Involutions for Rogers-Ramanujan-Gordon Type Identities with Parity Restrictions

William Y.C. Chen[1], Doris D. M. Sang[2], and Diane Y. H. Shi[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

[1]chen@nankai.edu.cn, [2]sdm@cfc.nankai.edu.cn, [3]yahuishi@gmail.com

Abstract. We find involutions for three Rogers-Ramanujan-Gordon type identities obtained by Andrews on the generating functions for partitions with part difference and parity restrictions.

1   Introduction

This paper is concerned with combinatorial interpretations of three Rogers-Ramanujan-Gordon type identities obtained by Andrews on the generating functions of partitions with parity restrictions. Recall that Gordon [5] found a combinatorial generalization of the Rogers-Ramanujan identities [3, 6], which has been called the Rogers-Ramanujan-Gordon identity, see Andrews [1].

Theorem 1.1 *Let $B_{k,a}(n)$ denote the number of partitions of n for the form $b_1 + b_2 + \cdots + b_j$, where $b_i \geq b_i + 1$, $b_i - b_{i+k-1} \geq 2$ and at most $a - 1$ of the b_i are equal to 1 and $1 \leq a \leq k$. Let $A_{k,a}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm a \pmod{2k+1}$. Then for all $n \geq 0$,*

$$A_{k,a}(n) = B_{k,a}(n). \tag{1.1}$$

It is easy to derive the generating function for $A_{k,a}(n)$,

$$\sum_{n=0}^{\infty} A_{k,a}(n)q^n = \frac{(q^{2k+1}; q^{2k+1})_\infty (q^a; q^{2k+1})_\infty (q^{2k+1-a}; q^{2k+1})_\infty}{(q; q)_\infty}. \tag{1.2}$$

In order to prove (1.1), Gordon has shown that $B_{k,a}$ has the same generating function as $A_{k,a}$, that is,

$$\sum_{n=0}^{\infty} B_{k,a}(n)q^n = \frac{(q^{2k+1}; q^{2k+1})_\infty (q^a; q^{2k+1})_\infty (q^{2k+1-a}; q^{2k+1})_\infty}{(q; q)_\infty}. \tag{1.3}$$

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In fact, Gordon established the above identity by giving an involution for the following reformulation of (1.3) by multiplying both sides by $(q;q)_\infty$ and applying the Jacobi triple product identity to the right hand side,

$$(q;q)_\infty \sum_{n=0}^{\infty} B_{k,a}(n)q^n = \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+\frac{1}{2})n^2 + (k-a+\frac{1}{2})n}. \qquad (1.4)$$

We shall give a description of Gordon's involution in Section 2. Andrews [2] found a generating function proof of the Rogers-Ramanujan-Gordon identity (1.3) in the following equivalent form:

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_a+N_{a+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} = \frac{(q^{2k+1};q^{2k+1})_\infty(q^a;q^{2k+1})_\infty(q^{2k+1-a};q^{2k+1})_\infty}{(q;q)_\infty},$$

where

$$N_j = n_j + n_{j+1} + \cdots + n_{k-1},$$

and

$$(a;q)_n = (a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Andrews [4] further considered the parity restrictions on partitions in connection with the Rogers-Ramanujan-Gordon identity. He has derived three identities in the spirit of the relation (1.3), which we call Rogers-Ramanujan-Gordon type identities. The three identities depend on the parities of k and a.

Theorem 1.2 *Suppose $k \geq a \geq 1$ are integers such that k and a are both even. Let $W_{k,a}(n)$ denote the number of partitions enumerated by $B_{k,a}(n)$ with further restriction that even parts appear an even number of times. Then we have*

$$\sum_{n\geq 0} W_{k,a}(n)q^n = \frac{(-q;q^2)_\infty(q^a;q^{2k+2})_\infty(q^{2k+2-a};q^{2k+2})_\infty(q^{2k+2};q^{2k+2})_\infty}{(q^2;q^2)_\infty}. \qquad (1.5)$$

