The Generating Function for the Dirichlet Series $L_m(s)$

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Abstract. The Dirichlet series $L_m(s)$ are of fundamental importance in number theory. Shanks defined the generalized Euler and class numbers in connection with these Dirichlet series, denoted by $\{s_{m,n}\}_{n\geq 0}$. We obtain a formula for the exponential generating function $s_m(x)$ of $s_{m,n}$, where m is an arbitrary positive integer. In particular, for $m > 1$, say, $m = bu^2$, where b is square-free and $u > 1$, we prove that $s_m(x)$ can be expressed as a linear combination of the four functions $w(b,t)\sec(btx)(\pm\cos((b-p)tx) \pm \sin(ptx))$, where p is an integer satisfying $0 \leq p \leq b$, $t|u^2$ and $w(b,t) = K_b t/u$ with K_b being a constant depending on b. Moreover, the Dirichlet series $L_m(s)$ can be easily computed from the generating function formula for $s_m(x)$. Finally, we show that the main ingredient in the formula for $s_{m,n}$ has a combinatorial interpretation in terms of the m-signed permutations defined by Ehrenborg and Readdy. In principle, this answers a question posed by Shanks concerning a combinatorial interpretation for the numbers $s_{m,n}$.

Keywords: Dirichlet series, generalized Euler and class number, Λ-alternating augmented m-signed permutation, r-cubical lattice, Springer number

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1 Introduction

The Dirichlet series

$$L_m(s) = \sum_{\substack{l>0 \\ \text{odd } l}} \left(\frac{-m}{l}\right)\frac{1}{l^s}, \tag{1.1}$$

where $(-m/l)$ is the Jacobi symbol, originate in the distribution of primes into arithmetic progressions, the class number of binary quadratic forms, as well as the distribution of Legendre and Jacobi symbols. They play a crucial role in the computation of certain number-theoretic constants, see [3, 6, 12, 14]. Several approaches have been developed for the computation of $L_m(s)$, see, for example, Shanks [15, 16, 18].

The generalized Euler and class numbers were introduced by Shanks for the computation of the Dirichlet series $L_m(s)$ [15, 17]. These numbers are also related to derivative polynomials and Euler polynomials, see Hoffman [7] and Shanks [17].

In this paper, we obtain the generating functions for the generalized Euler and class numbers. Let us recall the definition of the generalized Euler and class numbers $s_{m,n}$ $(m \geq 1, n \geq 0)$, introduced by Shanks,

$$s_{m,n} = \begin{cases} c_{m,\frac{n}{2}} & \text{if } n \text{ is even,} \\ d_{m,\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

where $c_{m,n}$ and $d_{m,n}$ are given by

$$c_{m,n} = (2n)!L_m(2n+1)(K_m\sqrt{m})^{-1}\left(\frac{\pi}{2m}\right)^{-2n-1}, \tag{1.2}$$

$$d_{m,n} = (2n-1)!L_{-m}(2n)(K_m\sqrt{m})^{-1}\left(\frac{\pi}{2m}\right)^{-2n}, \tag{1.3}$$

in which $K_m = \frac{1}{2}$ if $m = 1$ and 1 otherwise, and the Dirichlet series $L_m(s)$ are defined by (1.1).

Set

$$c_m(x) = \sum_{n\geq 0} c_{m,n}\frac{x^{2n}}{(2n)!},$$
$$d_m(x) = \sum_{n\geq 1} d_{m,n}\frac{x^{2n-1}}{(2n-1)!},$$
$$s_m(x) = \sum_{n\geq 0} s_{m,n}\frac{x^n}{n!}.$$

Clearly,

$$s_m(x) = c_m(x) + d_m(x).$$

By the definitions (1.2) and (1.3), we have

$$\sum_{n\geq 0} L_m(2n+1)x^{2n} = \frac{K_m\pi\sqrt{m}}{2m}c_m\left(\frac{\pi}{2m}x\right),$$
$$\sum_{n\geq 1} L_{-m}(2n)x^{2n-1} = \frac{K_m\pi\sqrt{m}}{2m}d_m\left(\frac{\pi}{2m}x\right).$$

Therefore, if we set

$$\hat{L}_m(s) = \begin{cases} L_{-m}(s+1) & \text{if } s \text{ is odd;} \\ L_m(s+1) & \text{if } s \text{ is even,} \end{cases}$$

then

$$\sum_{s\geq 0} \hat{L}_m(s)x^s = \frac{K_m\pi\sqrt{m}}{2m}s_m\left(\frac{\pi}{2m}x\right). \tag{1.4}$$

It follows from (1.4) that $\hat{L}_m(s)$ is determined by $s_m(x)$. In other words, the generating function $s_m(x)$ leads to a quick way to compute $\hat{L}_m(s)$.

Consider $c_{m,n}$ and $d_{m,n}$ as entries of the infinite matrices C and D, respectively. Then the first column of C forms the sequence of class numbers in connection with primitive binary quadratic forms, and the first row of C forms the sequence of secant numbers, corresponding to up-down permutations of even length. Meanwhile, the first row of D forms the sequence of tangent numbers corresponding to up-down permutations of odd length. Recall that both secant numbers and tangent numbers are called Euler numbers. This is why the numbers $s_{m,n}$ are called generalized Euler and class numbers.

Shanks [17] found recurrence relations for $c_{m,n}$ and $d_{m,n}$ over the index n, from which it follows that $c_{m,n}$ and $d_{m,n}$ are integers. For example, we have

$$\sum_{i=0}^{n}(-4)^i\binom{2n}{2i}c_{2,n-i}=(-1)^n$$

and

$$\sum_{i=0}^{n-1}(-4)^i\binom{2n-1}{2i}d_{2,n-i}=(-1)^{n-1}.$$

In fact, due to the well-known Euler product of the Dirichlet series $L_m(s)$ (see [8, 11]), it can be easily shown that $c_{m,n}$ and $d_{m,n}$ are positive.

Shanks [17] raised the following question: Whether all of the generalized Euler and class numbers may have some combinatorial interpretation? The combinatorial interpretations of $s_{m,n}$ for $m = 1, 2, 3, 4$ have been found. Let $(s_{m,n})_{n\geq 0}$ denote the sequence

$$(c_{m,0}, d_{m,1}, c_{m,1}, d_{m,2}, c_{m,2}, d_{m,3}, \ldots).$$

For $m = 1$, the sequence $(1, 1, 1, 2, 5, 16, \ldots)$ is listed as $A000111$ in the datbase of Sloane [19], which is called the sequence of Euler numbers, enumerating the number of alternating permutations on $[n] = \{1, 2, \ldots, n\}$.

