

# Identities Derived from Noncrossing Partitions of Type $B$

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**Abstract.** Based on weighted noncrossing partitions of type  $B$ , we obtain type  $B$  analogues of Coker's identities on the Narayana polynomials. A parity reversing involution is given for the alternating sum of Narayana numbers of type  $B$ . Moreover, we find type  $B$  analogues of the refinements of Coker's identities due to Chen, Deutsch and Elizalde. By combinatorial constructions, we provide type  $B$  analogues of three identities of Mansour and Sun also on the Narayana polynomials.

**Keywords:** noncrossing partition of type  $B$ , Narayana polynomial of type  $B$ , bijection.

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## 1 Introduction

The objective of this paper is to give type  $B$  analogues of combinatorial identities on the Narayana polynomials [4, 9, 10, 21, 22]

$$N_n(x) = \sum_{k=1}^n N_{n,k} x^k, \quad n \geq 1,$$

where

$$N_{n,k} = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}$$

are called the Narayana numbers. Let

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

be the  $n$ -th Catalan number. Using generating functions, Coker [9, 10] has derived the following identities

$$\sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1+x)^{n-2k-1}, \quad (1.1)$$

$$\sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} (1+x)^{2(n-1-k)} = \sum_{k=0}^{n-1} C_{k+1} \binom{n-1}{k} x^k (1+x)^k. \quad (1.2)$$

Chen, Yan and Yang [9] have given combinatorial interpretations of these two identities based on weighted Dyck paths and 2-Motzkin paths in answer to a question raised by Coker. It should be noticed that the identity (1.1) can also be derived from the following identity due to Simion and Ullman [19], see also Chen, Deng and Du [5]:

$$\frac{1}{n} \binom{n}{k} \binom{n}{k+1} = \sum_{r=0}^{k-1} \binom{n-1}{2r} \binom{n-2r-1}{k-1-r} C_r.$$

Recently, Mansour and Sun [16] have established the following three identities on the Narayana polynomials and have given both algebraic and combinatorial proofs:

$$x^{\frac{n}{2}+1} C_{\frac{n}{2}} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} N_{k+1}(x) (1+x)^{n-k}, \quad (1.3)$$

$$x^{n+2} C_{n+1} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} N_{k+1}(x^2) (1-x)^{2(n-k)}, \quad (1.4)$$

$$C_n = \sum_{k=0}^n \frac{2k+1}{2n+1} \binom{2n+1}{n-k} N_k(x) (1-x)^{n-k}, \quad (1.5)$$

where  $C_{\frac{n}{2}}$  is treated as zero if  $n$  is odd.

We obtain type  $B$  analogues of the above identities of Coker, and Mansour and Sun, based on the structure of type  $B$  noncrossing partitions. Recall that a type  $B$  partition of  $[n]$  is a partition of the set  $\{1, 2, \dots, n, -1, -2, \dots, -n\}$ , which may have a unique block, if it exists, called the zero block in which  $i$  and  $-i$  appear in pairs, such that for any block  $B$  of  $\pi$ , the set  $-B$ , obtained by negating the elements of  $B$ , is also a block of  $\pi$ , see, for example, [2, 3, 17, 20]. Evidently, the zero block is a union of antipodal pairs  $\{i, -i\}$  if it exists. Moreover, there does not exist any other block  $B$  such that  $B = -B$ . A type  $B$  partition  $\pi$  can be represented by a diagram, with the elements  $1, 2, \dots, n, -1, -2, \dots, -n$  drawn on a horizontal line in the following order

$$1 < 2 < \dots < n < -1 < -2 < \dots < -n. \quad (1.6)$$

Accordingly, one can list the elements of a block in the above order. Suppose that  $B = \{i_1, i_2, \dots, i_k\}$  is a nonzero block of a type  $B$  partition  $\pi$ . One may represent this block by a path from  $i_1$  to  $i_k$  with arcs drawn above the horizontal line from  $i_1$  to  $i_2$ ,  $i_2$  to  $i_3$ , and so on. A block with one element is called a singleton block. Such a diagram is called the linear representation of a type  $B$  partition. A type  $B$  partition is said to be noncrossing if its diagram contains no crossing arcs, see, Athanasiadis [3]. It is worth mentioning that we may also place the elements  $1, 2, \dots, n, -1, -2, \dots, -n$  on a circle, and use a cycle to represent a block. This is called the cyclic representation of a type  $B$  partition. This leads to an equivalent definition of noncrossing partitions of type  $B$ , see, Reiner [17]. An illustration of these two representations of a type  $B$  noncrossing partition is given in Figure 1, where

$$\pi = \{1, -7\}\{7, -1\}\{2, 4, -6\}\{6, -2, -4\}\{3\}\{-3\}\{5, -5\}\{8\}\{-8\}.$$

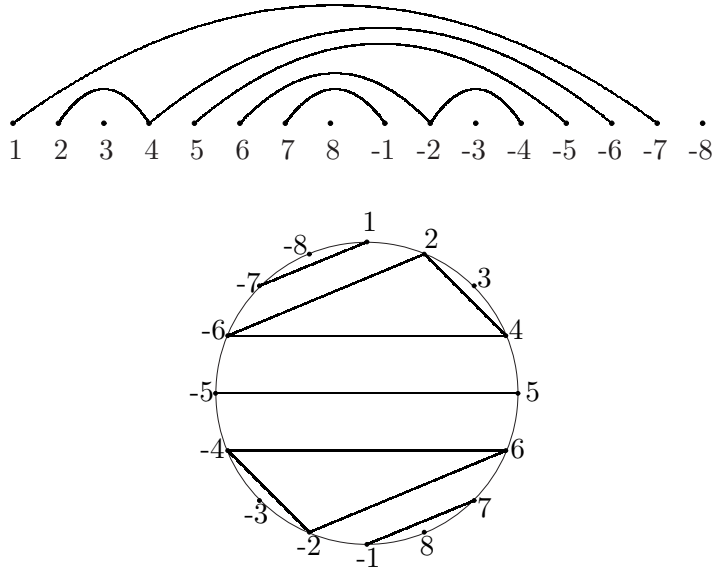


Figure 1: The linear and cyclic representations of a type  $B$  noncrossing partition.

