(5z)}\right)^2 \\ &= \frac{\eta^9(z)\eta(25z)\eta(50z)}{\eta(2z)\eta^2(5z)}. \end{aligned}$$

The two conditions (2.1) and (2.2) can be expressed as follows,

$$\begin{aligned} \sum_{\delta|50}\delta r_\delta &= 9-2-5\times 2+25+50 \equiv 0 \pmod{24}, \\ \sum_{\delta|50}\frac{50}{\delta}r_\delta &= 50\times 9-25-10\times 2+2+1\equiv 0 \pmod{24}. \end{aligned}$$

To make $g(z)$ a modular form, it remains to compute the order of $g(z)$ at cusps. By Proposition 2.2, it is easily verified that the order of $g(z)$ at the cusps of $\Gamma_0(50)$ are nonnegative, that is, for any $d|50$,

$$\sum_{\delta|50}\frac{(d,\delta)^2 r_\delta}{\delta}\geq 0.$$

So we have $g(z)\in M_4(\Gamma_0(50),\chi_1)$, where

$$\chi_1(d)=\left(\frac{25}{d}\right)=\left(\frac{5}{d}\right)^2=1$$

for $(d, 50) = 1$. This implies that $g(z) \in M_4(\Gamma_0(50))$.

Since we have proved that $g(z)$ satisfies the second condition, the following congruence is valid

$$a(25n + 22) \equiv 0 \pmod 5 \tag{3.4}$$

provided that for all $n \geq 0$,

$$b(25n + 25) \equiv 0 \pmod 5. \tag{3.5}$$

Let us rewrite (3.5) as

$$\sum_{n \geq 1} b(25n) q^n \equiv 0 \pmod 5. \tag{3.6}$$

By Proposition 2.5, we see that the summation on the left hand side of (3.6) is a modular form, that is,

$$g(z)|U(25) = \sum_{n \geq 1} b(25n) q^n \in M_4(\Gamma_0(50)).$$

Hence, by Proposition 2.3, we find that (3.6) is valid if (3.5) holds for

$$0 \leq n \leq \frac{4 \times 50}{12}(1 + \frac{1}{2})(1 + \frac{1}{5}) + 1 = 31.$$

Applying Lemma 3.1 with $p = 5$ to (3.3), we have

$$\sum_{n \geq 3} b(n) q^n \equiv (q^{25}; q^{25})_\infty (q^{50}; q^{50})_\infty \sum_{n=0}^{\infty} a(n) q^{n+3} \pmod 5. \tag{3.7}$$

Using the above relation and Proposition 2.4 (1), we see that the verification of (3.5) on $b(n)$ for $0 \leq n \leq 31$ can be reduced to the verification of (3.4) on $a(n)$ for a the same range $0 \leq n \leq 31$. It is readily checked that (3.4) holds for $0 \leq n \leq 31$. This completes the proof. ∎

Now, we turn to the proof of Theorem 1.4, namely,

$$a(49n + 15) \equiv a(49n + 29) \equiv a(49n + 36) \equiv a(49n + 43) \equiv 0 \pmod 7. \tag{3.8}$$

Proof of Theorem 1.4. Following the above procedure in the proof of Theorem 1.3, by the generating function of $a(n)$, we construct an eta-quotient

$$\begin{aligned} h(z) &= \frac{\eta(49z)\eta(98z)}{\eta(z)\eta(2z)} \left(\frac{\eta^7(z)}{\eta(7z)} \right)^2 \\ &= \frac{\eta^{13}(z)\eta(49z)\eta(98z)}{\eta(2z)\eta^2(7z)} \\ &= \left(\frac{(q; q)_\infty^7}{(q^7; q^7)_\infty} \right)^2 (q^{49}; q^{49})_\infty (q^{98}; q^{98})_\infty \sum_{n=0}^{\infty} a(n) q^{n+6}. \end{aligned} \tag{3.9}$$