For $m = 2$, the sequence $(1, 1, 3, 11, 57, 361, \ldots)$ is numbered $A001586$ in [19], which is also called the sequence of Springer numbers which arise in the work of Springer on the theory of Weyl group.

For $m = 3$, the sequence $(1, 2, 8, 46, 352, 3362, \ldots)$ is referred to $A007289$ in [19], and we call it the sequence of Ehrenborg and Readdy numbers because of their discovery of a combinatorial interpretation in terms of alternating 3-signed permutations on $[n]$, see [4].

For $m = 4$, a combinatorial interpretation of the sequence $(1, 4, 16, 128, 1280, 16384, \ldots)$ has been given implicitly by Ehrenborg and Readdy [5] in terms of non-augmented André 4-signed permutations on $[n]$.

For $m \geq 5$, we compute the generating functions $s_m(x)$. For $m = 1, 2, 3, 4$, it is known that

$$\begin{aligned} s_1(x) &= \sec x + \tan x, \\ s_2(x) &= \frac{\cos x + \sin x}{\cos 2x}, \\ s_3(x) &= \frac{\sin 2x + \cos x}{\cos 3x}, \\ s_4(x) &= \sec 4x + \tan 4x. \end{aligned}$$

From our general formula, we get the following expressions for $m = 5, 6, 7$,

$$\begin{aligned}
s_5(x) &= \frac{\cos 4x + \sin x}{\cos 5x} + \frac{\cos 2x + \sin 3x}{\cos 5x}, \\
s_6(x) &= \frac{\cos 5x + \sin x}{\cos 6x} + \frac{\cos x + \sin 5x}{\cos 6x}, \\
s_7(x) &= \frac{\cos 3x + \sin 4x}{\cos 7x} + \frac{\cos x + \sin 6x}{\cos 7x} - \frac{\cos 5x + \sin 2x}{\cos 7x}.
\end{aligned}$$

Our paper is organized as follows. In Section 2, we compute the generating function $s_m(x)$ when m is square-free, while in Section 3 we consider the case when m is not square-free. Section 4 is devoted to the combinatorial interpretation of the numbers $s_{m,n}$ in terms of m-signed permutations as introduced by Ehrenborg and Readdy.

2 Computation for $s_m(x)$ when $m > 1$ is square-free

In this section, we compute the generating function $s_m(x)$ when $m > 1$ is square-free. For $0 \le p \le m$, we follow the following notation used in [4],

$$\Lambda_{m,p}(x) := \frac{\cos((m-p)x) + \sin(px)}{\cos(mx)}. \tag{2.5}$$

When m is square-free, we shall not encounter the case that m is a multiple of 4. We shall have three formulas for $s_m(x)$ depending on the residue of m modulo 4.

Theorem 2.1 *Assume that m is square-free and $m = 4t + 3$. Then*

$$s_m(x) = \sum_{k=1}^{t} \left(\frac{k}{m}\right) \Lambda_{m,4k}(x) + \sum_{k=t+1}^{2t+1} \left(\frac{k}{m}\right) \Lambda_{m,2m-4k}(x). \tag{2.6}$$

Theorem 2.2 *Assume that m is square-free and $m = 4t + 1$. Then*

$$s_m(x) = \sum_{k=1}^{t} \left(\frac{k}{m}\right) \Lambda_{m,m-4k}(x) - \sum_{k=t+1}^{2t} \left(\frac{k}{m}\right) \Lambda_{m,4k-m}(x). \tag{2.7}$$

Theorem 2.3 *Assume that m is square-free and $m = 4t + 2$. Then*

$$s_m(x) = \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1} \left(\frac{-m}{k}\right) \Lambda_{m,k}(x). \tag{2.8}$$

To prove the above theorems, let us first recall the following formula of $L_m(2n+1)$ obtained by Shanks [15, 17].

Lemma 2.4 *Suppose that $m > 1$ is square-free. Then $L_m(2n+1)$ can be expressed as a linear combination of the Fourier series $S_{2n+1}(x)$ as follows*

$$L_m(2n+1) = \frac{2}{\sqrt{m}} \sum_k \epsilon_k S_{2n+1}(y_k),$$

where the Jacobi symbols ϵ_k and rational numbers y_k are uniquely determined by m, and $S_{2n+1}(x)$ is defined by

$$S_{2n+1}(x) = \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^{2n+1}}.$$

Furthermore, we have

$$c_m(x) = \frac{1}{\cos(mx)} \sum_k \epsilon_k \cos(mx(1-4y_k)).$$

In fact, Shanks has given an explicit procedure to determine the constants ϵ_k and y_k. To compute ϵ_k and y_k, we use the definition (1.1) of the series $L_m(s)$ and express the Jacobi symbol $\left(\frac{-m}{l}\right)$ as a linear combination of sines according to the following expansion, see [9].

Proposition 2.5 *Assume that l is odd and m satisfies the following two conditions: $m \equiv 1 \pmod 4$ or $m \equiv 8$ or $12 \pmod{16}$ and $p^2 \nmid m$ for any odd prime p. Then we have*

$$\left(\frac{m}{l}\right) = \frac{1}{\sqrt{m}} \sum_{r=1}^{|m|} \left(\frac{m}{r}\right) e^{2\pi i l r/|m|}. \tag{2.9}$$

In particular, when $m \equiv 3 \pmod 4$, then we have $-m \equiv 1 \pmod 4$ and we can use the above expansion for $\left(\frac{-m}{l}\right)$. Similarly, when $m \equiv 1 \pmod 4$, we have $-4m \equiv 12 \pmod{16}$ so that we can compute $\left(\frac{-4m}{l}\right)$ by using the above formula. Finally, when $m \equiv 2 \pmod 4$, we have $-4m \equiv 8 \pmod{16}$ so that we can compute $\left(\frac{-4m}{l}\right)$. Note that when l is odd, we have

$$\left(\frac{-4m}{l}\right) = \left(\frac{-m}{l}\right).$$

Thus, the Jacobi symbol $\left(\frac{-m}{l}\right)$ can be determined by the above procedure for $m > 1$.

On the other hand, Shanks [15, 17] provided the following formula for $L_{-m}(2n)$.