In this paper, we shall adopt the linear representation of type  $B$  noncrossing partitions. The set of type  $B$  noncrossing partitions on  $[n]$  will be denoted by  $NC^B(n)$ . It is well known that the cardinality of  $NC^B(n)$  equals  $\binom{2n}{n}$ , see, for example, Reiner [17, Proposition 6]. Furthermore, the set of type  $B$  noncrossing partitions of  $[n]$  having  $k$  pairs of nonzero blocks will be denoted by  $NC^B(n, k)$ . The cardinality of  $NC^B(n, k)$ , which is known to be  $\binom{n}{k}^2$ , has been referred to as the Narayana number of type  $B$  by Fomin and Reading [12]. It also equals the rank-size of the lattice of the noncrossing partitions of type  $B$  of  $[n]$  with the refinement order [20]. The polynomials

$$P_n(x) = \sum_{k=0}^n \binom{n}{k}^2 x^k, \quad n \geq 1$$

will be called the Narayana polynomials of type  $B$ .

This paper is organized as follows. In Section 2, we give type  $B$  analogues of Coker's identities and combinatorial proofs in terms of weighted type  $B$  noncrossing partitions. We find an involution for the case of the alternating sum of the Narayana numbers of type  $B$ . We also provide refinements of Coker's identities of type  $B$ . Section 3 is devoted to type  $B$  analogues of three identities due to Mansour and Sun [16].

## 2 Type $B$ Analogues of Coker's Identities

This section is concerned with type  $B$  analogues of Coker's identities. We shall use weighted type  $B$  noncrossing partitions as the underlying combinatorial structure.

Roughly speaking, we shall assign a weight to each point, and the weight of a partition is the product of the weights of all the points. To be precise, we shall assign weights only to half of the elements appearing in the canonical representation of a type  $B$  noncrossing partition. The following two identities can be regarded as type  $B$  analogues of Coker's identities.

**Theorem 2.1.** *For  $n \geq 0$ ,*

$$\sum_{k=0}^n \binom{n}{k}^2 x^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \binom{2k}{k} x^k (1+x)^{n-2k}, \quad (2.1)$$

$$\sum_{k=0}^n \binom{n}{k}^2 x^{2k} (1+x)^{2(n-k)} = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} x^k (1+x)^k. \quad (2.2)$$

Note that identity (2.1) was first derived by Riordan [18] by using generating functions. Before presenting the combinatorial proof of the above theorem, we recall a basic property of the linear representation of a type  $B$  noncrossing partition, as observed by Athanasiadis [3]. By a pure block we mean a block that contains only positive elements or only negative elements. A block is called a mixed block if it is not pure.

**Proposition 2.2.** *Let  $\pi$  be a type  $B$  noncrossing partition. In the linear representation of  $\pi$ , for any pair of antipodal blocks  $B$  and  $-B$ , either one lies entirely on the left of the other, or one is completely covered (or nested) by an arc of the other.*

*Proof.* When  $B$  is a pure block, the assertion is obvious. We now assume that  $B$  is a mixed block. Let  $i$  be the maximum positive element and  $-j$  be the minimum negative element of  $B$  according to the order (1.6). So  $(i, -j)$  is an arc in the linear representation of  $\pi$ . If  $j > i$ , it is easily seen that  $-B$  is nested by the arc  $(i, -j)$ . Similarly, in the case  $j < i$ ,  $B$  is nested by an arc of  $-B$ . This completes the proof. ■

In light of the above property, for a pair of antipodal blocks  $B$  and  $-B$ , we need only one of them to represent this pair. We shall choose  $B$  such that either it is on the left of  $-B$  or it is nested by an arc of  $-B$ . Moreover, we shall list the representative blocks  $B_1, B_2, \dots, B_k$  in increasing order of their minimum elements. In particular, we use  $B_0$  to denote the set of positive elements of the zero block. We shall call  $B_0/B_1/B_2/\dots/B_k$  the canonical representation of  $\pi$ . It is clear that the elements appearing in the canonical representation of a type  $B$  partition of  $[n]$  form a set in which either  $i$  or  $-i$  appears, but not both, for any  $i \in [n]$ . As an example, the representative blocks of the noncrossing partition  $\pi$  in Figure 1 can be read as  $B_0 = \{5\}, B_1 = \{3\}, B_2 = \{6, -2, -4\}, B_3 = \{7, -1\}$  and  $B_4 = \{8\}$ .

From now on, we shall use the above canonical representation  $\pi = B_0/B_1/\dots/B_k$  for a type  $B$  noncrossing partition. The elements appearing in the canonical representation, namely, the elements of  $B_0, B_1, \dots, B_k$ , will be classified into five types. Let  $i \in B_j$ . Then we say that

1.  $i$  is a *zero point* if  $i \in B_0$ ;
2.  $i$  is a *singleton* if  $B_j$  is a nonzero singleton block, that is,  $|B_j| = 1$ ;
3.  $i$  is a *transient point* if  $i$  is neither the smallest nor the largest element of  $B_j$ , according to the order (1.6);
4.  $i$  is a *departure point* if  $i$  is the smallest element of  $B_j$  and  $|B_j| > 1$ ;
5.  $i$  is a *destination point* if  $i$  is the largest element of  $B_j$  and  $|B_j| > 1$ .

