The Extended Zeilberger's Algorithm with Parameters

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Dedicated to Professor Wen-Tsun Wu
on the occasion of his ninetieth birthday

Abstract

For a hypergeometric series $\sum_k f(k,a,b,\ldots,c)$ with parameters $a,b,\ldots,c$, Paule has found a variation of Zeilberger's algorithm to establish recurrence relations involving shifts on the parameters. We consider a more general problem concerning several similar hypergeometric terms $f_1(k,a,b,\ldots,c)$, $f_2(k,a,b,\ldots,c)$, $\ldots$, $f_m(k,a,b,\ldots,c)$. We present an algorithm to derive a linear relation among the sums $\sum_k f_i(k,a,b,\ldots,c)$ $(1 \le i \le m)$. Furthermore, when the summand f_i contains the parameter x, we can require that the coefficients be x-free. Such relations with x-free coefficients can be used to determine whether a polynomial sequence satisfies the three term recurrence and structure relations for orthogonal polynomials. The q-analogue of this approach is called the extended q-Zeilberger's algorithm, which can be employed to derive recurrence relations on the Askey-Wilson polynomials and the q-Racah polynomials.

Keywords: Zeilberger's algorithm, the Gosper algorithm, hypergeometric series, orthogonal polynomials

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1. Introduction

Based on Gosper's algorithm, Zeilberger [14, 16] has developed a powerful theory for proving identities on hypergeometric series and basic hypergeometric series. Let $F(n,k)$ be a double hypergeometric term, namely, $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are both rational functions of n and k. Zeilberger's algorithm is devised to find a double hypergeometric term $G(n,k)$ and polynomials $a_0(n), a_1(n), \ldots, a_m(n)$ which are independent of k such that

$$a_0(n)F(n,k)+a_1(n)F(n+1,k)+\cdots+a_m(n)F(n+m,k) = G(n,k+1)-G(n,k). \tag{1.1}$$

Writing

$$S(n) = \sum_{k=0}^{\infty} F(n,k).$$

Summing (1.1) over k, we deduce that

$$a_0(n)S(n)+a_1(n)S(n+1)+\cdots+a_m(n)S(n+m) = G(n,\infty)-G(n,0). \tag{1.2}$$

Thus the identity

$$\sum_{k=0}^{\infty} F(n,k) = f(n) \tag{1.3}$$

can be justified by verifying that $f(n)$ also satisfies (1.2) and both sides of (1.3) share the same initial values.

Paule [12] extended Zeilberger's algorithm to the multi-variable case and found many applications. Let $\boldsymbol{n}$ denote the vector of variables $(n_1, \ldots, n_r)$ and $F(\boldsymbol{n}, k)$ be a multi-variable hypergeometric term, that is,

$$\frac{F(n_1+1, n_2, \ldots, n_r, k)}{F(n_1, n_2, \ldots, n_r, k)}, \ldots, \frac{F(n_1, n_2, \ldots, n_r+1, k)}{F(n_1, n_2, \ldots, n_r, k)}, \frac{F(n_1, n_2, \ldots, n_r, k+1)}{F(n_1, n_2, \ldots, n_r, k)}$$

are all rational functions of $\boldsymbol{n}$ and k. Given m shifts $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_m \in \mathbb{Z}^r$ of the variables $\boldsymbol{n}$, he found that one can use a similar procedure to Zeilberger's algorithm to find a multi-variable hypergeometric term $G(\boldsymbol{n}, k)$ and coefficients $\alpha_1(\boldsymbol{n}), \ldots, \alpha_m(\boldsymbol{n})$ which are independent of k such that

$$\sum_{i=1}^{m} \alpha_i(\boldsymbol{n}) F(\boldsymbol{n}+\boldsymbol{\gamma}_i, k) = G(\boldsymbol{n}, k+1) - G(\boldsymbol{n}, k). \tag{1.4}$$

The main idea of this paper is the observation that Paule's approach can be further extended to a more general telescoping problem. Let $f_1(k, a, b, \ldots, c)$, $\ldots$, $f_m(k, a, b, \ldots, c)$ be m similar hypergeometric terms of k with parameters $a, b, \ldots, c$, namely, the ratios

$$\frac{f_i(k, a, b, \ldots, c)}{f_j(k, a, b, \ldots, c)} \quad \text{and} \quad \frac{f_i(k+1, a, b, \ldots, c)}{f_i(k, a, b, \ldots, c)}$$

are all rational functions of k and $a, b, \ldots, c$. Find a hypergeometric term $g(k, a, b, \ldots, c)$, that is, the ratio $g(k+1, a, b, \ldots, c)/g(k, a, b, \ldots, c)$ is a rational function of k and $a, b, \ldots, c$, and polynomial coefficients $a_1(a, b, \ldots, c)$, $a_2(a, b, \ldots, c)$, $\ldots$, $a_m(a, b, \ldots, c)$ which are independent of k such that

$$a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) = g(k+1) - g(k). \tag{1.5}$$