Lemma 2.6 *Suppose that $m > 1$ is square-free. Then $L_{-m}(2n)$ can be expressed as a linear combination of the Fourier series $C_{2n}(x)$ as follows*

$$L_{-m}(2n) = \frac{2}{\sqrt{m}} \sum_k \epsilon'_k C_{2n}(y'_k),$$

where the Jacobi symbols ϵ'_k and rational numbers y'_k are uniquely determined by m, and $C_{2n}(x)$ is defined by

$$C_{2n}(x) = \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^{2n}}.$$

Furthermore, we have

$$d_m(x) = \frac{1}{\cos(mx)} \sum_k \epsilon'_k \sin(mx(1-4y'_k)).$$

Similarly, Shanks has shown how to compute the constants ϵ'_k and y'_k. In order to compute ϵ'_k and y'_k, we use the definition of the series $L_{-m}(s)$

$$L_{-m}(s) = \sum_{\substack{l>0 \\ \text{odd } l}} \left(\frac{m}{l}\right) \frac{1}{l^s},$$

and express the Jacobi symbol $\left(\frac{m}{l}\right)$ as a linear combination of cosines resorting to proposition 2.5.

For the case $m \equiv 3 \pmod 4$, we have $4m \equiv 12 \pmod{16}$ so that we can use the above expansion for $\left(\frac{4m}{l}\right)$. When $m \equiv 1 \pmod 4$, we can also compute $\left(\frac{m}{l}\right)$ by using the above expansion. Finally, when $m \equiv 2 \pmod 4$, we have $4m \equiv 8 \pmod{16}$ and $\left(\frac{4m}{l}\right)$ can be determined in the same manner. Note that when l is odd, we have

$$\left(\frac{4m}{l}\right) = \left(\frac{m}{l}\right).$$

Thus, the Jacobi symbol $\left(\frac{m}{l}\right)$ can be determined for $m > 1$.

Keep in mind that m is assumed to be square-free. Set

$$\hat{c}_m(x) = \cos(mx)c_m(x), \quad \hat{d}_m(x) = \cos(mx)d_m(x), \quad \hat{s}_m(x) = \cos(mx)s_m(x).$$

Proof of Theorem 2.1. If $m \equiv 3 \pmod 4$, by using the expansion (2.9) for $\left(\frac{-m}{l}\right)$ and $\left(\frac{4m}{l}\right)$, we have

$$\epsilon_k = \left(\frac{k}{m}\right), \quad y_k = \frac{k}{m}, \quad \epsilon'_k = \left(\frac{m}{k}\right), \quad y'_k = \frac{k}{4m},$$

which imply that

$$L_m(2n+1) = \frac{2}{\sqrt{m}} \sum_{k=1}^{(m-1)/2} \left(\frac{k}{m}\right) S_{2n+1}\left(\frac{k}{m}\right),$$

$$L_{-m}(2n) = \frac{2}{\sqrt{m}} \sum_{\text{odd } k<m} \left(\frac{m}{k}\right) C_{2n}\left(\frac{k}{4m}\right).$$

Therefore, by Lemmas 2.4 and 2.6, we have

$$\hat{s}_m(x) = \hat{c}_m(x) + \hat{d}_m(x) = \sum_{k=1}^{(m-1)/2} \left(\frac{k}{m}\right) \cos(m-4k)x + \sum_{\text{odd } k<m} \left(\frac{m}{k}\right) \sin(m-k)x.$$

Suppose that $m = 4t+3$. It follows that

$$\hat{c}_m(x) = \sum_{k=1}^{t} \left(\frac{k}{m}\right) \cos(m-4k)x + \sum_{k=t+1}^{2t+1} \left(\frac{k}{m}\right) \cos(m-4k)x$$
$$= \sum_{k=1}^{t} \left(\frac{k}{m}\right) \cos(m-4k)x + \sum_{k=t+1}^{2t+1} \left(\frac{k}{m}\right) \cos(4k-m)x,$$
$$\hat{d}_m(x) = \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1} \left(\frac{m}{k}\right) \sin(m-k)x.$$

Thus we obtain

$$\hat{s}_m(x) = \sum_{k=1}^{t} \left(\left(\frac{k}{m}\right) \cos(m-4k)x + \left(\frac{m}{m-4k}\right) \sin(4k)x \right)$$
$$+ \sum_{k=t+1}^{2t+1} \left(\left(\frac{k}{m}\right) \cos(4k-m)x + \left(\frac{m}{4k-m}\right) \sin(2m-4k)x \right).$$

It remains to verify that

$$\left(\frac{k}{m}\right) = \left(\frac{m}{m-4k}\right) \tag{2.10}$$

for $1 \le k \le t$ and

$$\left(\frac{k}{m}\right) = \left(\frac{m}{4k-m}\right) \tag{2.11}$$

for $t+1 \le k \le 2t+1$.

With respect to (2.10), since both m and $m-4k$ are odd positive numbers, by the law of quadratic reciprocity, we find

$$\left(\frac{m}{m-4k}\right) = -\left(\frac{m-4k}{m}\right) = -\left(\frac{-4k}{m}\right) = -\left(\frac{-4}{m}\right)\left(\frac{k}{m}\right)$$
$$= -\left(\frac{-1}{m}\right)\left(\frac{4}{m}\right)\left(\frac{k}{m}\right) = \left(\frac{2}{m}\right)^2\left(\frac{k}{m}\right) = \left(\frac{k}{m}\right).$$

Similarly, (2.11) can be checked via the following steps,

$$\left(\frac{m}{4k-m}\right) = \left(\frac{4k-m}{m}\right) = \left(\frac{-(m-4k)}{m}\right) = \left(\frac{-1}{m}\right)\left(\frac{m-4k}{m}\right)$$
$$= -\left(\frac{m-4k}{m}\right) = -\left(\frac{-4k}{m}\right) = \left(\frac{k}{m}\right).$$

Hence we deduce that

$$s_m(x) = \sum_{k=1}^{t} \left(\frac{k}{m}\right) \frac{\cos(m-4k)x + \sin(4k)x}{\cos mx}$$
$$+ \sum_{k=t+1}^{2t+1} \left(\frac{k}{m}\right) \frac{\cos(4k-m)x + \sin(2m-4k)x}{\cos mx}.$$