For example, suppose that  $\pi$  is the partition in Figure 1. Then 5 is a zero point; 3 and 8 are singletons;  $-2$  is a transient point; 6 and 7 are departure points;  $-1$  and  $-4$  are destination points. Before proving Theorem 2.1, we first present a formula that will be used later.

**Proposition 2.3.** *The number of partitions in  $NC^B(n)$  with exactly  $k$  pairs of nonzero blocks but no singletons equals to*

$$\binom{n}{2k} \binom{2k}{k}.$$

*Proof.* From the correspondence by Reiner [17] between type  $B$  noncrossing partitions and pairs  $(L, R)$  of  $k$ -subsets of  $[n]$ , it is not hard to see that a point  $i$  is a singleton of  $\pi$  if and only if  $i$  appears in both  $L$  and  $R$ . Thus a type  $B$  noncrossing partition  $\pi$  without singletons is uniquely determined by a pair  $(L, R)$  of disjoint subsets of  $[n]$  with equal cardinality, namely  $L, R \subseteq [n]$ ,  $L \cap R = \emptyset$  and  $|L| = |R|$ . Moreover, the number of pairs of nonzero blocks equals the cardinality of  $|L|$  and  $|R|$ . Clearly, there are  $\binom{n}{2k} \binom{2k}{k}$  ways to choose  $(L, R)$ . This completes the proof. ■

To make this paper self-contained, we give a more detailed description of the procedure to generate a type  $B$  noncrossing partition  $\pi$  without singletons from a pair  $(L, R)$  of disjoint  $k$ -subsets of  $[n]$  by using the beautiful construction of Reiner [17]. First, put  $2n$  points on a horizontal line with labels  $1, 2, \dots, n, -1, -2, \dots, -n$  from left to right in accordance with the order (1.6). If  $l_i \in L$ , we replace the points  $l_i$  and  $-l_i$  each by a left parenthesis; if  $r_i \in R$ , replace the points  $r_i$  and  $-r_i$  each by a right parenthesis. Thus at the positions of the elements  $1, 2, \dots, n, -1, -2, \dots, -n$ , the pair  $(L, R)$  corresponds to a sequences of  $2k$  left parentheses,  $2k$  right parentheses and  $2n - 4k$  points.

It is now important to recall a property of the  $2k$  parentheses in the positions  $1, 2, \dots, n$ , as discovered by Greene and Kleitman [13] in the construction of the symmetric chain decomposition of the posets of subsets of a finite set. To be more precise, any sequence of left parentheses and right parentheses consists of well-parenthesized segments separated by parentheses which can be read from left to right as  $) \cdots ) ( \cdots ($ . This is to say that the unpaired right parentheses appear before the unpaired left parentheses. Moreover, there is no left or right parentheses between any well-paired parentheses.

Thus any sequence of parentheses can be decomposed into well-parenthesized segments, separated by a sequence of right parentheses followed by left parentheses. For example, the sequence  $)((()))((()$  has the following decomposition into well-parenthesized segments.

$$\begin{array}{ccccccc} ) & ( & ( & ) & ) & ( & ( & ) & ( \\ & \underbrace{\hspace{1.5cm}} & & & & & \underbrace{\hspace{1.5cm}} & & \end{array}$$

For the well-paired segments at the positive positions, or at the negative positions, we can easily construct pure blocks. For a pair of the form  $(\dots)$ , that is, a pair of parentheses for which there is no parentheses between them, we simply form a pure block by selecting the corresponding elements of the parentheses and the points between them. After such a block is formed, we may delete the underlying elements and continue the above procedure until all well-paired parentheses at the positive positions or at the negative positions are processed.

Upon the deletion of the elements of all pure blocks, the remaining parentheses have the following form

$$\underbrace{\dots)\dots)\dots)\dots(\dots(\dots(\dots| \dots)\dots)\dots)}_{+} \underbrace{\dots(\dots(\dots(\dots)}_{-}$$

The positive left parentheses and negative right parentheses can be well-paired, which will lead to mixed blocks. It can be shown that a mixed block  $B$  formed in the above procedure must be nested by its antipodal block. If  $(i, -j)$  is a paired parentheses which yields a mixed block  $B$ , then  $j$  is the largest positive element of  $-B$  and  $-i$  is the smallest negative element of  $-B$ . Clearly, the mixed block  $B$  is nested by the arc  $(j, -i)$  of  $-B$  since  $j < i$ . Moreover, one readily sees that the block  $B$  forms a consecutive segment with respect to the order (1.6) after removing the pure blocks, whereas the block  $-B$  occupies two consecutive segments at both ends. This implies that one can continue this process to get noncrossing blocks. When all the pure and mixed blocks are obtained, if there are still some elements left, we collect them together to form the zero block.

Conversely, given a type  $B$  noncrossing partition, the pair of subsets  $(L, R)$  can be easily determined by the absolute values of the departure points and the destination points. Thus we have obtained the desired bijection. An illustration of this correspondence is given in Figure 2.

*Combinatorial Proof of (2.1).* We assign the weight  $x$  to departure points and singletons and the weight 1 to other points (in the canonical representation). Thus the left-hand side of (2.1) equals the total weight of the set  $NC^B(n)$ .