Theorem 1.3 *Suppose $k \geq a \geq 1$ are integers such that k and a are both odd. Let $W_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with further restriction that even parts appear an even number of times. Then for all $n \geq 0$, we have*

$$\sum_{n\geq 0} W_{k,a}(n)q^n = \frac{(q^2;q^4)_\infty(q^a;q^{2k+2})_\infty(q^{2k+2-a};q^{2k+2})_\infty(q^{2k+2};q^{2k+2})_\infty}{(q;q)_\infty}. \qquad (1.6)$$

Theorem 1.4 *Suppose $k \geq a \geq 1$ and k odd and a even. Let $\overline{W}_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with further restriction that odd parts appear an even number of times. Then for all $n \geq 0$, we have*

$$\sum_{n\geq 0} \overline{W}_{k,a}(n)q^n = \frac{(q^a;q^{2k+2})_\infty(q^{2k+2-a};q^{2k+2})_\infty(q^{2k+2};q^{2k+2})_\infty}{(-q;q^2)_\infty(q;q)_\infty}. \qquad (1.7)$$

Andrews [4] gave algebraic proofs of the above three identities. In this paper, we shall give three involutions for Andrews' identities. Gordon's involution will play a role in our involutions for the purpose of partial cancelations.

2 Gordon's involution

In this section we give an overview of Gordon's involution for the Rogers-Ramanujan-Gordon identity. Recall that $B_{k,a}(N)$ denotes the number of partitions of N for the form $b_1+b_2+\cdots+b_m$, where $b_i \geq b_i + 1$, $b_i - b_{i+k-1} \geq 2$ and at most $a-1$ of the b_i are equal to 1 and $1 \leq a \leq k$. Gordon's involution is concerned with the following relation

$$(q;q)_\infty \sum_{N=0}^{\infty} B_{k,a}(N)q^N = \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+1/2)n^2+(k-a+1/2)n}. \tag{2.8}$$

Let us consider the set $\mathcal{P}_{k,a}$ of pairs of partitions $(A|B)$, where A is a signed partition with distinct parts and B is a partition enumerated by $B_{k,a}$. To be precise, each part in the partition A carries a minis sign, and B is a partition of the form $N = b_1 + b_2 + \cdots + b_m$, where $b_i \geq b_{i+1}$, $b_i \geq b_{i+k-1}+2$ and $b_{m-a+1} \geq 2$. Clearly, the set $\mathcal{P}_{k,a}$ corresponds to the left hand side of (2.8). Gordon's involution is based on the set $\mathcal{P}_{k,a}$. As will be shown, the generating function of the fixed points of Gordon's involution equals

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(k+\frac{1}{2})n^2+(k-a+\frac{1}{2})n},$$

which is the right hand side of (2.8).

Gordon's involution consists of the following steps.

Step 1. We compare the largest parts of A and B, if $a_1 > b_1$ then set

$$(A'|B') = (b_1, a_1, \ldots, a_l|b_2, \ldots, b_m).$$

In this case, the resulting pair $(A'|B')$ is still in $\mathcal{P}_{k,a}$. In the case $a_1 \leq b_1$, if there exists $1 \leq i \leq k$ such that

$$a_1 = b_1 = \cdots = b_{i-1} = b_i + 1 = \cdots = b_{k-1} + 1, \tag{2.9}$$

then go to Step 2. Otherwise, let

$$(A'|B') = (a_2, \ldots, a_l|a_1, b_1, b_2, \ldots, b_m).$$

It can be checked that the resulting pair $(A'|B')$ is in $P_{k,a}$

Step 2. Let U_i denote the set of partition pairs $(A|B)$ for which the condition (2.9) holds, and let U denote the union of U_i. We shall further classify the sets U_i. Let us define the three numbers p, q and r as follows:

(1) $p = a_l$.