For brevity, from now on we may omit the parameters $a, b, \ldots, c$ and write $f_i(k)$ for $f_i(k, a, b, \ldots, c)$, a_i for $a_i(a, b, \ldots, c)$, and $g(k)$ for $g(k, a, b, \ldots, c)$. Once the telescoping relation (1.5) is established, summing over k often leads to a homogenous relation among the sums $\sum_k f_1(k)$, $\ldots$, $\sum_k f_m(k)$:

$$a_1 \sum_k f_1(k) + a_2 \sum_k f_2(k) + \cdots + a_m \sum_k f_m(k) = 0.$$

Let $F(\boldsymbol{n}, k)$ be a multi-variable hypergeometric term and $\boldsymbol{\gamma}_i \in \mathbb{Z}^r$. Then $f_i(k, \boldsymbol{n}) = F(\boldsymbol{n} + \boldsymbol{\gamma}_i, k)$ are similar hypergeometric terms of k with parameters $n_1, \ldots, n_r$. Therefore, Paule's equation (1.4) is a special case of (1.5). However, we should note that the extended Zeilberger's algorithm is very much in the spirit of the original algorithm of Zeilberger, and it should be regarded as a variation as well because the implementation of the extended algorithm is essentially the same as the original algorithm.

As an application of our algorithm, one can determine whether a given hypergeometric series satisfies the recurrence relation and the structure relations for orthogonal polynomials. Meanwhile, we obtain the coefficients in these relations. For instance, let $P_n(x) = \sum_k P_{n,k}(x)$ be the hypergeometric representation of the Jacobi polynomials as given in (3.3). Set

$$f_1(k) = P_{n,k}(x), \; f_2(k) = P'_{n+1,k}(x), \; f_3(k) = P'_{n,k}(x), \; f_4(k) = P'_{n-1,k}(x),$$

where $P'_{n,k}(x)$ denotes the derivative of $P_{n,k}(x)$ with respect to x. The extended Zeilberger's algorithm enables us to find the structure relation for $P_n(x)$

$$P_n(x) = \tilde{a}_n P'_{n+1}(x) + \tilde{b}_n P'_n(x) + \tilde{c}_n P'_{n-1}(x). \tag{1.6}$$

It is worth mentioning that neither the original Zeilberger's algorithm nor the variation of Paule is directly applicable to the above relation (1.6) involving derivatives.

Furthermore, it is important to impose an additional requirement that the coefficients $a_1, \ldots, a_m$ in (1.5) are not only independent of k but also independent of some other parameters such as the variable x occurring as the variable of orthogonal polynomials. For example, $\tilde{a}_n, \tilde{b}_n$ and $\tilde{c}_n$ in (1.6) are required to be independent of the variable x. Based on this parameter-free property of the coefficients, Chen and Sun [6] have developed a computer algebra approach to proving identities on Bernoulli polynomials and Euler polynomials.

We should notice that Koepf and Schmersau [10] have shown that one can derive the recurrence relation and structure relations for orthogonal polynomials by variations of Zeilberger's algorithm. For each of the three kinds of relations, they have provided an algorithm. The extended Zeilberger's algorithm serves as a unification of their algorithms and applies to more general cases. For instance, our algorithm can also be used to derive recurrence relations for the connection coefficients between two classes of Meixner polynomials with different parameters.

In another direction, the extended Zeilberger's algorithm can be adapted to deal with basic hypergeometric terms. Using the q-analogue of this algorithm, we can recover the three term recurrence relations for the Askey-Wilson polynomials and the q-Racah polynomials.

Let us recall some terminology and notation. A function $t(k)$ is called a hypergeometric term if $t(k+1)/t(k)$ is a rational function of k. A hypergeometric series is defined by

$$ {}_rF_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| \, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}, $$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the raising factorial. The q-shifted factorial is given by $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$ and we write

$$ (a_1, \ldots, a_m; q)_k = (a_1;q)_k \cdots (a_m;q)_k. $$

Then a basic hypergeometric series is defined by

$$ {}_r\phi_s \left[\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q; z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots b_s; q)_k} \frac{z^k}{(q;q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{s-r+1}. $$

2. The Extended Zeilberger's Algorithm

Let $f_1(k), f_2(k), \ldots, f_m(k)$ be similar hypergeometric terms with parameters $a, b, \ldots, c$. Recall that two hypergeometric terms $f(k)$ and $g(k)$ are said to be similar if their ratio is a rational function of k and the parameters. We assume that

$$ \frac{f_1(k+1)}{f_1(k)} = \frac{u(k)}{v(k)} \quad \text{and} \quad \frac{f_i(k)}{f_1(k)} = \frac{p_i(k)}{Q(k)}, \ i = 1, 2, \ldots, m, \tag{2.1} $$

where $u(k), v(k), p_i(k), Q(k)$ are polynomials in k and the parameters $a, b, \ldots, c$. Suppose that $f_i(k)$ satisfy (2.1). Then

$$ \frac{f_i(k+1)}{f_i(k)} = \frac{f_i(k+1)/f_1(k+1)}{f_i(k)/f_1(k)} \frac{f_1(k+1)}{f_1(k)} = \frac{p_i(k+1)Q(k)u(k)}{p_i(k)Q(k+1)v(k)} $$

and

$$ \frac{f_i(k)}{f_j(k)} = \frac{f_i(k)/f_1(k)}{f_j(k)/f_1(k)} = \frac{p_i(k)}{p_j(k)} $$

are rational functions of k and $a, b, \ldots, c$. Thus (2.1) is equivalent to the statement that $f_1(k), f_2(k), \ldots, f_m(k)$ are similar hypergeometric terms.