To give a combinatorial interpretation of the right-hand side, let  $S_k$  denote the set of partitions in  $NC^B(n)$  with exactly  $k$  pairs of nonzero blocks but no singletons, and let  $T_k$  denote the set of partitions in  $NC^B(n)$  with exactly  $k$  pairs of nonzero blocks which

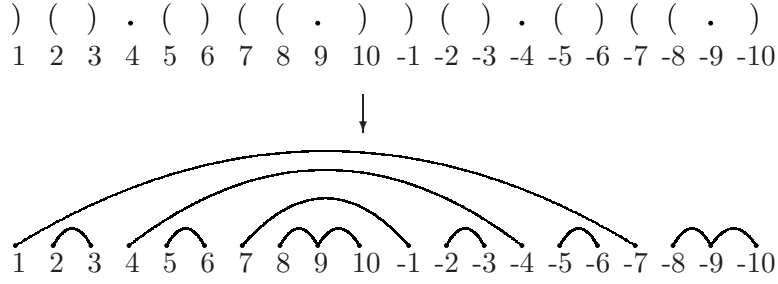


Figure 2: The correspondence between a pair  $(L, R)$  and a noncrossing partition.

have at least two elements. Clearly, every nonzero block of a partition in  $S_k$  contains at least two elements. By Proposition 2.3, the total weight of  $S_k$  equals  $\binom{n}{2k} \binom{2k}{k} x^k$ . From the above description of the construction for Proposition 2.3, it is clearly seen that a zero point (in the canonical representation) can not be covered by an arc of a nonzero block, or more precisely, a zero point cannot appear between a pair of departure point and destination point. Furthermore, a partition in  $T_k$  can be obtained by changing some zero points and transient points to singletons. Conversely, given a partition in  $T_k$ , there is only one way to change it back to a partition in  $S_k$  by switching every singleton to either a zero point or a transient point. This implies that the total weight of  $T_k$  equals the total weight of  $S_k$  multiplied by the factor  $(1+x)^{n-2k}$ . This completes the proof.  $\blacksquare$

Setting  $x = 1$ , the identity (2.1) takes the form

$$\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \binom{2k}{k} 2^{n-2k} = \binom{2n}{n}, \quad (2.3)$$

which can be regarded as a type  $B$ -analogue of Touchard's formula

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{2k} C_k 2^{n-1-2k} = C_n.$$

Simion [20] has given a combinatorial interpretation of (2.3) by means of a symmetric boolean decomposition of the lattice  $NC^B(n)$  with the refinement order. Our combinatorial interpretation of (2.2) will be based on the set  $V_n$  of colored type  $B$  noncrossing partitions, which consists of all type  $B$  noncrossing partitions of  $[n]$  in which a pair of antipodal singletons may be colored with two colors, say, black and white.

*Combinatorial Proof of (2.2).* We assign the weight  $x^2$  to departure points and singletons, and assign the weight  $(1+x)^2$  to zero points, transient points, and destination points. In this way, the left-hand side of (2.2) equals the total weight of the set  $NC^B(n)$ .

As far as the right-hand side is concerned, we need to classify the set  $NC^B(n)$  as follows. Given a partition  $\pi \in NC^B(n)$ , let  $L_\pi$  and  $R_\pi$  denote the sets of departure points and destination points respectively in the canonical representation of  $\pi$ . Two partitions  $\pi$  and  $\sigma$  are in the same class if  $(L_\pi, R_\pi) = (L_\sigma, R_\sigma)$ . Suppose that  $(L, R)$  is a pair of feasible sets of departure points and destination points, namely, there exists  $\pi$  such that  $(L_\pi, R_\pi) = (L, R)$ . Let  $G(L, R)$  be the set of partitions  $\pi \in NC^B(n)$  such that  $(L_\pi, R_\pi) = (L, R)$ . We further assume that both  $L$  and  $R$  contain  $k$  elements. Since the departure points and destination points always appear in pairs and  $x^2(1+x)^2 = (x(1+x))^2$ , the above weight assignment is equivalent to the effect of assigning the weight  $x(1+x)$  to both departure points and destination points. By the same argument in the proof (2.1), the noncrossing property implies that the other points can be either singletons or non-singletons (transient points or zero points), we deduce that the total weight of  $G(L, R)$  equals

$$(x(1+x))^{2k}(x^2 + (1+x)^2)^{n-2k}. \quad (2.4)$$

We next proceed to give an alternative interpretation of the total weight (2.4) in terms of colored type  $B$  noncrossing partitions in  $V_n$ . Assign the weight  $x(1+x)$  to black singletons, zero points, transient points, departure points, destination points and the weight 1 to white singletons. Let us define  $H(L, R)$  for  $V_n$  in the same way as we have defined  $G(L, R)$ , that is, the set of colored type  $B$  noncrossing partitions  $\pi$  such that  $(L_\pi, R_\pi) = (L, R)$  and  $|L| = |R| = k$ . By the same argument we see that the total weight of  $H(L, R)$  equals

$$(x(1+x))^{2k}(1 + x(1+x) + x(1+x))^{n-2k}, \quad (2.5)$$

which apparently coincides with (2.4). This implies that the right-hand side of (2.2) can be reinterpreted in terms of partitions in  $V_n$ . It is necessary to show that the total weight of  $V_n$  equals the right-hand side of (2.2). To construct a partition of  $V_n$  with  $n - k$  white singletons, we may first choose  $n - k$  white singletons from  $[n]$  in  $\binom{n}{k}$  ways. Observe that white singletons have weight 1 and the remaining  $2k$  elements can form a type  $B$  noncrossing partition with each element (in the canonical representation) having weight  $x(1+x)$ . Clearly, there are  $\binom{2k}{k}$  choices of such partitions and the weight of each equals  $x^k(1+x)^k$ . This completes the proof. ■

When  $x = -1$ , the identity (2.1) becomes

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^2 = \begin{cases} (-1)^r \binom{2r}{r}, & \text{if } n = 2r; \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

We shall provide an involution for the identity (2.6). It should be noted that (2.6) is a type  $B$  analogue of the identity on the alternating sum of Narayana numbers,

$$\sum_{k=0}^n (-1)^k \frac{1}{n} \binom{n}{k-1} \binom{n}{k} = \begin{cases} (-1)^{r+1} C_r, & \text{if } n = 2r + 1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.7)$$



The above identity (2.7) was first discovered by Bonin, Shapiro and Simion [4] in their study of Schröder paths. It also has been studied by Coker [10], Klazar [14], Eu, Liu and Yeh [11]. A combinatorial proof of (2.7) has been given by Chen, Shapiro and Yang [8] by using plane trees and 2-Motzkin paths.