This completes the proof. ∎

Proof of Theorem 2.2. If $m \equiv 1 \pmod 4$, applying the expansion (2.9) to $\left(\frac{-4m}{l}\right)$ and $\left(\frac{m}{l}\right)$, we get

$$\epsilon_k = \left(\frac{-m}{k}\right), \quad y_k = \frac{k}{4m}, \quad \epsilon'_k = \left(\frac{k}{m}\right), \quad y'_k = \frac{k}{m}.$$

It follows that

$$L_m(2n+1) = \frac{2}{\sqrt{m}} \sum_{\text{odd } k<m} \left(\frac{-m}{k}\right) S_{2n+1}\left(\frac{k}{4m}\right),$$

$$L_{-m}(2n) = \frac{2}{\sqrt{m}} \sum_{k=1}^{(m-1)/2} \left(\frac{k}{m}\right) C_{2n}\left(\frac{k}{m}\right).$$

In view of Lemmas 2.4 and 2.6, we find

$$\hat{s}_m(x) = \hat{c}_m(x) + \hat{d}_m(x) = \sum_{\text{odd } k<m} \left(\frac{-m}{k}\right) \cos(m-k)x + \sum_{k=1}^{(m-1)/2} \left(\frac{k}{m}\right) \sin(m-4k)x.$$

Writing $m = 4t+1$, we obtain

$$\begin{aligned} \hat{d}_m(x) &= \sum_{k=1}^{t} \left(\frac{k}{m}\right) \sin(m-4k)x + \sum_{k=t+1}^{2t} \left(\frac{k}{m}\right) \sin(m-4k)x \\ &= \sum_{k=1}^{t} \left(\frac{k}{m}\right) \sin(m-4k)x - \sum_{k=t+1}^{2t} \left(\frac{k}{m}\right) \sin(4k-m)x, \\ \hat{c}_m(x) &= \sum_{\substack{k=1 \\ \text{odd } k}}^{4t-1} \left(\frac{-m}{k}\right) \cos(m-k)x. \end{aligned}$$

It follows that

$$\begin{aligned} \hat{s}_m(x) =& \sum_{k=1}^{t} \left(\left(\frac{k}{m}\right) \sin(m-4k)x + \left(-\frac{m}{m-4k}\right) \cos(4k)x \right) \\ &+ \sum_{k=t+1}^{2t} \left(-\left(\frac{k}{m}\right) \sin(4k-m)x + \left(-\frac{m}{4k-m}\right) \cos(2m-4k)x \right). \end{aligned}$$

Finally, we need to show that

$$\left(\frac{k}{m}\right) = \left(-\frac{m}{m-4k}\right) \tag{2.12}$$

for $1 \le k \le t$ and

$$-\left(\frac{k}{m}\right) = \left(-\frac{m}{4k-m}\right) \tag{2.13}$$

for $t+1 \le k \le 2t$.

To confirm (2.12), since both m and $m-4k$ are odd positive numbers, we may employ the law of quadratic reciprocity to deduce

$$\begin{aligned}\left(-\frac{m}{m-4k}\right) &= \left(\frac{-1}{m-4k}\right)\left(\frac{m}{m-4k}\right) = \left(\frac{m}{m-4k}\right) = \left(\frac{m-4k}{m}\right) = \left(\frac{-4k}{m}\right) \\ &= \left(\frac{-4}{m}\right)\left(\frac{k}{m}\right) = \left(\frac{-1}{m}\right)\left(\frac{2}{m}\right)^2\left(\frac{k}{m}\right) = \left(\frac{k}{m}\right).\end{aligned}$$

Moreover, (2.13) can be checked as follows

$$\begin{aligned}-\left(-\frac{m}{4k-m}\right) &= -\left(\frac{-1}{4k-m}\right)\left(\frac{m}{4k-m}\right) = \left(\frac{m}{4k-m}\right) = \left(\frac{4k-m}{m}\right) \\ &= \left(\frac{-(m-4k)}{m}\right) = \left(\frac{-1}{m}\right)\left(\frac{m-4k}{m}\right) = \left(\frac{-4k}{m}\right) = \left(\frac{k}{m}\right).\end{aligned}$$

So we arrive at

$$\begin{aligned}s_m(x) =& \sum_{k=1}^{t}\left(\frac{k}{m}\right)\frac{\cos(4k)x+\sin(m-4k)x}{\cos mx} \\ &- \sum_{k=t+1}^{2t}\left(\frac{k}{m}\right)\frac{\cos(2m-4k)x+\sin(4k-m)x}{\cos mx}.\end{aligned}$$

This completes the proof. ∎

Proof of Theorem 2.3. If $m \equiv 2 \pmod 4$, by using the expansion (2.9) for $\left(\frac{-4m}{l}\right)$ and $\left(\frac{4m}{l}\right)$, we obtain

$$\epsilon_k = \left(\frac{-m}{k}\right), \quad y_k = \frac{k}{4m}, \quad \epsilon'_k = \left(\frac{m}{k}\right), \quad y'_k = \frac{k}{4m}.$$

Consequently,

$$\begin{aligned}L_m(2n+1) &= \frac{2}{\sqrt{m}}\sum_{\text{odd } k<m}\left(\frac{-m}{k}\right)S_{2n+1}\left(\frac{k}{4m}\right), \\ L_{-m}(2n) &= \frac{2}{\sqrt{m}}\sum_{\text{odd } k<m}\left(\frac{m}{k}\right)C_{2n}\left(\frac{k}{4m}\right).\end{aligned}$$

By Lemmas 2.4 and 2.6, we see that

$$\hat{s}_m(x) = \hat{c}_m(x) + \hat{d}_m(x) = \sum_{\text{odd } k<m}\left(\frac{-m}{k}\right)\cos(m-k)x + \sum_{\text{odd } k<m}\left(\frac{m}{k}\right)\sin(m-k)x.$$

Writing $m = 4t+2$, since

$$\left(\frac{4t+2}{2t+1}\right) = 0,$$

we obtain

$$\hat{s}_m(x) = \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1}\left(\left(\frac{-m}{k}\right)\cos(m-k)x + \left(\frac{m}{m-k}\right)\sin(k)x\right).$$

Finally, it follows from $-k \equiv m - k \pmod{m}$ that

$$\left(\frac{-m}{k}\right) = \left(\frac{m}{m-k}\right) \tag{2.14}$$

for $1 \le k \le 4t + 1$. Hence we have reached the conclusion

$$s_m(x) = \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1} \left(\frac{-m}{k}\right) \frac{\cos(m-k)x + \sin(k)x}{\cos mx}.$$

This completes the proof. ∎

For $m = 5, 6, 7$, the generating function $s_m(x)$ has been given in the introduction.