Our aim is to find coefficients $a_1, \ldots, a_m$ as rational functions in the parameters $a, b, \ldots, c$ but independent of k (called *k-free* coefficients) such that

$$ a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) = g(k+1) - g(k) \tag{2.2} $$

for some hypergeometric term $g(k)$ with parameters $a, b, \ldots, c$. By the similarity of $f_1(k), \ldots, f_m(k)$,

$$ t_k = a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) \tag{2.3} $$

is a hypergeometric term of k with parameters $a, b, \ldots, c$. So we can apply Gosper's algorithm [8] to find $g(k)$ such that $t_k = g(k+1) - g(k)$. Notice that we always have a trivial solution $a_1 = a_2 = \cdots = a_m = 0$ and $g(k) = 0$.

Notice that by multiplying the common denominator, the coefficients $a_1, \ldots, a_m$ in (2.2) become polynomials in the parameters $a, b, \ldots, c$. If no confusion arises, we may not mention the parameters $a, b, \ldots, c$. In the usual case n is the parameter for identities on finite sums.

It follows from (2.3) that

$$\begin{aligned}\frac{t_{k+1}}{t_k} &= \frac{f_1(k+1)}{f_1(k)} \frac{\sum_{i=1}^m a_i f_i(k+1)/f_1(k+1)}{\sum_{i=1}^m a_i f_i(k)/f_1(k)} \\ &= \frac{u(k)Q(k)}{v(k)Q(k+1)} \frac{\sum_{i=1}^m a_i p_i(k+1)}{\sum_{i=1}^m a_i p_i(k)}.\end{aligned}$$

Suppose that

$$\frac{u(k)Q(k)}{v(k)Q(k+1)} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}$$

is a Gosper representation, i.e., $a(k), b(k), c(k)$ are polynomials such that $\gcd(a(k), b(k+h)) = 1$ for all non-negative integers h. Then a Gosper representation of t_{k+1}/t_k is given by

$$\frac{t_{k+1}}{t_k} = \frac{a(k)}{b(k)} \frac{c(k+1)P(k+1)}{c(k)P(k)},$$

where

$$P(k) = \sum_{i=1}^m a_i p_i(k). \tag{2.4}$$

Gosper's algorithm states that $g(k)$ exists if and only if there exists a polynomial $x(k)$ such that

$$a(k)x(k+1) - b(k-1)x(k) = c(k)P(k). \tag{2.5}$$

Moreover, the degree bound d for $x(k)$ can be estimated by $a(k)$ and $b(k)$. Suppose

$$x(k) = \sum_{i=0}^d c_i k^i.$$

By comparing the coefficients of each power of k on both sides, we obtain a system of linear equations on $a_1, \ldots, a_m$ and $c_0, c_1, \ldots, c_d$. Solving the system of linear equations, we find coefficients $a_1, \ldots, a_m$ and

$$g(k) = \frac{b(k-1)x(k)}{c(k)Q(k)} f_1(k).$$

The extended Zeilberger's algorithm can be described in terms of the following steps.

Input: m similar hypergeometric terms $f_1(k), \ldots, f_m(k)$.
Output: k-free coefficients $a_1, a_2, \ldots, a_m$ and a hypergeometric term $g(k)$ satisfying (2.2).

1. Compute the rational functions

$$r_i(k) = \frac{f_i(k)}{f_1(k)} \quad \text{and} \quad r(k) = \frac{f_1(k+1)}{f_1(k)}.$$

   Set $Q(k)$ to be the common denominator of $r_1(k), \ldots, r_m(k)$,

$$p_i(k) = r_i(k)Q(k),$$

   and let $P(k)$ be given by (2.4).

2. Compute a Gosper representation of

$$r(k)\frac{Q(k)}{Q(k+1)} = \frac{a(k)}{b(k)}\frac{c(k+1)}{c(k)}.$$

3. Compute the degree bound d for $x(k)$ and solve the equation (2.5) by the method of undetermined coefficients to obtain the k-free coefficients $a_1, \ldots, a_m$ and the polynomial $x(k)$.

4. The hypergeometric term $g(k)$ is then given by

$$g(k) = \frac{b(k-1)x(k)}{c(k)Q(k)}f_1(k).$$

Suppose that $F(n,k)$ is a double hypergeometric term. Let $f_i(k) = F(n+i-1,k)$. Then the extended Zeilberger's algorithm reduces to the original Zeilberger's algorithm. More generally, suppose that $\boldsymbol{n} = (n_1, \ldots, n_r)$, $F(\boldsymbol{n},k)$ is a multi-variable hypergeometric term and $\boldsymbol{\gamma}_i \in \mathbb{Z}^r$. The specialization $f_i(k) = F(\boldsymbol{n}+\boldsymbol{\gamma}_i, k)$ reduces to Paule's variation.

As will be seen, in some applications it is necessary to require that the coefficients $a_1, \ldots, a_m$ be independent of some parameters, say, the parameter

a. More precisely, let $f_1(k, a, b, \ldots, c)$, $\ldots$, $f_m(k, a, b, \ldots, c)$ be m similar multi-variable hypergeometric terms, that is,

$$\frac{f_i(k+1, a, b, \ldots, c)}{f_i(k, a, b, \ldots, c)} \quad \text{and} \quad \frac{f_i(k, a, b, \ldots, c)}{f_1(k, a, b, \ldots, c)} \tag{2.6}$$

are rational functions of $k, a, b, \ldots, c$. We aim to find $a_1, a_2, \ldots, a_m$ not only independent of k but also independent of the parameter a such that (2.2) holds.