(2) q is the largest integer such that

$$a_1 - a_2 = a_2 - a_3 = \cdots = a_{q-1} - a_q = 1.$$

(3) r is the largest integer such that

$$b_{k-1} - b_{2(k-1)} = b_{2(k-1)} - b_{3(k-1)} = \cdots = b_{(r-1)(k-1)} - b_{r(k-1)} = 2.$$

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For $U_2, \ldots, U_{k-1}$, there is a fourth parameter number s, which is defined to be the largest integer such that

$$b_{i-1} - b_{i-1+(k-1)} = b_{i-1+(k-1)} - b_{i-1+2(k-1)} = \cdots = b_{i-1+(s-2)(k-1)} - b_{i-1+(s-1)(k-1)} = 2.$$

Now we divide each of U_1, U_k into three classes, and each of $U_2, \ldots, U_{k-1}$ into four classes. For U_1, U_k, we consider the minimal of the three numbers p, q and r, and let $n = min\{p, q, r\}$. For $2 \leq i \leq k - 1$, we consider the minimal of the four numbers p, q, r and s, and let $n = min\{p, q, r, s\}$. We now divide the sets U_i into classes according to the conditions on the parameters p, q, r and s:

$$U_1^1 : \quad p = n, \ q \geq n, r \geq n,$$
$$U_1^2 : \quad p > n, \ q = n, \ r \geq n,$$
$$U_1^3 : \quad p > n, \ q > n, \ r = n,$$
$$U_k^1 : \quad p = n, \ q \geq n, \ r \geq n,$$
$$U_k^2 : \quad p > n, \ q \geq n, \ r = n,$$
$$U_k^4 : \quad p > n, \ q = n, \ r > n,$$
$$U_i^1 : \quad p = n, \ q \geq n, \ r \geq n,$$
$$U_i^2 : \quad p > n, \ q \geq n, \ r \geq n, s = n$$
$$U_i^3 : \quad p > n, \ q > n, \ r = n, \ s > n,$$
$$U_i^4 : \quad p > n, \ q = n, \ r \geq n, \ s > n.$$

Step 3. We now define three maps α, β, γ and their inverses. Let

$$\pi = (A|B) = (a_1, \ldots, a_l | b_1, \ldots, b_m) \in U_1^2,$$

define
$$\alpha(\pi) = (a_1 - 1, a_2 - 1, \ldots, a_n - 1, a_{n+1}, \ldots, a_l, n | b_1, b_2, \ldots, b_m),$$

It is clear that $\alpha(\pi)$ belongs U_k^1, unless $l = n$, $a_n - 1 = n$, and π of the following form

$$(2n, 2n - 1, \ldots, n + 1 | (2n - 1)^{k-1} \ (2n - 3)^{k-1} \ldots \ 1^{k-1}).$$

In this case, the weight of π equals $(k + \frac{1}{2})n^2 + \frac{1}{2}n$.

Conversely, if $\pi \in U_k^1$, we have

$$\alpha^{-1}(\pi) = (a_1 + 1, \ldots, a_n + 1, a_{n+1}, \ldots, a_{l-1} | b_1, \ldots, b_m),$$

which belongs U_1^2, unless $l = n$. For the case $l = n$ we have

$$\pi = (2n - 1, 2n - 2, \ldots, n | (2n - 1)^{k-1} \ (2n - 3)^{k-1} \ldots \ 1^{k-1}),$$

which has weight $(k + \frac{1}{2})n^2 - \frac{1}{2}$. In this case, it can be seen that the part 1 appears $k - 1$ times. On the other hand, by the condition in the theorem, the element 1 may appear at most $a - 1$ times and $a \leq k$, and so the two exceptional cases can happen only when $a = k$.

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If $\pi \in U_i^1$ for some $1 \leq i \leq k-1$, we put

$$\beta(\pi) = (a_1, \ldots, a_{l-1}|b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1}, \ldots, b_{i-1+(k-1)},$$
$$b_{i+(k-1)} + 1, \ldots, b_{i+(n-1)(k-1)} + 1, \ldots).$$

Clearly, $\beta(\pi) \in U_{i+1}^2$.

Conversely, if $\pi \in U_i^2$ for some $2 \leq i \leq k$, then we have

$$\beta^{-1}(\pi) = (a_1, \ldots, a_l, n|b_1, \ldots, b_{i-1}, b_i - 1, \ldots, b_{i+(n-1)(k-1)} - 1, \ldots) \tag{2.10}$$

belongs U_{i-1}^1 unless $i = k - a$. In the case of $i = k - a$, the element 1 appears more than $a - 1$ times in the partition in (2.10). We deduce that

$$\pi = (2n, 2n - 1, \ldots, n + 1|(2n)^{k-a} \, (2n - 1)^{a-1} \, (2n - 2)^{k-a} \ldots),$$

which has weight $(k + \frac{1}{2})n^2 + (k - a + \frac{1}{2})n$.