3 Computation for $s_m(x)$ when m is not square-free

In this section, we obtain an expression for $s_m(x)$ for the case when m is not square-free. Assume that m can be divided by a square $u^2 > 1$, say, $m = bu^2$, where b is square-free. Below is the formula for this case.

Theorem 3.1 *Suppose that $m = bu^2$ as given above. Then we can express $s_m(x)$ as a linear combination of the four terms*

$$w(b,t)\sec(btx)(\pm\cos((b-p)tx) \pm \sin(ptx)),$$

where p is an integer satisfying $0 \le p \le b$ and $t|u^2$, and the coefficient $w(b,t) = K_b t/u$ is uniquely determined for any given t.

The idea of the proof is to establish two recursive relations (3.16) and (3.17) between $s_{m,n}$ and $s_{b,n}$. Then we express $s_m(x)$ as a linear combination of the two terms $c_b(tx)$ and $d_b(tx)$ by considering the two cases according to whether there exist odd prime factors u_i of u with residues 3 modulo 4. Since b is square-free, $c_b(tx)$ and $d_b(tx)$ can be evaluated by using the formulas in the previous section.

Proof. Let us start with the following relation given by Shanks [15]

$$L_m(s) = L_b(s) \prod_{u_i|u} \left(1 - \left(\frac{-b}{u_i}\right)\frac{1}{u_i^s}\right), \tag{3.15}$$

where the product ranges over odd primes u_i (if any) that divide u. To be precise, in case there are no odd prime factors, the empty product is assumed to be one by convention. From the definitions (1.2) and (1.3) it follows that

$$c_{m,n} = K_b u(u^2)^{2n} \prod_i \left(u_i^{2n+1} - \left(\frac{-b}{u_i}\right)\right) \left(\prod_i \frac{1}{u_i}\right)^{2n+1} c_{b,n}, \tag{3.16}$$

$$d_{m,n} = K_b u(u^2)^{2n-1} \prod_i \left(u_i^{2n} - \left(\frac{b}{u_i}\right)\right) \left(\prod_i \frac{1}{u_i}\right)^{2n} d_{b,n}. \tag{3.17}$$

For the purpose of computing $s_m(x)$ for the case when m is not square-free, we need to consider the two cases according to whether there exist $u_i \equiv 3 \pmod 4$ among the k odd factors $u_1, u_2, \ldots, u_k$ of u.

Case 1: $u_i \equiv 1 \pmod 4$ for $1 \le i \le k$. In this case, we see that

$$\left(\frac{-b}{u_i}\right) = \left(\frac{b}{u_i}\right).$$

Suppose that among the factors $u_1, u_2, \ldots, u_k$ there are k_1 primes $u_1, u_2, \ldots, u_{k_1}$ satisfying $\left(\frac{b}{u_i}\right) = 1$ for $1 \le i \le k_1$, k_2 primes $u_{k_1+1}, u_{k_1+2}, \ldots, u_{k_1+k_2}$ satisfying $\left(\frac{b}{u_{k_1+j}}\right) = -1$ for $1 \le j \le k_2$, and k_3 primes $u_{k_1+k_2+1}, u_{k_1+k_2+2}, \ldots, u_{k_1+k_2+k_3}$ satisfying $\left(\frac{b}{u_{k_1+k_2+l}}\right) = 0$ for $1 \le l \le k_3$, where $k_1 + k_2 + k_3 = k$. By (3.16), we get

$$c_{m,n} = K_b u (u^2)^{2n} \frac{\prod_{i=1}^{k_1}(u_i^{2n+1} - 1) \times \prod_{j=1}^{k_2}(u_{k_1+j}^{2n+1} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} c_{b,n}. \tag{3.18}$$

Let

$$f_c = \frac{\prod_{i=1}^{k_1}(u_i^{2n+1} - 1) \times \prod_{j=1}^{k_2}(u_{k_1+j}^{2n+1} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}}.$$

In this notation, (3.18) can be rewritten as

$$c_{m,n} = K_b u (u^2)^{2n} f_c c_{b,n},$$

which implies that

$$c_m(x) = K_b u \sum_{n \ge 0} f_c c_{b,n} \frac{(u^2 x)^{2n}}{(2n)!}. \tag{3.19}$$

Since

$$\prod_{i=1}^{k_1}(u_i^{2n+1} - 1) = \prod_{i=1}^{k_1} u_i^{2n+1} - \sum_{i=1}^{k_1}\left(\prod u_1 \cdots u_{i-1} u_{i+1} \cdots u_{k_1}\right)^{2n+1} + \cdots + (-1)^{k_1}$$

and

$$\prod_{j=1}^{k_2}(u_{k_1+j}^{2n+1} + 1) = \prod_{j=1}^{k_2} u_{k_1+j}^{2n+1} + \sum_{j=1}^{k_2}\left(\prod u_{k_1+1} \ldots u_{k_1+j-1} u_{k_1+j+1} \cdots u_{k_1+k_2}\right)^{2n+1} + \cdots + 1,$$

we can expand f_c as follows

$$\begin{aligned} f_c =& 1 + \sum_{j=1}^{k_2} \frac{1}{u_{k_1+j}^{2n+1}} + \cdots + \frac{1}{\prod_{j=1}^{k_2} u_{k_1+j}^{2n+1}} - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n+1}} - \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \frac{1}{u_i^{2n+1} u_{k_1+j}^{2n+1}} + \cdots \\ & - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n+1} \prod_{j=1}^{k_2} u_{k_1+j}^{2n+1}} + \cdots + \frac{(-1)^{k_1}}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}}. \end{aligned} \tag{3.20}$$

Plugging (3.20) into (3.19), we find that $c_m(x)$ is a linear combination of the terms $c_b(tx)$, where $t | u^2$ and the coefficient of $c_b(tx)$ in the linear combination is equal to $K_b t/u$.