Since the solutions $(a_1, \ldots, a_m, g(k))$ of (2.2) form a linear vector space, we may use the following form as the output

$$\begin{aligned} & a_1 = v_1, \ldots, a_r = v_r, \\ & a_{r+1} = h_{r+1}(v_1, \ldots, v_r), \ldots, a_m = h_m(v_1, \ldots, v_r), \\ & g(k) = h(v_1, \ldots, v_r) f_1(k), \end{aligned} \tag{2.7}$$

where $v_1, \ldots, v_r$ are free variables and $h_{r+1}, \ldots, h_m, h$ are linear combinations of $v_1, \ldots, v_r$. For this purpose, we should first ignore the independence of a and apply the extended Zeilberger's algorithm to find the solution (2.7). By (2.6), the functions $h_{r+1}, \ldots, h_m$ are rational functions of the parameters $a, b, \ldots, c$ and thus can be written as

$$h_i = p_i(a, b, \ldots, v_1, \ldots, v_r)/q_i(a, b, \ldots, c),$$

where p_i, q_i are relatively prime polynomials. Now consider the additional requirement that $a_1, a_2, \ldots, a_m$ are independent of the parameters a. Therefore, all the coefficients of

$$p_i(a, b, \ldots, c, v_1, \ldots, v_r) - a_i q_i(a, b, \ldots, c)$$

in variable a must be zero. This gives a system of linear equations on $a_1, \ldots, a_m$ and $v_1, \ldots, v_r$. Upon solving these equations, we eventually find $a_1, a_2, \ldots, a_m$ which are independent of k and the parameter a. The above version of the extended Zeilberger's algorithm will still be called the extended Zeilberger's algorithm.

Let us take the Hermite polynomials as the first example to show how to use the above algorithm to derive linear relations on sums of similar hypergeometric terms with parameters.

Example 2.1 The Hermite polynomials $H_n(x)$ are given by

$$H_n(x) = (2x)^n \ {}_2F_0 \left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} \middle| -\frac{1}{x^2} \right), \tag{2.8}$$

see [2, Section 6.1]. We aim to find a three term recurrence

$$xH_n(x) = \alpha_n H_{n+1}(x) + \beta_n H_n(x) + \gamma_n H_{n-1}(x)$$

with coefficients $\alpha_n, \beta_n, \gamma_n$ being independent of x. Let

$$H_{n,k}(x) = (2x)^n \frac{\left(-\frac{n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k}{k!} \left(-\frac{1}{x^2}\right)^k$$

be the summand in (2.8). We first ignore the x-freeness requirement and apply the extended Zeilberger's algorithm to the four similar hypergeometric terms with parameters n and x

$$f_1(k) = xH_{n,k}(x),\ f_2(k) = H_{n+1,k}(x),\ f_3(k) = H_{n,k}(x),\ f_4(k) = H_{n-1,k}(x).$$

We find that

$$a_1 = v_1,\ a_2 = v_2,\ a_3 = -x(v_1 + 2v_2),\ a_4 = 2nv_2, \tag{2.9}$$

and

$$g(k) = \frac{-4kv_2}{n+1-2k} xH_{n,k}(x). \tag{2.10}$$

Now it is time to impose the x-freeness condition to give an additional equation

$$v_1 + 2v_2 = 0.$$

Hence we obtain

$$a_1 = v_1,\ a_2 = -\frac{v_1}{2},\ a_3 = 0,\ a_4 = -nv_1,\ g(k) = \frac{2kv_1}{n+1-2k} xH_{n,k}(x).$$

It follows that

$$v_1 xH_{n,k}(x) - \frac{v_1}{2} H_{n+1,k}(x) - nv_1 H_{n-1,k}(x) = g(k+1) - g(k).$$

Summing over k, we deduce that

$$xH_n(x) = \frac{1}{2} H_{n+1}(x) + nH_{n-1}(x).$$

3. Orthogonal Polynomials

Using the extended Zeilberger's algorithm, we can determine whether a hypergeometric series satisfies a three term relation and the structure relations for a sequence of orthogonal polynomials. In other words, one can verify the orthogonality of a terminating hypergeometric series by using the extended Zeilberger's algorithm.

The method to derive the relation in Example 2.1 is in fact valid in the general case. Given a hypergeometric series $P_n(x)$, we can compute the coefficients for the following recurrence relation

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x).$$

Let $P_n(x) = \sum_k P_{n,k}(x)$, where $P_{n,k}(x)$ is a hypergeometric term of k with parameters n and x. Set

$$f_1(k) = xP_{n,k}(x),\ f_2(k) = P_{n+1,k}(x),\ f_3(k) = P_{n,k}(x),\ f_4(k) = P_{n-1,k}(x).$$

Clearly, f_1, f_2, f_3 and f_4 are similar hypergeometric terms. So we can use the extended Zeilberger's algorithm to find a_1, a_2, a_3, a_4 which are independent of k and x such that

$$a_1 xP_{n,k}(x) + a_2 P_{n+1,k}(x) + a_3 P_{n,k}(x) + a_4 P_{n-1,k}(x) = g(k+1) - g(k).$$