Setting $N = 98$, we see that $h(z)$ satisfies the conditions (2.1) and (2.2) in Proposition 2.1, namely,

$$\begin{aligned}\sum_{\delta|98} \delta r_\delta &= 13 - 2 - 2 \times 7 + 49 + 98 \equiv 0 \pmod{24},\\ \sum_{\delta|98} \frac{98}{\delta} r_\delta &= 98 \times 13 - 49 - 14 \times 2 + 2 + 1 \equiv 0 \pmod{24}.\end{aligned}$$

Moreover, it is not difficult to verify that the order of $h(z)$ at the cusps of $\Gamma_0(98)$ are non-negative by Proposition 2.2, that is, for any $d|98$,

$$\sum_{\delta|98} \frac{(d,\delta)^2 r_\delta}{\delta} \geq 0.$$

Hence we deduce that $h(z)$ is a modular form, i.e., $h(z) \in M_6(\Gamma_0(98), \chi_2)$. Moreover,

$$\chi_2(d) = \left(\frac{49}{d}\right) = \left(\frac{7}{d}\right)^2 = 1$$

for $(d, 98) = 1$. This implies that $h(z) \in M_6(\Gamma_0(98))$.

Write

$$h(z) = \sum_{n \geq 6} c(n) q^n.$$

Applying Lemma 3.1 with $p = 7$, (3.9) becomes

$$\sum_{n \geq 6} c(n) q^n \equiv (q^{49}; q^{49})_\infty (q^{98}; q^{98})_\infty \sum_{n \geq 6} a(n-6) q^n \pmod{7}. \tag{3.10}$$

Since $(q^{49}; q^{49})_\infty (q^{98}; q^{98})_\infty$ can be expanded as series in q^{49} with constant term 1, we can make use of Proposition 2.4 (2) to deduce that the four congruences in (3.8) can be derived from the corresponding congruences for $c(n)$, i.e., for $n \geq 0$,

$$c(49n + 21) \equiv c(49n + 35) \equiv c(49n + 42) \equiv c(49n + 49) \equiv 0 \pmod{7}. \tag{3.11}$$

Observing that the arithmetic progressions $49n + 21, 49n + 35, 49n + 42, 49n + 49$ in the above congruences are divided by 7, we construct another function based on $c(n)$ as follows

$$u(z) = \sum_{n \geq 1} d(n) q^n = \sum_{n \geq 1} c(7n) q^n.$$

By Proposition 2.5, we have $u(z) \in M_6(\Gamma_0(98))$. Obviously, (3.11) can be restated as

$$d(7n + 3) \equiv d(7n + 5) \equiv d(7n + 6) \equiv d(7n + 7) \equiv 0 \pmod{7}. \tag{3.12}$$

As will be seen, one can combine the above four congruences into a single congruence relation. Define

$$v(z) = \sum_{n \geq 1} e(n) q^n = \sum_{n \geq 1} d(n) q^n - \sum_{n \geq 1} \left(\frac{n}{7}\right) d(n) q^n. \tag{3.13}$$

Since $\left(\frac{n}{7}\right) = 1$ for $n \equiv 1, 2, 4 \pmod 7$, $\left(\frac{n}{7}\right) = -1$ for $n \equiv 3, 5, 6 \pmod 7$ and $\left(\frac{n}{7}\right) = 0$ for $n \equiv 0 \pmod 7$, $v(z)$ can be expressed as

$$v(z) = \sum_{\left(\frac{n}{7}\right)=-1} 2d(n)q^n + \sum_{n \equiv 0 \pmod 7} d(n)q^n. \tag{3.14}$$

To prove (3.12), it suffices to show that $v(z) \equiv 0 \pmod 7$.