In the case of  $x = -\frac{1}{2}$ , (2.2) reduces to

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 4^{n-k} = \binom{2n}{n}. \quad (2.8)$$

This identity can be found in Riordan [18] and it was derived by means of generating functions.

We now give a combinatorial interpretation of (2.6). Let  $NC_e^B(n)$  (resp.  $NC_o^B(n)$ ) denote the number of type  $B$  noncrossing partitions of  $[n]$  into even (resp. odd) pairs of nonzero blocks. We give a parity reversing involution on type  $B$  noncrossing partitions which implies the following formulation of (2.6):

$$NC_e^B(n) - NC_o^B(n) = \begin{cases} (-1)^r \binom{2r}{r}, & \text{if } n = 2r; \\ 0, & \text{otherwise.} \end{cases} \quad (2.9)$$

Let  $A_n$  denote the set of type  $B$  noncrossing partitions of  $[n]$  without the zero block such that every nonzero block contains exactly two elements. Notice that  $A_n$  is empty if  $n$  is odd and

$$|A_{2n}| = \binom{2n}{n}.$$

Define the parity of a type  $B$  noncrossing partition as the parity of the number of pairs of nonzero blocks. Moreover, we define the sign of a partition as  $-1$  if it is odd, and as  $1$  if it is even.

**Theorem 2.4.** *There is a parity reversing involution  $\rho$  on the set  $NC^B(n) \setminus A_n$ .*

*Proof.* Let  $\pi = B_0/B_1/\cdots/B_k \in NC^B(n) \setminus A_n$  be the canonical representation of  $\pi$ . We first list the elements in  $B_0, B_1, \dots, B_k$  in the increasing order of their absolute values. Then we define the critical point  $i$  of  $\pi$  as the first element in the above order that is neither a departure point nor a destination point, or equivalently, is either a zero point, or a transient point, or a singleton. We now conduct the following map  $\rho$  based on two cases concerning the critical point:

If the critical point  $i$  is a zero point or a transient point, we take the element  $i$  out of the block and form a singleton block  $\{i\}$ .

If the critical point  $i$  is a singleton, we need to determine whether  $i$  should be put into the zero block or a nonzero block. We may pay attention to the arc between  $i$  and  $-i$  in the linear representation of  $\pi$ . If this arc does not cross any nonzero block, then

we put  $i$  into the zero block. Otherwise, put  $i$  into the nonzero block which has an arc covering  $i$  and there are no arcs of other blocks covered by this arc.

It is not hard to see that the above map changes the number of blocks by one. Moreover, the critical point remains unchanged under the above map. Thus we can infer that  $\rho$  is a parity reversing involution.  $\blacksquare$

Figure 3 gives an example of the involution  $\rho$ , where the italic 1 is the critical point.

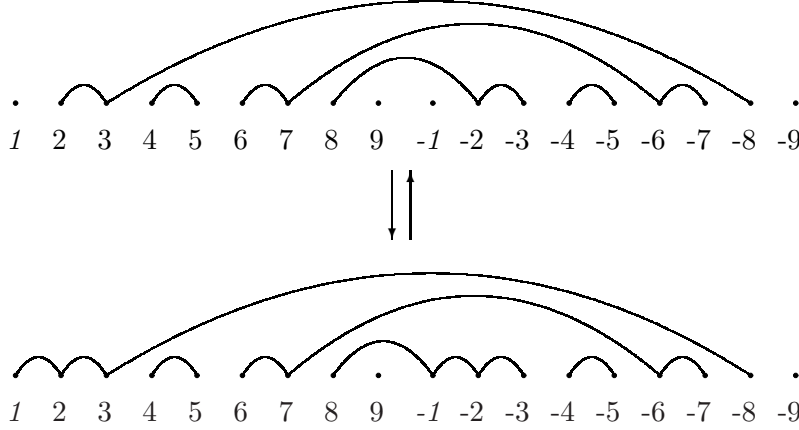


Figure 3: Involution  $\rho$  on a type  $B$  noncrossing partition.

Since  $A_{2n+1}$  is empty and any partition in  $A_{2n}$  has  $n$  pairs of nonzero blocks, the involution  $\rho$  leads to a combinatorial proof of (2.9).

Appealing to the correspondence between plane trees and 2-Motzkin paths, Chen, Deutsch and Elizalde [6] obtained the following refinements of Coker's identities:

$$\sum_{i=1}^n \sum_{j=0}^{n-2i+1} \frac{1}{n} \binom{n}{i} \binom{n-1}{j} \binom{n-i-j}{i-1} x^{i-1} y^j = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1+y)^{n-2k-1}, \quad (2.10)$$

$$\sum_{i=1}^n \sum_{j=0}^{n-2i+1} \frac{1}{n} \binom{n}{i} \binom{n-1}{j} \binom{n-i-j}{i-1} x^{2(i-1)} y^j z^{n-2i-j+1} = \sum_{k=0}^{n-1} C_{k+1} \binom{n-1}{k} x^k (y+z-2x)^{n-1-k}. \quad (2.11)$$

The following refinements of (2.1) and (2.2) can be treated as type  $B$  analogues of (2.10) and (2.11).