Then summing over k leads to

$$\alpha_n = -a_2/a_1,\ \beta_n = -a_3/a_1,\ \gamma_n = -a_4/a_1.$$

By this method, we can recover the three term recurrences for the Laguerre polynomials, the Jacobi polynomials, the Charlier polynomials, the Meixner polynomials, the Kravchuk polynomials and the Hahn polynomials, as listed in the following table. Notice that we have adopted the notation in [9].

monic OPs	coefficients ($\alpha_n = 1$)
Laguerre $L_n^{(a)}(x)$	$\beta_n = a + 2n + 1,\ \gamma_n = n(a+n)$
Jacobi $P_n^{(a,b)}(x)$	$\begin{cases} \beta_n = \frac{-(a-b)(a+b)}{(2n+2+a+b)(2n+a+b)} \\ \gamma_n = \frac{4(n+b)(a+n)(n+a+b)n}{(2n+a+b+1)(2n+a+b-1)(2n+a+b)^2} \end{cases}$
Charlier $C_n(x;a)$	$\beta_n = a + n,\ \gamma_n = an$
Meixner $M_n(x;b,c)$	$\beta_n = \dfrac{cb+nc+n}{1-c},\ \gamma_n = \dfrac{nc(b+n-1)}{(c-1)^2}$
Krawtchouk $K_n(x;p,N)$	$\begin{cases} \beta_n = Np - 2np + n, \\ \gamma_n = pn(1-p)(N-n+1) \end{cases}$
Hahn $Q_n(x;a,b,N)$	$\begin{cases} \beta_n = \frac{(b+a+2n^2+2n+2nb+2na+a^2+ab)N-n(a-b)(n+a+b+1)}{(2n+a+b)(2n+2+a+b)} \\ \gamma_n = \frac{n(N-n+1)(n+b)(a+n)(n+a+b)(a+b+N+1+n)}{(2n+a+b+1)(2n+a+b-1)(2n+a+b)^2} \end{cases}$

Example 3.1 The Wilson polynomials $W_n(x)$ are given by

$$\frac{W_n(x^2)}{(a+b)_n(a+c)_n(a+d)_n} = {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+xi, a-xi \\ a+b, a+c, a+d \end{matrix} \middle| 1 \right),$$

where $i = \sqrt{-1}$, see [15]. Let $W_{n,k}(x^2)$ be the summand of the right hand side multiplied by the factor $(a+b)_n(a+c)_n(a+d)_n$. Applying the the extended Zeilberger's algorithm to the similar terms

$$f_1(k) = x^2 W_{n,k}(x^2),\ f_2(k) = W_{n+1,k}(x^2),\ f_3(k) = W_{n,k}(x^2),\ f_4(k) = W_{n-1,k}(x^2),$$

we obtain the following relation, see also [9, p. 24]

$$xW_n(x) = \alpha_n W_{n+1}(x) + \beta_n W_n(x) + \gamma_n W_{n-1}(x),$$

where

$$\alpha_n = -\frac{a+b+c+d+n-1}{(a+b+c+d+2n)(a+b+c+d+2n-1)},$$

$$\beta_n = \frac{4n^2 - 4(1-a-b-c-d)n + (a+b+c+d)^2 - 2(a^2+b^2+c^2+d^2)}{8}$$
$$+ \frac{(a+b+c+d-2)(a+b-c-d)(a+c-b-d)(a+d-b-c)}{8(a+b+c+d+2n)(a+b+c+d+2n-2)},$$

and

$$\gamma_n = -(a+b+n-1)(a+c+n-1)(a+d+n-1)$$
$$\times \frac{(b+c+n-1)(b+d+n-1)(c+d+n-1)n}{(a+b+c+d+2n-1)(a+b+c+d+2n-2)}.$$

Example 3.2 The Racah polynomials $R_n(x)$ are given by

$$R_n(x(x+c+d+1)) = {}_4F_3\left(\begin{matrix} -n, n+a+b+1, -x, x+c+d+1 \\ a+1, b+d+1, c+1 \end{matrix} \middle| 1 \right),$$

see [4]. The extended Zeilberger's algorithm gives the recurrence relation of the Racah polynomials [9, p. 27]

$$xR_n(x) = \alpha_n R_{n+1}(x) + \beta_n R_n(x) + \gamma_n R_{n-1}(x),$$

where

$$\alpha_n = \frac{(a+n+1)(c+n+1)(a+b+n+1)(d+b+n+1)}{(a+b+2n+1)(a+b+2n+2)},$$

$$\beta_n = \frac{-4n^2 - 4(a+b+1)n + (a-2d-b-2-2c)(a-b)}{8}$$
$$- \frac{(c+1)(b+d+1)}{2} - \frac{(a-b)(a+b)(a-2d-b)(a-2c+b)}{8(a+b+2n)(a+b+2n+2)},$$

and

$$\gamma_n = \frac{(b+n)(a-d+n)(a+b-c+n)n}{(a+b+2n)(a+b+2n+1)}.$$

Example 3.3 Askey and Ismail [3] provided two hypergeometric representations for the Pollaczek polynomials:

$$P_n(x) \quad = \quad \eta^n \, {}_2F_1\left(\begin{matrix} -n, b(x-\eta)/\xi, \\ b/a \end{matrix} \middle| -\frac{\xi}{a\eta} \right)$$

$$= \zeta^n\, {}_2F_1\left(\begin{array}{c} -n, b(\zeta - x)/\xi, \\ b/a \end{array} \middle| \frac{\xi}{a\zeta} \right),$$

where

$$\xi = \sqrt{(1+a)^2x^2 - 4a}, \quad \eta = ((1+a)x - \xi)/2a, \quad \zeta = ((1+a)x + \xi)/2a.$$