Denote the second summation in (3.13) by

$$w(z) = \sum_{n\geq 1} \left(\frac{n}{7}\right) d(n)q^n.$$

By Proposition 2.6, we see that $w(z)$ is a modular form. In other words, $w(z) \in M_6(\Gamma_0(4802))$. Since $u(z)$ is also a modular form, we obtain

$$v(z) = u(z) - w(z) \in M_6(\Gamma_0(4802).$$

By Proposition 2.3, we find that $v(z) \equiv 0 \pmod 7$ can be verified by a finite number of cases. To be precise, we need to check that

$$e(n) \equiv 0 \pmod 7$$

holds for $0 \leq n \leq 4117$. In view of (3.14), we only need to verify that (3.12) holds for $0 \leq n \leq \lceil \frac{4117-1}{7} \rceil - 1 = 587$. Since $d(n) = c(7n)$, it suffices to check (3.11) for $0 \leq n \leq 587$. Finally, using (3.10) and Proposition 2.4 (1), it is necessary to verify Theorem 1.4 holds only for $0 \leq n \leq 587$, which is an easy task. This completes the proof. ∎

4 The Parity of $a(n)$

In this section, we show that the function $a(n)$ takes infinitely many even values and infinitely many odd values. For the partition function $p(n)$, it has been conjectured by Parkin and Shanks [19] that the parities of $p(1), p(2), \ldots, p(N)$ are equidistributed when N tends to infinity. Using the Euler's recurrence formula for $p(n)$,

$$p(n) + \sum_{0<\omega_j\leq n} (-1)^j p(n - \omega(j)) = 0, \tag{4.1}$$

where $\omega(j) = j(3j-1)/2, -\infty < j < \infty$, Kolberg [13] proved that $p(n)$ takes both even and odd values infinitely often.

To prove the analogous property for $a(n)$, we need Jacobi's identity

$$\sum_{n=0}^{\infty} (-1)^n(2n+1)q^{n(n+1)} = (q^2; q^2)_\infty^3 \tag{4.2}$$

and Gauss's identity

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}. \tag{4.3}$$

We now have the following recurrence relation modulo 2.

Theorem 4.1.

$$a(n)+\sum_{0<k+k^2\leq n} a(n-k-k^2)\equiv \Delta(n)\ (\mathrm{mod}\ 2), \tag{4.4}$$

where $\Delta(n)=1$*, if* $n=s(s+1)/2$ *for some integer* s *and* $\Delta(n)=0$*, otherwise.*

Proof. Multiplying both sides of (1.9) by $(q^2;q^2)_\infty^3$, we get

$$(q^2;q^2)_\infty^3\sum_{n=0}^{\infty} a(n)q^n=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{4.5}$$

Substituting (4.2) and (4.3) into both sides of (4.5), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} q^{n(n+1)/2} &= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}\sum_{n=0}^{\infty} a(n)q^n\\ &\equiv \sum_{n=0}^{\infty} q^{n(n+1)}\sum_{n=0}^{\infty} a(n)q^n \quad (\mathrm{mod}\ 2).\end{aligned} \tag{4.6}$$

Equating coefficients of q^n on both sides of (4.6) gives (4.4). This completes the proof. ∎

With the aid of the formula (4.4) and following the idea of Kolberg [13] for $p(n)$, we obtain the following theorem for $a(n)$.

Theorem 4.2. *There are infinitely many integers* n *such that* $a(n)$ *is even and there are infinitely many integers* n *such that* $a(n)$ *is odd.*

Proof. We prove by contradiction. Assume that there exists m such that $a(n)$ is odd for any $n\geq m$. Without loss of generality, we may assume that m is an even integer greater than 2. It is easy to show that there exists an integer $m^2+2m\leq t\leq m^2+3m+1$ such that $\Delta(t)=0$. Setting $t=m^2+2m+\delta$, where $0\leq\delta\leq m+1$. Substituting $n=t$ into (4.4) yields

$$a(m^2+2m+\delta)+a(m^2+2m+\delta-2)+\cdots+a(m+\delta)\equiv 0\ (\mathrm{mod}\ 2). \tag{4.7}$$

But the left hand side of (4.7) is the sum of $m+1$ odd numbers, so it is also odd since m is even. This leads to a contradiction with parity of the right hand side of (4.7).