**Theorem 2.5.** *For  $n \geq 1$ , we have*

$$\sum_{i=0}^n \sum_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \binom{n}{i} \binom{n-i}{j} \binom{n-j-i}{j} x^j y^i = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \binom{2k}{k} x^k (1+y)^{n-2k}, \quad (2.12)$$

$$\sum_{i=0}^n \sum_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \binom{n}{i} \binom{n-i}{j} \binom{n-j-i}{j} x^{2j} y^i z^{n-2j-i} = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} x^k (y+z-2x)^{n-k}. \quad (2.13)$$

*Proof.* We first consider (2.12) and shall also use weighted type  $B$  noncrossing partitions to give the combinatorial interpretations of both sides of (2.12). The weight assignment is almost the same as in the proof of identity (2.1) except that a singleton is endowed with the weight  $y$ . Suppose that  $\pi$  is a type  $B$  noncrossing partition with  $i$  singletons (in the canonical representation, to be precise). The remaining  $n-i$  elements form a type  $B$  noncrossing partition  $\sigma$  such that each nonzero block contains at least two elements (in the canonical representation as well). Assume that  $\sigma$  contains  $j$  nonzero blocks (in the canonical representation). By Proposition 2.3, there are  $\binom{n-i}{j} \binom{n-j-i}{j} = \binom{n-i}{2j} \binom{2j}{j}$  choices for  $\sigma$ . According to the weight assignment,  $\sigma$  has weight  $x^j$ . In view of the number of singletons, we see that the left-hand side of (2.12) equals the sum of the weights of all type  $B$  noncrossing partitions on  $[n]$ .

Regarding the right-hand side, let  $S_k$  be the set of the partitions in  $NC^B(n)$  with  $k$  pairs of nonzero blocks but no singletons, and let  $T_k$  be the set of partitions in  $NC^B(n)$  with exactly  $k$  pairs of nonzero and nonsingleton blocks. The total weight of any partition in  $S_k$  is  $x^k$ . Meanwhile, there are  $\binom{n}{2k} \binom{2k}{k}$  partitions in  $S_k$ . Using the same argument as in the proof of (2.1), we see that the total weight of  $T_k$  equals the total weight of  $S_k$  multiplied by the factor  $(1+y)^{n-2k}$ . Thus the right-hand side of (2.12) also equals the total weight of  $NC^B(n)$ . Hence (2.12) is proved.

We now turn to the proof of (2.13) using a different weight assignment. Assign the weight  $x$  to departure points and destination points, the weight  $y$  to singletons and the weight  $z$  to zero points, transient points. Suppose that  $\pi$  is a partition with  $i$  singletons and  $j$  nonzero blocks with at least two elements. Since the departure points and destination points appear in pairs, there are  $n-2j-i$  other elements with weight  $z$ . From Proposition 2.3 it follows that the left-hand side of (2.13) equals the total weight of the set  $NC^B(n)$ .

On the other hand, we may consider the set  $V_n$  of colored type  $B$  noncrossing partitions, as defined before. Assign the weight  $x$  to black singletons, zero points, transient points, departure points, destination points and the weight  $y+z-2x$  to white singletons. Let  $G(L, R)$  and  $H(L, R)$  denote the subsets of  $NC^B(n)$  and  $V_n$  respectively as in the proof of the identity (2.2). According to the weight assignment, the departure points and destination points have weight  $x$ , singletons have weight  $y$  and the other points have weight  $z$ , thus the total weight of the set  $G(L, R)$  is  $x^{2k}(y+z)^{n-2k}$ . Similarly, the total weight of  $H(L, R)$  equals  $x^{2k}(x+x+y+z-2x)^{n-2k}$ , which coincides with the total weight of  $G(L, R)$ . Now it suffices to show that the total weight of  $V_n$  agrees with the right-hand side of (2.2). In order to compute the total weight of  $V_n$ , we can first choose  $n-k$  white singletons from  $[n]$  in  $\binom{n}{k}$  ways. These white singletons have total weight  $(y+z-2x)^{n-k}$ . The remaining  $2k$  elements form a type  $B$  noncrossing

partition with each kind of points having weight  $x$ . There are  $\binom{2k}{k}$  choices of such partitions and the weight of each partition equals  $x^k$ . This completes the proof. ■

### 3 Type $B$ Analogues of the Identities of Mansour and Sun

In this section, we provide type  $B$  analogues along with combinatorial proofs of the identities (1.3), (1.4) and (1.5) due to Mansour and Sun [16]. We begin with the following identity which can be considered as a type  $B$  analogue of (1.3).

**Theorem 3.1.** *For  $n \geq 0$ , we have*

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} P_k(x) (1+x)^{n-k} = \begin{cases} x^r \binom{2r}{r}, & \text{if } n = 2r; \\ 0, & \text{otherwise.} \end{cases} \quad (3.1)$$

Setting  $x = 1$ , (3.1) reduces to the following identity of Dawson, see, Riordan [18, p.71],

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{2k}{k} 2^{n-k} = \begin{cases} \binom{2r}{r}, & \text{if } n = 2r; \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

Andrews [1] has given a proof of (3.2) by employing Gauss's second summation theorem, which is stated as

$${}_2F_1 \left[ \begin{matrix} a, b \\ 1/2 + a/2 + b/2 \end{matrix} ; 1/2 \right] = \frac{\Gamma(1/2)\Gamma(1/2 + a/2 + b/2)}{\Gamma(1/2 + a/2)\Gamma(1/2 + b/2)}.$$

*Combinatorial Proof of Theorem 3.1.* The left-hand side of (3.1) can be interpreted in terms of weighted type  $B$  colored partitions in  $V_n$ . We use the following weight assignment. Transient points, zero points and destination points are given the weight 1; black singletons and departure points are given the weight  $x$ ; white singletons are given the weight  $-1 - x$ , which is equivalent to the assignment of either  $-x$  or  $-1$ .