Using the extended Zeilberger's algorithm, we derive the following three term recurrence from either representation [3]:

$$xP_n(x) = \frac{an+b}{(1+a)n+b} P_{n+1}(x) + \frac{n}{(1+a)n+b} P_{n-1}(x).$$

We continue to show that the extended Zeilberger's algorithm can be employed to express the derivatives of orthogonal polynomials in terms the original polynomials, and vice versa. Let $P_{n,k}(x)$ be the summand of the hypergeometric representation of $P_n(x)$ and $P'_{n,k}(x)$ be the derivative of $P_{n,k}(x)$. It is easily seen that $P'_{n,k}(x)$ is similar to $P_{n,k}(x)$. This enables us to derive the three term recurrence for $P'_n(x)$ and the structure relations for $P_n(x)$ as given below

$$\sigma(x)P'_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \tag{3.1}$$

and

$$P_n(x) = \bar{a}_n P'_{n+1}(x) + \bar{b}_n P'_n(x) + \bar{c}_n P'_{n-1}(x), \tag{3.2}$$

where $\sigma(x)$ is a polynomials in x of degree less than or equal to 2 and $a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n$ are constants not depending on x. To derive (3.1), we set

$$f_1(k) = \sigma(x)P'_{n,k}(x), f_2(k) = P_{n+1,k}(x),\ f_3(k) = P_{n,k}(x),\ f_4(k) = P_{n-1,k}(x).$$

To establish (3.2), we set

$$f_1(k) = P_{n,k}(x), f_2(k) = P'_{n+1,k}(x),\ f_3(k) = P'_{n,k}(x),\ f_4(k) = P'_{n-1,k}(x).$$

Example 3.4 The monic Jacobi polynomials are given by

$$P_n(x) = \frac{(a+1)_n 2^n}{(n+a+b+1)_n}\, {}_2F_1\left(\begin{array}{c} -n, n+a+b+1 \\ a+1 \end{array} \middle| \frac{1-x}{2} \right). \tag{3.3}$$

Let $P_{n,k}(x)$ denote the summand. Its derivative with respect to x equals

$$P'_{n,k}(x) = -\frac{(a+1)_n 2^n}{(n+a+b+1)_n} \frac{(-n)_k (n+a+b+1)_k}{2(a+1)_k (k-1)!} \left(\frac{1-x}{2}\right)^{k-1}.$$

Consider the four similar terms

$$f_1(k) = xP'_{n,k}(x),\ f_2(k) = P'_{n+1,k}(x),\ f_3(k) = P'_{n,k}(x),\ f_4(k) = P'_{n-1,k}(x).$$

By the extended Zeilberger's algorithm with parameters n and x, we find that

$$\begin{aligned} xP'_n(x) &= \frac{n}{n+1}P'_{n+1}(x) - \frac{(a+2+b)(a-b)}{(2n+2+a+b)(2n+a+b)}P'_n(x) \\ &\quad + \frac{4n(b+n)(a+n)(n+a+b+1)}{(2n+a+b+1)(2n+a+b-1)(2n+a+b)^2}P'_{n-1}(x), \end{aligned}$$

$$\begin{aligned} (1-x^2)P'_n(x) &= -nP_{n+1}(x) + \frac{2n(a-b)(n+a+b+1)}{(2n+2+a+b)(2n+a+b)}P_n(x) \\ &\quad + \frac{4(n+b)(a+n)(n+a+b+1)(n+a+b)n}{(2n+a+b+1)(2n+a+b-1)(2n+a+b)^2}P_{n-1}(x). \end{aligned}$$

and

$$\begin{aligned} P_n(x) &= \frac{1}{n+1}P'_{n+1}(x) + \frac{2(a-b)}{(2n+2+a+b)(2n+a+b)}P'_n(x) \\ &\quad - \frac{4(n+b)(a+n)n}{(2n+a+b+1)(2n+a+b-1)(2n+a+b)^2}P'_{n-1}(x). \end{aligned}$$

The following example is concerned with expressing the shifts of orthogonal polynomials with parameters in terms of the original polynomials and their derivatives.

Example 3.5 Let

$$P_n^{(a,b)}(x) = \frac{(a+1)_n 2^n}{(n+a+b+1)_n}\ {}_2F_1\left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix} \middle| \frac{1-x}{2}\right)$$

be the Jacobi polynomials, see [2,9]. By applying the extended Zeilberger's algorithm to $f_1(k) = P_{n,k}^{(a+1,\,b)}(x)$ ($f_1(k) = P_{n,k}^{(a,\,b+1)}(x)$, respectively) and

$$f_2(k) = P_{n+1,k}^{(a,\,b)}{}'(x),\;\; f_3(k) = P_{n,k}^{(a,\,b)}{}'(x),\;\; f_4(k) = P_{n-1,k}^{(a,\,b)}{}'(x),$$

we are led to the known relations due to Koepf and Schmersau [10]

$$P_n^{(a+1,\,b)}(x) = \frac{1}{n+1} P_{n+1}^{(a,\,b)}{}'(x) + \frac{2(a+1+n)}{(2n+2+a+b)(2n+a+b+1)} P_n^{(a,\,b)}{}'(x)$$

and

$$P_n^{(a,\,b+1)}(x) = \frac{1}{n+1} P_{n+1}^{(a,\,b)}{}'(x) - \frac{2(b+1+n)}{(2n+2+a+b)(2n+a+b+1)} P_n^{(a,\,b)}{}'(x).$$