If $\pi \in U_i^4$ for some $2 \leq i \leq k$, we define

$$\gamma(\pi) = (b_1, a_1 - 1, \ldots, a_n - 1, a_{n+1}, \ldots, a_l|$$
$$b_2, \ldots, b_{k-1}, b_k + 1, b_{k+1}, \ldots, b_{2k-1} + 1, \ldots, b_{k+(n-1)(k-1)} + 1, \ldots).$$

Then $\gamma(\pi) \in U_{i-1}^3$.

Conversely if $\pi \in U_i^3$, for some $1 \leq i \leq k - 1$, then

$$\gamma^{-1}(\pi) = (a_2 + 1, \ldots, a_{n+1} + 1, \ldots, a_l|a_1, b_1, \ldots, b_{k-1} - 1, \ldots, b_{(n-1)(k-1)} - 1, \ldots), \tag{2.11}$$

belongs U_i^4 unless $a = i$. When $a = i$, the element 1 appears more than $a - 1$ times in the partition in (2.11). So we are led to the fixed point

$$\pi = (2n - 1, 2n - 2, \ldots, n|(2n - 1)^{a-1} \, (2n - 2)^{k-a} \, (2n - 3)^{a-1} \ldots),$$

which has weight $(k + \frac{1}{2})n^2 - (k - a + \frac{1}{2})n$.

In summary, we get two types of fixed points

$$(2n, 2n - 1, \ldots, n + 1|(2n)^{k-a} \, (2n - 1)^{a-1} \, (2n - 2)^{k-a} \ldots)$$

and

$$(2n - 1, 2n - 2, \ldots, n|(2n - 1)^{a-1} \, (2n - 2)^{k-a} \, (2n - 3)^{a-1} \ldots),$$

where $n \geq 0$.

For example, let $k = a = 3$ and $N = 17$. Then the correspondence that is not covered by Step 1 consists of the following maps.

(1) $\alpha \colon U_1^2 \to U_3^1$:

$$
\begin{array}{lcl}
(6 \mid 5, 5, 1) & \xrightarrow{\ \alpha\ } & (5 \mid 5, 4, 3) \\
(5, 3 \mid 4, 4, 1) & & (4, 3, 1 \mid 4, 4, 1) \\
(5, 2 \mid 4, 4, 2) & & (4, 2, 1 \mid 4, 4, 2) \\
(5, 2 \mid 4, 4, 1, 1) & & (4, 2, 1 \mid 4, 4, 1, 1) \\
(5 \mid 4, 4, 2, 2) & & (4, 1 \mid 4, 4, 2, 2).
\end{array}
$$

(2) $\beta\colon U_1^1 \to U_2^2$:

$$
\begin{array}{lcl}
(6,1 \mid 5,5) & \xrightarrow{\ \beta\ } & (6 \mid 6,5) \\
(5,3,1 \mid 4,4) & & (5,3 \mid 5,4) \\
(5,2,1 \mid 4,4,1) & & (5,2 \mid 5,4,1) \\
(5,1 \mid 4,4,2,1) & & (5 \mid 5,4,2,1) \\
(4,3,2,1 \mid 3,3,1) & & (4,3,2 \mid 4,3,1).
\end{array}
$$

(3) $\beta\colon U_2^1 \to U_3^2$:

$$
\begin{array}{lcl}
(5,2,1 \mid 5,4) & \xrightarrow{\ \beta\ } & (5,2 \mid 5,5) \\
(5,1 \mid 5,4,2) & & (5 \mid 5,5,2) \\
(5,2 \mid 4,4,2) & & (5 \mid 5,5,1,1) \\
(4,3,2,1 \mid 4,3) & & (4,3,2 \mid 4,4) \\
(4,3,1 \mid 4,3,2) & & (4,3 \mid 4,4,2) \\
(4,3,1 \mid 4,3,1,1) & & (4,3 \mid 4,4,1,1) \\
(4,2,1 \mid 4,3,2,1) & & (4,2 \mid 4,4,2,1).
\end{array}
$$

(4) $\gamma\colon U_2^4 \to U_1^3$:

$$
(5 \mid 5,4,3) \qquad \xrightarrow{\ \gamma\ } \qquad (5,4 \mid 4,4).
$$

In this example, there are no fixed points.