To construct a partition in  $V_n$ , we first choose a subset  $S$  of  $n - k$  elements from  $[n]$  to form white singletons with weight  $-1 - x$ . There are  $\binom{n}{k}$  ways to choose  $S$ , and these white singletons will contribute a factor  $(-1 - x)^{n-k}$  to the weight. On the other hand, the remaining elements constitute a noncrossing partition on  $2k$  elements with the same weight assignment except that the singletons are assumed to be black. It follows that such partitions have a total weight  $P_k(x)$ . Thus the left-hand side of (3.1) equals the total weight of the set  $V_n$ .

We now aim to construct a sign reversing involution  $\theta$  on  $V_n \setminus A_n$ , where  $A_n$  is defined in the preceding section. By the definition of  $A_n$ , we may regard the partitions

in  $A_n$  as noncrossing partitions in  $V_n$  with no zero block, no singletons and no transient points. The involution  $\theta$  can be described as follows.

Given a type  $B$  colored noncrossing partition  $\pi \in V_n \setminus A_n$ , in the increasing order of the absolute values, we search the first point  $i$  in the canonical representation of  $\pi$  that is neither a departure point nor a destination point of  $\pi$ . Then we conduct the following operations:

- If  $i$  is a black singleton, then we change it to a white singleton with weight  $-x$ ;
- If  $i$  is a transient point or a zero point, then we change it to a white singleton with weight  $-1$ ;
- If  $i$  is a white singleton with weight  $-x$ , then we change it to a black singleton;
- If  $i$  is a white singleton with weight  $-1$ , then we change it to a transient point or a zero point by the same criterion in the proof of Theorem 2.4.

Evidently,  $\theta$  is weight preserving and sign reversing. It is also easily seen the critical point after the map is neither a departure point nor a destination point. Moreover, in the increasing order of the absolutes, the elements before  $i$  stay unchanged since the above operations do not cause additional departure point or destination point. Therefore, the critical point remains the same. Thus  $\theta$  is a sign reversing and weight preserving involution.

Our final task is to compute the total weight of  $A_n$ . First consider the case when  $n$  is odd. For a type  $B$  noncrossing partition  $\pi$  on  $[n]$ , it is clear that there exists at least one point which is neither a departure point nor a destination point. By the involution  $\theta$  we deduce that the sum of weights of type  $B$  colored noncrossing partitions on  $[n]$  equals zero. For  $n = 2r$ , the total weight of  $V_n$  equals to the total weight of  $A_n$ , which equals  $x^r \binom{2r}{r}$ . This completes the proof.  $\blacksquare$

Figure 4 is an illustration of the involution  $\theta$ , where a circle stands for a white singleton.

We now turn to a type  $B$  analogue of the identity (1.4).

**Theorem 3.2.** *For  $n \geq 0$ , we have*

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} P_k(x^2) (1-x)^{2(n-k)} = x^n \binom{2n}{n}. \quad (3.3)$$

It should be noted that setting  $x = -1$  in (3.3), we arrive at (2.8) again.

*Combinatorial Proof of (3.3).* The proof is similar to that of (3.1). We still consider type  $B$  colored noncrossing partitions with a different weight assignment. Destination

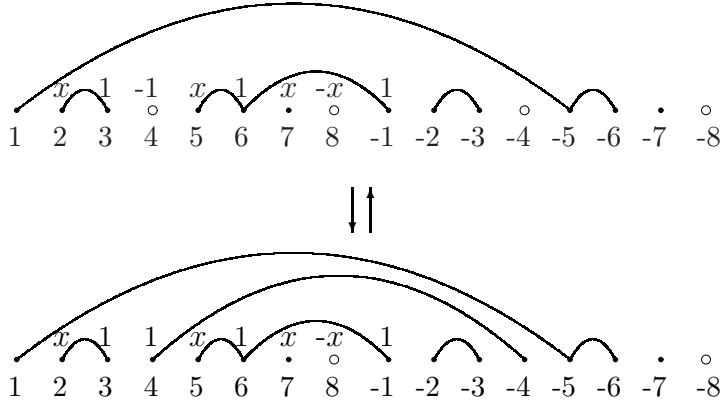


Figure 4: The involution  $\theta$  on a type  $B$  colored noncrossing partition.

points, zero points and transient points are given the weight 1; departure points and black singletons are given the weight  $x^2$ ; white singletons are given the weight  $-1 + 2x - x^2$ , or, equivalently, the weight of a white singleton can be either  $-1$ , or  $2x$  or  $-x^2$ . Then the left-hand side of (3.3) can be interpreted as the total weight of the set  $V_n$ .

Let  $D_n$  be the set of colored noncrossing partitions in  $V_n$  that have only two types of points (in the canonical representation): (1) departure points or destination points. (2) white singletons with weight  $2x$ . In other words, in any partition in  $D_n$ , the following four types of points are not allowed: (1) zero points; (2) transient points; (3) black singletons; (4) white singletons with weight  $-1$  or  $-x^2$ . To prove (3.3), we proceed to construct a sign reversing and weight preserving involution  $\eta$  on the set  $V_n \setminus D_n$ .

Given a type  $B$  colored noncrossing partition  $\pi \in V_n \setminus D_n$ , we seek the first point  $i$  in the increasing order of the absolute values which is neither a departure point nor a destination point. As usual,  $i$  is called the critical point. The map  $\eta$  is defined by the following operations:

- If  $i$  is a black singleton, then set  $i$  to be a white singleton with weight  $-x^2$ ;
- If  $i$  is a transient point or a zero point, then set  $i$  to be a white singleton with weight  $-1$ ;
- If  $i$  is a white singleton with weight  $-x^2$ , then set  $i$  to be a black singleton;
- If  $i$  is a white singleton with weight  $-1$ , then set  $i$  to be a transient point or a zero point according to the criterion as given before.