Moreover, we can deduce the following relations which seem to be new:

$$\begin{aligned} P_n^{(a+1,\,b-1)}(x) &= \frac{1}{n+1} P_{n+1}^{(a,\,b)}{}'(x) + \frac{4(a+1+n)}{(2n+2+a+b)(2n+a+b)} P_n^{(a,\,b)}{}'(x) \\ &\quad + \frac{4(a+1+n)(a+n)n}{(2n+a+b-1)(2n+a+b+1)(2n+a+b)^2} P_{n-1}^{(a,\,b)}{}'(x), \end{aligned}$$

and

$$\begin{aligned} P_n^{(a-1,\,b+1)}(x) &= \frac{1}{n+1} P_{n+1}^{(a,\,b)}{}'(x) - \frac{4(b+1+n)}{(2n+2+a+b)(2n+a+b)} P_n^{(a,\,b)}{}'(x) \\ &\quad + \frac{4(b+1+n)(b+n)n}{(2n+a+b-1)(2n+a+b+1)(2n+a+b)^2} P_{n-1}^{(a,\,b)}{}'(x). \end{aligned}$$

The extended Zeilberger's algorithm can also be employed to compute the connection coefficients of two sequences of orthogonal polynomials. Ronveaux [13] developed an approach to computing recurrence relations for the connection coefficients. The extended Zeilberger's algorithm serves this purpose without resorting to the properties of the orthogonal polynomials. As an example, let us consider the connection coefficients of two classes of Meixner polynomials with different parameters.

Example 3.6 Let $M_n^{(\gamma,\,\mu)}(x)$ be the monic Meixner polynomials defined by

$$M_n^{(\gamma,\,\mu)}(x) = (\gamma)_n \left(\frac{\mu}{\mu-1}\right)^n {}_2F_1\left(\begin{matrix} -n, -x \\ \gamma \end{matrix} \,\middle|\, 1 - \frac{1}{\mu} \right),$$

see [9, p. 45]. We wish to find a recurrence relation for the connection coefficients $C_m(n)$ defined by

$$M_n^{(\gamma,\,\mu)}(x) = \sum_{m=0}^{n} C_m(n) M_m^{(\delta,\,\nu)}(x). \tag{3.4}$$

To this end, we first find a difference operator which eliminates $M_n^{(\gamma,\,\mu)}(x)$. This goal can be achieved by applying the extended Zeilberger's algorithm to the similar terms

$$f_1(k) = M_{n,k}^{(\gamma,\,\mu)}(x),\;\; f_2(k) = M_{n,k}^{(\gamma,\,\mu)}(x+1),\;\; \text{and}\;\; f_3(k) = M_{n,k}^{(\gamma,\,\mu)}(x-1),$$

where

$$M_{n,k}^{(\gamma,\,\mu)}(x) = (\gamma)_n \left(\frac{\mu}{\mu-1}\right)^n \frac{(-n)_k(-x)_k}{(\gamma)_k k!} \left(1-\frac{1}{\mu}\right)^k.$$

From the telescoping relation generated by the extended Zeilberger's algorithm, we deduce that

$$(x\mu+\mu\gamma+x-n+n\mu)M_n^{(\gamma,\,\mu)}(x)-\mu(\gamma+x)M_n^{(\gamma,\,\mu)}(x+1)-xM_n^{(\gamma,\,\mu)}(x-1) = 0.$$

Let

$$S_m(x) = (x\mu+\mu\gamma+x-n+n\mu)M_m^{(\delta,\,\nu)}(x)-\mu(\gamma+x)M_m^{(\delta,\,\nu)}(x+1)-xM_m^{(\delta,\,\nu)}(x-1),$$

which can be used to establish a linear relation on the connection coefficients $C_m(n)$. Indeed, it follows from (3.4) that

$$\sum_{m=0}^{n} C_m(n) S_m(x) = 0. \tag{3.5}$$

Suppose that we can express $S_m(x)$ in terms of a suitable basis $\{P_m(x)\}$:

$$S_m(x) = a_m P_{m+1}(x) + b_m P_m(x) + c_m P_{m-1}(x), \tag{3.6}$$

where a_m, b_m and c_m are independent of x. Substituting (3.6) into (3.5), by the linear independence of $P_m(x)$ for $m = 0, 1, 2, \ldots$, that is, the coefficients of $P_i(x)$ are all zeros, we find

$$a_{m-1}C_{m-1}(n) + b_m C_m(n) + c_{m+1}C_{m+1}(n) = 0. \tag{3.7}$$

Thus the question has become how to find the polynomials $P_m(x)$ in order to determine the coefficients a_m, b_m and c_m. In view of the relation (3.6), we consider a hypergeometric term $P_m(x)$ that is similar to $S_m(x)$ so that we can solve the equation

$$S_m(x) - a_m P_{m+1}(x) - b_m P_m(x) - c_m P_{m-1}(x) = 0$$

by using the extended Zeilberger's algorithm. In fact, we may choose

$$P_m(x) = \Delta(M_m^{(\delta,\nu)}(x)) = M_m^{(\delta,\nu)}(x+1) - M_m^{(\delta,\nu)}(x).$$