3 Combinatorial proof of Theorem 1.2

In this section, we give an involution for Andrews' Theorem 1.2. By the relation

$$
\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{(q^2;q^4)_\infty}{(q;q)_\infty},
$$

and the Jacobi triple product identity, Theorem 1.2 can be restated as

$$
(q;q)_\infty \sum_{n=0}^{\infty} W_{k,a}(n)q^n = (q^2;q^4)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+1)n^2 + (k+1-a)n}. \tag{3.12}
$$

As the first step, we shall transform the partition pairs $(A|B)$ corresponding the left hand side of identity (3.12) into triples of partitions $(A'|B'|E)$ such that partition pairs $(A'|B')$ satisfy the conditions for Gordon's involution for some k and a. Then we apply Gordon's involution on $(A'|B')$ and keep E unchanged. It will be shown that the right hand side of (3.12) equals the generating function for the fixed points $(A'|B'|E)$ of this involution.

Combinatorial Proof of Theorem 1.2. The left hand side of (3.12) can be interpreted as the generating function of pairs of partitions $(A|B)$, where $A = (a_1, a_2, \ldots)$ is a signed partition

with distinct parts and $B = (b_1, b_2, \ldots)$ is a partition enumerated by $W_{k,a}$. We denote the set of such partition pairs by $\mathcal{G}_{k,a}$

We shall construct an involution on $\mathcal{G}_{k,a}$ which leads to a combinatorial proof of (3.12). It consists of five steps. The objective of the first two steps is transform a pair of partitions $(A|B)$ in $\mathcal{G}_{k,a}$ into a triple of partitions $(A'|B'|E)$ such that E is a signed partition with parts congruent to 2 modulo 4 and B' is a partition with odd parts distinct. Step 3 gives an involution to cancel out certain partition pairs so that we can apply Gordon's involution to the remaining partition pairs, which is Step 4. In Step 5, we still need an involution to reach the fixed points.

Step 1. Let $(A|B)$ be a partition pair in $\mathcal{G}_{k,a}$ as given above. If b_i is an odd part in B and there is a part in A that equals b_i, then we remove b_i from B and the equal part from A to form a part in E that is congruent to 2 modulo 4. Since the part in A has a minus sign, the resulting part in E also has a minus sign. Repeating this procedure until there are no common odd parts in A and B.

Step 2. Combining two equal parts into one part in B' until we get a partition with odd parts distinct. For the sake of notational convenience, we still use B' to denote the resulting partition. We denote the set of the triples of partitions $(A'|B'|E)$ by $\mathcal{G}'_{k,a}$.

Step 3. We compare the largest parts in A' and B', namely, a'_1 and b'_1. If $b'_1 > a'_1$, then we move b'_1 to A'. This operation leads to a triple of partitions in $\mathcal{G}'_{k,a}$. If $b'_1 \le a'_1$, we move a'_1 to B', except for the following case. To describe the exceptional case, let us rewrite B' as a pair of partitions $(C|D)$, where $C = (c_1, c_2, \ldots)$ consists of the odd parts of B' and $D = (d_1, d_2, \ldots)$ consists of the even parts of B'. The conditions are as follows

(1) $a_1 = d_1 = d_2 = \cdots = d_{i-1} = d_i + 2 = \cdots = d_{\frac{k-2}{2}} + 2$, or

(2) $a_1 = d_1 = d_2 = \cdots = d_{i-1} = d_i + 2 = \cdots = d_{\frac{k-4}{2}} + 2$, and there is an odd part c_j such that $c_j = a_1/2$ or $a_1/2 - 1$.

It can be checked that if either (1) or (2) holds, then the partition triples obtained from $(A'|B'|E)$ by moving a'_1 to B no longer belongs $\mathcal{G}'_{k,a}$. Moreover, one sees that if neither (1) nor (2) holds, then the triple of partitions obtained by the above operation is still in $\mathcal{G}'_{k,a}$.