On the other hand, assume that $a(n)$ is even for any $n\geq m$, where $m\geq 5$. It is easy to verify the following inequalities for $k\geq 10$,

$$k(k-1)<\frac{\sqrt{2}k\big(\sqrt{2}k-1\big)}{2}<\frac{\sqrt{2}k\big(\sqrt{2}k+1\big)}{2}<k(k+1).$$

Therefore, for $k\geq 10$, there are no integers $e(e+1)$ in the following interval with $\lceil\sqrt{2}k\rceil$ elements,

$$\left[\frac{\lceil\sqrt{2}k\rceil\big(\lceil\sqrt{2}k\rceil-1\big)}{2},\frac{\lceil\sqrt{2}k\rceil\big(\lceil\sqrt{2}k\rceil+1\big)}{2}\right].$$

Choose k such that $d=\lceil\sqrt{2}k\rceil>m$ and set

$$t=\frac{d(d+1)}{2}.$$

Let r be the largest integer k such that $k^2 + k \leq t$. It follows that

$$t - r - r^2 > d > m.$$

Substituting $n = t$ into (4.4), we obtain that

$$a(t) + a(t-2) + \cdots + a(t - r - r^2) \equiv \Delta(t) \equiv 1 \pmod 2.$$

This is impossible since the left hand side of above congruence are a sum of even numbers. This completes the proof. ∎

Acknowledgments. We wish to thank Lisa H. Sun for helpful comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] G.E. Andrews and B.C. Berndt, Ramanujan's Lost Notebook, Part I, Springer, New York, 2005.

[2] S. Alhgren, The partition function modulo composite integers M, Math. Ann., 318 (2000), 795–803.

[3] S. Alhgren and M. Boylan, Arithmetic properties of the partition function, Invent. Math., 153 (2003), 487–502.

[4] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.

[5] B.C. Berndt, Number Theory in the Spirit of Ramanujan, Amer. Math. Soc., Providence, 2006.

[6] H.-C. Chan, Ramanujan's cubic continued fraction and a generalization of his "most beautiful identity", Int. J. Number Theory, to appear.

[7] H.-C. Chan, Ramanujan's cubic continued fraction and Ramanujan type congruences for a ceratin partition function, Int. J. Number Theory, to appear.

[8] H.-C. Chan, Distribution of a certain partition function modulo powers of primes, preprint, 2008.

[9] H.H. Chan, On Ramanujan's cubic continued fraction, Acta Arith., 73 (1995), 343–355.

[10] M.D. Hirschhorn and D.C. Hunt, A simple proof of the Ramanujan conjecture for powers of 5, J. Reine Angew. Math., 326 (1981), 1–17.

[11] B. Kim, A crank analog on a certain kind of partition function arising from the cubic continued fraction, preprint, 2008.

[12] I. Kiming and J. Olsson, Congruences like Ramanujan's for powers of the partition function, Arch. Math., 59 (1992), 348–360.

[13] O. Kolberg, Note on the parity of the partition function, Math. Scand., 7 (1959), 377–378.

[14] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag, 1984.

[15] J. Lovejoy, Ramanujan-type congruences for three colored Frobenius partitions, J. Number Theory, 86 (2000), 283–290.

[16] K. Ono, Congruences for Frobenius partitions, J. Number Theory, 57 (1996), 170–180.

[17] K. Ono, Distribution of the partition function modulo m, Ann. Math., 151 (2000), 293–307.

[18] K. Ono, The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series, CBMS Regional Conference Series in Mathematics, 102, Amer. Math. Soc., Province, RI, 2004.

[19] T.R. Parkin and D. Shanks, On the distribution of parity in the partition function, Math. Comp., 21 (1967), 466–480.

[20] S. Ramanujan, Congruence propertities of partitions, Proc. Lond. Math. Soc. (2), 19 (1919), 207–210.

[21] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delpi, 1988.

[22] J. Sinick, Ramanujan congruences for a class of eta quotients, arXiv:math.NT/0810.1931.

[23] J. Sturm, On the congruence of modular forms, Lecture Notes in Math. 1240 (1984), 275–280.