It is easily checked that $P_m(x)$ satisfies (3.6) and the corresponding coefficients are given by

$$a_m = \frac{(\mu-1)(n-m)}{m+1}, \quad c_m = \frac{(\nu-\mu)(\delta+m-1)m\nu}{(1-\nu)^2},$$

$$b_m = \frac{-\nu\mu m - m\mu + 2m\nu + \nu\mu\gamma - \nu n + \nu\delta - \nu + \mu - \nu\mu\delta - \mu\gamma + \nu n\mu}{1-\nu}.$$

Hence we have derived a recurrence relation (3.7) for the connection coefficients $C_m(n)$.

4. q-Orthogonal Polynomials

The extended Zeilberger's algorithm can be readily adapted to basic hypergeometric terms t_k with parameters $a, b, \ldots, c$, that is, the ratio of two consecutive terms is a rational function of q^k and the parameters. The q-analogue of the extended Zeilberger's algorithm will be called the extended q-Zeilberger's algorithm. Let $f_1(k)$, $f_2(k), \ldots, f_m(k)$ be similar q-hypergeometric terms, namely,

$$f_i(k)/f_j(k) \quad \text{and} \quad f_i(k+1)/f_i(k) \ (1 \le i, j \le m)$$

are rational functions of q^k and the parameters. The objective of the extended q-Zeilberger's algorithm is to find a q-hypergeometric term $g(k)$ and coefficients a_1, $a_2, \ldots, a_m$ which are independent of k such that

$$a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) = g(k+1) - g(k). \tag{4.1}$$

The detailed description of the extended q-Zeilberger's algorithm is similar to that of the ordinary case, hence is omitted. We will only give examples to demonstrate how to use this method to compute the three term recurrences and structure relations for q-orthogonal polynomials.

Example 5.1 The discrete q-Hermite I polynomials are given by [1]

$$H_n(x) = q^{\binom{n}{2}}\ {}_2\phi_1 \left[\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; -qx \right].$$

Let $dH_n(x) = \frac{H(xq)-H(x)}{(q-1)x}$ be the q-difference of $H_n(x)$. We derive that

$$xdH_n(x) = \frac{1-q^n}{1-q^{n+1}} dH_{n+1}(x) + q^{n-2}(1-q^n)dH_{n-1}(x)$$

and

$$dH_n(x) = \frac{1-q^n}{1-q} H_{n-1}(x).$$

Example 5.2 The Askey-Wilson polynomials $p_n(x; a, b, c, d|q)$ are defined by

$$\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q; q \right], \ x = \cos\theta,$$

see [7, (7.5.2)]. Let $t_{n,k}(x)$ be the summand of the right hand side multiplied by $(ab, ac, ad; q)_n/a^n$. Applying the extended q-Zeilberger's algorithm to

$$f_1(k) = xt_{n,k}(x),\ f_2(k) = t_{n+1,k}(x),\ f_3(k) = t_{n,k}(x),\ f_4(k) = t_{n-1,k}(x),$$

we find that

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

where

$$\alpha_n = \frac{1 - abcdq^{n-1}}{2(1 - abcdq^{2n})(1 - abcdq^{2n-1})},$$

$$\beta_n = \frac{q^{n-1}(abcdq^{2n-1}+1)((a+b+c+d)q + bcd + acd + abd + abc)}{2(1 - abcdq^{2n})(1 - abcdq^{2n-2})}$$

$$-\frac{q^{2n-2}(1+q)((bcd+acd+abd+abc)q+abcd(a+b+c+d))}{2(1-abcdq^{2n})(1-abcdq^{2n-2})},$$

and

$$\gamma_n = (1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1}) \times \frac{(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{2(1-abcdq^{2n-1})(1-abcdq^{2n-2})}.$$

Example 5.3 The q-Racah polynomials $R_n(x;a,b,c,d|q)$ are given by

$$R_n(q^{-x}+cdq^{x+1};a,b,c,d|q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-x}, cdq^{x+1} \\ aq, bdq, cq \end{matrix}\middle| q;q\right],$$

see [9, p. 122]. The extended q-Zeilberger's algorithm gives the following recurrence relation first derived by Askey and Wilson [4] using a transformation formula on a ${}_8\phi_7$ series:

$$xR_n(x) = \alpha_n R_{n+1}(x) + \beta_n R_n(x) + \gamma_n R_{n-1}(x),$$

where

$$\alpha_n = \frac{(1-aq^{n+1})(1-abq^{n+1})(1-bdq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$

$$\beta_n = \frac{q^{n+1}(abq^{2n+1}+1)(c+bcd+dc+a+bd+ab+ca+abd)}{(1-abq^{2n})(1-abq^{2n+2})} - \frac{q^{2n+1}(1+q)(ab^2d+abcd+ca+bcd+abd+ab+abc+a^2b)}{(1-abq^{2n})(1-abq^{2n+2})},$$

and

$$\gamma_n = \frac{(1-q^n)(1-bq^n)(c-abq^n)(d-aq^n)q}{(1-abq^{2n})(1-abq^{2n+1})}.$$

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