Similarly, we have

$$d_{m,n} = K_b u (u^2)^{2n-1} \frac{\prod_{i=1}^{k_1}(u_i^{2n}-1) \times \prod_{j=1}^{k_2}(u_{k_1+j}^{2n}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} d_{b,n}. \tag{3.21}$$

Let

$$f_d = \frac{\prod_{i=1}^{k_1}(u_i^{2n}-1) \times \prod_{j=1}^{k_2}(u_{k_1+j}^{2n}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}},$$

then (3.21) can be rewritten as

$$d_{m,n} = K_b u (u^2)^{2n-1} f_d d_{b,n},$$

which leads to the following relation

$$d_m(x) = K_b u \sum_{n \geq 1} f_d d_{b,n} \frac{(u^2 x)^{2n-1}}{(2n-1)!}. \tag{3.22}$$

Again we may expand f_d as follows

$$\begin{aligned} f_d =& 1 + \sum_{j=1}^{k_2} \frac{1}{u_{k_1+j}^{2n}} + \cdots + \frac{1}{\prod_{j=1}^{k_2} u_{k_1+j}^{2n}} - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n}} - \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \frac{1}{u_i^{2n} u_{k_1+j}^{2n}} + \cdots \\ & - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n} \prod_{j=1}^{k_2} u_{k_1+j}^{2n}} + \cdots + \frac{(-1)^{k_1}}{\prod_{i=1}^{k_1+k_2} u_i^{2n}}. \end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22), we find again that $d_m(x)$ is a linear combination of the terms $d_b(tx)$, where $t|u^2$ the coefficient of $d_b(tx)$ in the linear combination is equal to $K_b t/u$. Furthermore, we see that $c_m(x)$ and $d_m(x)$ have the same coefficients for the linear combinations. In other words, the relation for $c_m(x)$ and $c_b(tx)$ is still valid after changing $c_m(x)$ and $c_b(tx)$ to $d_m(x)$ and $d_b(tx)$, respectively. Therefore, in this case, $s_m(x)$ can be expressed as a sum of the terms $w(b,t)(c_b(tx) + d_b(tx))$, where $t|u^2$ and the coefficient $w(b,t) = K_b t/u$.

Case 2: Among the k primes $u_1, u_2, \ldots, u_k$, there exists q primes $u_{i_1}, u_{i_2}, \ldots, u_{i_q}$ with residue 3 modulo 4. To compute $s_m(x)$, we first consider the case when $q = 1$, and then argue that the case $q > 1$ can be dealt with in the same way.

Since $q = 1$, we assume that the first $k-1$ odd primes $u_1, u_2, \ldots, u_{k-1}$ satisfy $u_i \equiv 1 \pmod 4$ for $1 \leq i \leq k-1$, and suppose that the last prime u_k satisfies $u_k \equiv 3 \pmod 4$, or

$$\left(\frac{-b}{u_k}\right) = -\left(\frac{b}{u_k}\right).$$

Similarly, we may define the indices k_1, k_2 and k_3 as in Case 1 except that $k_1+k_2+k_3 = k-1$ since we now have $k-1$ primes with residue 1 modulo 4. This leads us to consider two subcases according to whether $\left(\frac{-b}{u_k}\right)$ equals 0. Keep in mind that $q = 1$ in these two subcases.

On the one hand, in order to use the two recursive relations (3.16) and (3.17) between $s_{m,n}$ and $s_{b,n}$, we may assume that $\left(\frac{-b}{u_k}\right) = 0$. Therefore, the term u_k on the denominator and the same term on the numerator cancel each other in (3.16). This argument also applies to the relation (3.17). In other words, there is no need to consider the occurrence of the term u_k^{2n+1} in (3.16) and the terms u_k^{2n} in (3.17). In this sense, it remains to consider the other $k-1$ primes $u_1, u_2, \ldots, u_{k-1}$ such that $u_i \equiv 1 \pmod 4$ for $1 \le i \le k-1$. By the argument in Case 1, we see again that $s_m(x)$ can be expressed as a sum of the terms $w(b,t)(c_b(tx)+d_b(tx))$, where $t|u^2$ and $w(b,t) = K_b t/u$.

On the other hand, we should consider the case for $\left(\frac{-b}{u_k}\right) = 1$ or $\left(\frac{-b}{u_k}\right) = -1$. Since the proofs for these two situations are similar, we only give the proof of the case $\left(\frac{-b}{u_k}\right) = -1$. Assuming so, from (3.16) it follows that

$$
\begin{aligned}
c_{m,n} =& K_b u(u^2)^{2n} \frac{\prod_{i=1}^{k_1}(u_i^{2n+1}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n+1}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} \times \frac{u_k^{2n+1}+1}{u_k^{2n+1}} c_{b,n} \\
=& K_b u(u^2)^{2n} \left(\frac{\prod_{i=1}^{k_1}(u_i^{2n+1}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n+1}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} + \frac{\prod_{i=1}^{k_1}(u_i^{2n+1}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n+1}+1)}{u_k^{2n+1}\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} \right) c_{b,n}.
\end{aligned}
$$

Let

$$
g_b(x) = K_b u \sum_{n\ge 0} \left(\frac{\prod_{i=1}^{k_1}(u_i^{2n+1}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n+1}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} \right) c_{b,n} \frac{(u^2x)^{2n}}{(2n)!},
$$

then we have

$$
c_m(x) = g_b(x) + \frac{1}{u_k} g_b(x/u_k). \tag{3.24}
$$

In fact, by the argument in Case 1, we see that $g_b(x)$ is a linear combination of the terms $c_b(tx)$, where $t|u^2$ and the coefficient of $c_b(tx)$ in linear combination is equal to $K_b t/u$.

Similarly, by (3.17) we find

$$
\begin{aligned}
d_{m,n} =& K_b u(u^2)^{2n-1} \frac{\prod_{i=1}^{k_1}(u_i^{2n}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} \times \frac{u_k^{2n}-1}{u_k^{2n}} d_{b,n} \\
=& K_b u(u^2)^{2n-1} \left(\frac{\prod_{i=1}^{k_1}(u_i^{2n}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} - \frac{\prod_{i=1}^{k_1}(u_i^{2n}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n}+1)}{u_k^{2n}\prod_{i=1}^{k_1+k_2} u_i^{2n}} \right) d_{b,n}.
\end{aligned}
$$

Let

$$
h_b(x) = K_b u \sum_{n\ge 1} \left(\frac{\prod_{i=1}^{k_1}(u_i^{2n}-1)\prod_{j=1}^{k_2}(u_{k_1+j}^{2n}+1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} \right) d_{b,n} \frac{(u^2x)^{2n-1}}{(2n-1)!}.
$$

We get

$$
d_m(x) = h_b(x) - \frac{1}{u_k} h_b(x/u_k). \tag{3.25}
$$

Again, from the reasoning in Case 1 it follows that $h_b(x)$ is a linear combination of the terms $d_b(tx)$, where $t|u^2$ and the coefficient of $d_b(tx)$ in linear combination is equal to $K_b t/u$. Combining equations (3.24) and (3.25) yields the following formula

$$
s_m(x) = g_b(x) + h_b(x) + \frac{1}{u_k}(g_b(x/u_k) - h_b(x/u_k)).
$$

Thus we see that $s_m(x)$ can also be expressed as a linear combination of the terms $w(b,t)(\pm c_b(tx)\pm d_b(tx)$, where $t|u^2$ and $w(b,t) = K_b t/u$.