It can be verified that  $\eta$  is a sign reversing and weight preserving involution. Thus the total weight of  $V_n$  equals the total weight of  $D_n$ .

We now are left the task to show that the total weight of the set  $D_n$  equals the right-hand side of (3.3), namely,  $x^n \binom{2n}{n}$ . To construct a weighted type  $B$  colored

noncrossing partition in  $D_n$ , we can first choose  $2k$  elements from  $[n]$  to construct a type  $B$  noncrossing partition with  $k$  departure points and  $k$  destination points. There are  $\binom{n}{2k}\binom{2k}{k}$  choices. The remaining  $n - 2k$  points are taken to be white singletons with weight  $2x$ . Thus the total weight of the set  $D_n$  equals

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} x^{2k} (2x)^{n-2k},$$

which can be rewritten as

$$x^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} 2^{n-2k}.$$

Invoking the identity (2.3) gives (3.3), so the proof is complete.  $\blacksquare$

We remark in passing that (1.5) is related to the following identity obtained by Chen and Pang [7]

$$\sum_{k=0}^n \frac{1}{n} \binom{n}{k-1} \binom{n}{k} x^k (x+1)^{n-k} = \sum_{k=0}^n (-1)^{n-k} \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} (x+1)^k, \quad (3.4)$$

which was independently derived by Mansour and Sun [15] in a slightly different form

$$\sum_{k=0}^n \frac{1}{n} \binom{n}{k-1} \binom{n}{k} x^k = \sum_{k=0}^n \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} (x-1)^{n-k}. \quad (3.5)$$

Upon substituting  $x$  with  $x/(1-x)$ , we see that (3.4) can be recast in the form of (3.5), that is,

$$\frac{N_n(x)}{(x-1)^n} = \sum_{k=0}^n \binom{n+k}{n-k} C_k \frac{1}{(x-1)^k}. \quad (3.6)$$

Then the identity (1.5) follows from the Legendre inversion formula [18]

$$a_n = \sum_{k=0}^n \binom{n+k}{n-k} b_k \iff b_n = \sum_{k=0}^n (-1)^{n-k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} a_k.$$

Below is a type  $B$  analogue of (3.4), which yields a type  $B$  analogue of (1.5).

**Theorem 3.3.** *For  $n \geq 0$ , we have*

$$\sum_{k=0}^n \binom{n}{k}^2 x^k (x-1)^{n-k} = \sum_{k=0}^n (-1)^{n-k} \binom{n+k}{k} \binom{n}{k} x^k. \quad (3.7)$$

*Proof.* The weight assignment of the points of a type  $B$  noncrossing partition on  $[n]$  is given as follows. A departure point or a singleton is endowed with the weight  $x$ ,

whereas the other kinds of points are given the weight  $x - 1$ . In this way, the left-hand side of (3.7) equals the total weight of the set  $NC^B(n)$ .

We need an equivalent weight assignment, that is, departure points and singletons always have weight  $x$  and the weight  $x - 1$  for other types of points can be interpreted as the assignment of either  $x$  or  $-1$ . Using this weight assignment, we proceed to show that the summand of the right-hand side equals the sum of weights of type  $B$  noncrossing partitions on  $[n]$  with exactly  $k$  elements having weight  $x$ . Such partitions can be constructed as follows. We first choose a  $k$ -subset  $S$  of  $[n]$  under the condition that if an element  $i$  or  $-i$  has weight  $x$ , then  $i \in S$ . Then we choose  $m$  elements from  $S$  to be departure points or singletons, denoted by  $L = \{l_1, l_2, \dots, l_m\}$ . Meanwhile, we choose another set of  $m$  elements from the set  $[n]$ , denoted by  $R = \{r_1, r_2, \dots, r_m\}$ . Applying the bijection of Reiner [17] to the pair  $(L, R)$ , we get a noncrossing partition with  $m$  pairs of antipodal nonzero blocks.

It remains to compute the sum of weights of such partitions. Note that the departure points and singletons always have weight  $x$  and consequently belong to  $S$ . It suffices to determine which of the other types of points can be given weight  $x$ . The weight of the destination points, zero points and transient points can be either  $x$  or  $-1$  subject to the condition for the choice of  $S$ . If the absolute value of such an element belongs to the set  $S$ , then it has weight  $x$ ; otherwise, it has weight  $-1$ . That is to say that subject to the condition on the choice of  $S$ , the weight assignment for all the elements are uniquely determined. So the weight of a partition constructed by the above procedure equals  $(-1)^{n-k}x^k$ . On the other hand, the number of such partitions equals

$$\binom{n}{k} \sum_{m=0}^k \binom{k}{m} \binom{n}{m} = \binom{n}{k} \binom{n+k}{k}.$$

This completes the proof. ■

In virtue of the identity (3.7), we can express the central binomial coefficients in terms of the Narayana polynomials of type  $B$ .

**Theorem 3.4.** *For  $n \geq 0$ ,*

$$\binom{2n}{n} = \sum_{k=0}^n \frac{2k+1}{2n+1} \binom{2n+1}{n-k} P_k(x) (1-x)^{n-k}. \quad (3.8)$$

*Proof.* Substituting  $x$  with  $\frac{1}{1-x}$  into (3.7), and observing that

$$\binom{n+k}{k} \binom{n}{k} = \binom{n+k}{n-k} \binom{2k}{k},$$

we find

$$\sum_{k=0}^n \binom{n}{k}^2 x^k = \sum_{k=0}^n (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} (1-x)^{n-k}, \quad (3.9)$$



which serves as a type  $B$  analogue of (3.6). Rewriting (3.9) as

$$\frac{P_n(x)}{(x-1)^n} = \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} \frac{1}{(x-1)^k},$$

applying the Legendre inversion formula, we arrive at the expression (3.8). ■

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