Finally, as mentioned before we shall show that the justification for the above two subcases can be applied to the case for $q > 1$. Let us give an example for $q = 2$. Suppose that u_{k-1} and u_k are the last two primes such that $u_{k-1} \equiv 3 \pmod 4$, $u_k \equiv 3 \pmod 4$, and $\left(\frac{-b}{u_{k-1}}\right) = -1$, $\left(\frac{-b}{u_k}\right) = -1$. Then by using the same notation $g_b(x)$ and $h_b(x)$ as before, we can deduce the following formula for $s_m(x)$,

$$\begin{aligned} s_m(x) =& g_b(x) + h_b(x) + \frac{1}{u_{k-1}}(g_b(x/u_{k-1}) - h_b(x/u_{k-1})) + \frac{1}{u_k}(g_b(x/u_k) - h_b(x/u_k)) \\ &+ \frac{1}{u_{k-1}u_k}(g_b(x/u_{k-1}u_k) + h_b(x/u_{k-1}u_k)). \end{aligned}$$

Clearly, $s_m(x)$ can also be expressed as a linear combination of the terms $w(b,t)(\pm c_b(tx) \pm d_b(tx)$, where $t|u^2$ and $w(b,t) = K_b t/u$. For other assumptions on u_{k-1} and u_k, the computation can be similarly deduced.

In summary, $s_m(x)$ can be expressed as a linear combination of the terms $w(b,t)(\pm c_b(tx) \pm d_b(tx)$, where $t|u^2$ and $w(b,t) = K_b t/u$. Since b is square-free, $c_b(tx)$ and $d_b(tx)$ can be expressed as a linear combination of the functions $\sec(btx)\cos((b-p)tx)$ and $\sec(btx)\sin(ptx)$, respectively, by using the formulas in the previous section. Hence the theorem holds. ∎

Here we give three examples corresponding to the above three cases. For Case 1, suppose that $m = 3(5\times 13)^2 = 3(65)^2 = 3\times 4225 = 12675$. Then we have

$$s_{12675}(x) = 65 s_3(4225x) - 5s_3(325x) + 13 s_3(845x) - s_3(65x),$$

where

$$s_3(x) = \sec(3x)(\sin 2x + \cos x).$$

For the first subcase of Case 2, suppose that $m = 6(5\times 3)^2 = 3(15)^2 = 6\times 225 = 1350$. Then we get

$$s_{1350}(x) = 15 s_6(225x) - 3s_6(45x),$$

where

$$s_6(x) = \sec(6x)(\cos 5x + \sin x) + \sec(6x)(\cos x + \sin 5x).$$

For the second subcase of Case 2, assume that $m = 225 = (5\times 3)^2$. We find

$$\begin{aligned} 2S_{225}(x) =& 15(c_1(225x) + d_1(225x)) - 3(c_1(45x) + d_1(45x)) + 5(c_1(75x) - d_1(75x)) \\ &- (c_1(15x) - d_1(15x)), \end{aligned}$$

where

$$c_1(x) = \sec(x), \quad d_1(x) = \tan(x).$$

4 Combinatorial Interpretation for $s_{m,n}$

In this section, we aim to give a combinatorial interpretation for $s_{m,n}$ based on its generating function formula $s_m(x)$. Let us begin by recalling the known combinatorial interpretations of $s_{m,n}$ for $m = 1, 2, 3, 4$.

For $m = 1$, $(s_{1,n})_{n\geq 0}$ is called the sequence of Euler numbers. Let E_n be the n-th Euler number, that is, the number of alternating permutations or up-down permutations of $[n] = \{1, 2, \ldots, n\}$, which are also called snakes of type A_{n-1} by Arnol'd [2]. The following generating function is due to André [1]:

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x,$$

Note that Springer also gave an explanation of the Euler numbers in terms of the irreducible root system A_{n-1} and derived the generating function of André in this context.

For $m = 2$, the sequence $(s_{2,n})_{n\geq 0}$ turns out to be the sequence of Springer numbers of the irreducible root system B_n ([20]). Purtill [13] has found an interpretation of this sequence. Let P_n be the n-th entry of this sequence, whereas Purtill used the notation $E_n^{\pm}$. He has shown that P_n equals the number of André signed permutations on $[n]$. On the other hand, it has been shown by Arnol'd [2] that S_n also counts the number of snakes of type B_n. Hoffman [7] has derived the generating function of the number of snakes of type B_n by giving a direct combinatorial proof.

For $m = 3$, the sequence $(s_{3,n})_{n\geq 0}$ has been studied by Ehrenborg and Readdy ([4]). Let F_n denote the n-th Ehrenborg and Readdy number, which was denoted by $|ER_n|$, see [7]. It has been shown that F_n equals the number Λ-alternating augmented 3-signed permutations on $[n]$. Meanwhile, Hoffman [7] presented another combinatorial interpretation of the sequence in the case $m = 3$ in terms of ER_n-snakes in the spirit of the snakes of type A_{n-1} and B_n.

For $m = 4$, the sequence $(s_{4,n})_{n\geq 0}$ has also been studied by Ehrenborg and Readdy ([5]). They introduced the concept of non-augmented André 4-signed permutations on $[n]$ and proved that such permutations are counted by $s_{4,n}$.

In principle, we have a combinatorial interpretation of $s_{m,n}$ for $m \geq 5$ based on the generating function $s_m(x)$. Here are some definitions. In [4], Ehrenborg and Readdy defined a poset called the Sheffer poset, which can be viewed as a generalization of the binomial poset introduced by Stanley [21]. As an important example, they studied the r-cubical lattice, which is a set of ordered r-tuples $(A_1, A_2, \ldots, A_r)$ of subsets from an infinite set I together with the reverse inclusion order and a minimum element $\hat{0}$ adjoined. Note that the r-cubical lattice has been studied by Metropolis, Rota, Strehl and White [10]. Ehrenborg and Readdy further generalized the concept of R-labelings to linear edge-labelings. By considering the set of maximal chains in the interval $[\hat{0}, \hat{1}]$ on the Hasse diagram of the r-cubical lattice, they deduced a formula for the number of Λ-alternating augmented r-signed permutations.

To introduce the definition of these permutations, Ehrenborg and Readdy constructed a linear edge-labeling on the Hasse diagram of the r-cubical lattice. To be more specific, for an edge corresponding to the cover relation $A < B$ with $A \neq \hat{0}$, let (i, j) be its label where i equals the unique index such that $A_i \neq B_i$ and j takes the singleton element in $A_i - B_i$. Let G be the label of the edge corresponding to $\hat{0} < A$, whereas Ehrenborg and Readdy called it the special element. Then an augmented r-signed permutation is a list $(G, ((i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)))$, where $i_1, i_2, \ldots, i_n \in [r]$ and $(j_1, j_2, \ldots, j_n)$ forms a permutation on $[n]$. In other words, r-signed permutations are permutations on $[n]$ in which each element is assigned r signs.

To define the descent set of r-signed permutations, let Λ be the set of such labels of the edges on the Hasse diagram of the r-cubical lattice. It is easy to see that

$$\Lambda = ([r] \times [n]) \cup \{G\}.$$

Let p be an integer such that $0 \le p \le r$. For fixed r, n and p, we can define a linear order on Λ which satisfies the following conditions

$$(i, j) <_\Lambda G \Rightarrow i \le r - p, \tag{4.26}$$

and

$$(i, j) >_\Lambda G \Rightarrow i > r - p, \tag{4.27}$$

where (i, j) is the label of the edge corresponding to the cover relation $A < B$ such that A covers $\hat{0}$, and G is the special element. For the remaining labels, we may arrange them in the lexicographic order. The descent set of an augmented r-signed permutation $(G = g_0, g_1, \ldots, g_n)$ is defined as the set $\{k : g_{k-1} >_\Lambda g_k\}$, where $g_k = (i_k, j_k)$ for $1 \le k \le n$. Therefore, for an Λ-alternating augmented r-signed permutation, that is, permutation having descent set $\{2, 4, \ldots\}$, it is necessary to have the condition $G < (i_1, j_1)$, or $i_1 > r - p$. In other words, the labels above G in this ordering are those whose first coordinate may take p possible values from the set $\{r - p + 1, r - p + 2, \ldots, r\}$.

If we denote the number of Λ-alternating augmented r-signed permutations by $\Lambda_{r,p,n}$, then there are $\Lambda_{r,p,n}$ maximal chains with descent set $\{2, 4, \ldots\}$ in the interval $[\hat{0}, A]$ with A being an element of rank $n + 1$. Based on this observation, Ehrenborg and Readdy have derived the following generating function formula for $\Lambda_{r,p,n}$

$$\Lambda_{r,p}(x) = \sum_{n \ge 0} \Lambda_{r,p,n} \frac{x^n}{n!} = \frac{\cos((r-p)x) + \sin(px)}{\cos(rx)}. \tag{4.28}$$

Now we interpret the generalized Euler and class numbers $s_{m,n}$ as follows. When m is square-free, it is easily seen that the formula for $s_{m,n}$ can be expressed as a linear combination of the number $\Lambda_{r,p_2,n} - \Lambda_{r,p_1,n}$ for $p_2 > p_1$, and the number $\Lambda_{r,p_1,n} + \Lambda_{r,p_2,n}$. So we shall give combinatorial interpretations of these two numbers.

For the first case, we consider the function $\sec(rx)[\cos((r - p_2)x) + \sin(p_2 x) - \cos((r - p_1)x) - \sin(p_1 x)]$. It is the generating function for the number of Λ-alternating augmented r-signed permutations $(G = g_0, (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n))$ with $r - p_2 + 1 \le i_1 \le r - p_1$, or the number of maximal chains in the interval $[\hat{0}, \hat{1}]$ whose first non-special edge has the label (i_1, j_1) with $r - p_2 + 1 \le i_1 \le r - p_1$. Here the first non-special edge is the one corresponding to the cover relation $A < B$ such that A covers $\hat{0}$. This gives a combinatorial interpretation of the number $\Lambda_{r,p_2,n} - \Lambda_{r,p_1,n}$.

On the other hand, the numbers $\Lambda_{r,p_1,n} + \Lambda_{r,p_2,n}$ have the generating function $\sec(rx)[\cos((r - p_2)x) + \sin(p_2 x) + \cos((r - p_1)x) + \sin(p_1 x)]$. Its combinatorial interpretation can be described in terms of the r-cubical lattice, since the set of maximal chains in the interval $[\hat{0}, \hat{1}]$ constitutes a subposet of the r-cubical lattice. Let P_1 and P_2 denote these two subposets. Then P_1 consists of maximal chains in $[\hat{0}, \hat{1}]$ whose first non-special edge is labeled with (i_1, j_1), where $r - p_1 + 1 \leq i_1 \leq r$. Similarly, P_2 consists of the maximal chains in $[\hat{0}, \hat{1}]$ whose first non-special edge is labeled with (i_1, j_1), where $r - p_2 + 1 \leq i_1 \leq r$. Hence, $\Lambda_{r,p_1,n} + \Lambda_{r,p_2,n}$ equals the number of maximal chains in the Hasse diagram of the disjoint union $P_1 + P_2$.

Therefore, the above two numbers can be endowed a combinatorial interpretation in terms of Λ-alternating augmented r-signed permutations, or in terms of the maximal chains in the Hasse diagram of the r-cubical lattice.

For the case when m is not square-free, say, $m = bu^2$. We shall encounter the functions $\sec(bx)(\cos((b-p)x) - \sin(px))$ in the formulas for $s_{m,n}$. It is clear to see that these functions are the generating functions for the numbers $(-1)^n \Lambda_{b,p,n}$ after replacing x with $-x$ in (4.28)

$$\Lambda_{b,p}(-x) = \sum_{n \geq 0} (-1)^n \Lambda_{b,p,n} \frac{x^n}{n!} = \frac{\cos((b-p)x) - \sin(px)}{\cos(bx)},$$

so that we may give an analogous interpretation of the numbers $(-1)^n \Lambda_{b,p,n}$.

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