Step 4. We shall give an involution for triples of partitions in $\mathcal{G}'_{k,a}$ that satisfy the above conditions (1) or (2). Note that the odd parts in A' and C are all distinct. Let us compare the largest odd parts in A' and C. When the larger one is in A', we move it to C. Similarly, when the larger one is in C, we move the largest part of C to A'. To be precise, if only one of A' and C contains odd parts, then we move the largest odd part to the other partition. So there is only one case that it is impossible to make any move, that is, neither A' nor C contains odd parts.

Till now, we have left the triples of partitions in $\mathcal{G}'_{k,a}$ such that $(A'|B')$ is a pair of partitions with even parts.

Step 5. For remaining partition pairs $(A'|B')$, we apply Gordon's involution to partition pairs $(A'|B')$ with k and a replaced by $\frac{k}{2}$ and $\frac{a}{2}$ respectively and with every part doubled. Then we get two types of fixed points

$$(A'|B'|E) = (4n, 4n-2, \ldots, 2n+2|(4n)^{\frac{k-a}{2}}, (4n-2)^{\frac{a-2}{2}}, (4n-4)^{\frac{k-a}{2}} \ldots |E), \qquad (3.13)$$

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and

$$(A'|B'|E) = (4n - 2, \ldots, 2n|(4n - 2)^{\frac{a-2}{2}}, (4n - 4)^{\frac{k-a}{2}}, (4n - 6)^{\frac{a-2}{2}} \ldots |E). \qquad (3.14)$$

For the partition triples in (3.13), we have

$$|A'| + |B'| = (k + 1)n^2 + (k - a + 1)n,$$

whereas for the partition triples in (3.14), we have

$$|A'| + |B'| = (k + 1)n^2 - (k - a + 1)n.$$

Since E is any signed partition with distinct parts congruent to 2 modulo 4, we see that the generating function of the fixed points $(A'|B'|E)$ equals

$$(q^2; q^4)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+1)n^2 + (k+1-a)n}.$$

This completes the combinatorial proof of Theorem 1.2. ∎

For example, let $k = a = 6$ and $n = 44$, and let $(A|B) = (10, 8, 5 \mid 5, 4^4)$. The first two steps are as follows:

$$(10, 8, 5 \mid 5, 4^4) \Rightarrow (10, 8 \mid 4^4 \mid 5, 5) \Rightarrow (10, 8 \mid 8, 8 \mid 10).$$

Now we see that the partition $(10, 8 \mid 8, 8 \mid 10)$ belongs U_1^3 as defined in Section 2 if every part for Gordon's involution is doubled. Using the map γ^{-1} also with every part doubled, we obtain the partition $(10 \mid 10, 8, 6 \mid 10)$ from the partition $(10, 5 \mid 5, 5, 5, 4, 4, 3, 3)$ by the following steps

$$(10, 5 \mid 5, 5, 5, 4, 4, 3, 3) \Rightarrow (10 \mid 5, 5, 4, 4, 3, 3 \mid 5, 5) \Rightarrow (10 \mid 10, 8, 6 \mid 10).$$

4 Combinatorial proof of Theorem 1.3

In this section, we give a combinatorial proof of Theorem 1.3 which also involves Gordon's involution. Using the relation

$$\frac{(q^2; q^4)_\infty}{(q; q)_\infty} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty},$$

and the Jacobi triple product identity, Theorem 1.3 can be rewritten as follows

$$(q^2; q^2)_\infty \sum_{n=0}^{\infty} W_{k,a}(n)q^n = (-q; q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+1)n^2 + (k+1-a)n}. \qquad (4.15)$$

Suppose that $(A|B)$ is a partition pair corresponding to the left hand side of the identity (4.15). We first transform it into a triple of partitions $(A|C'|D')$. Then we establish an involution on the set of these triples such that the fixed points have the generating function equals the right hand side of (4.15).

Combinatorial proof of the Theorem 1.3. The left hand side of (4.15) can be interpreted as the generating function of pairs of partitions $(A|B)$, where $A = (a_1, a_2, \ldots)$ is a signed partition

with distinct even parts, and $B = (b_1, b_2, \ldots)$ is a partition enumerated by $W_{k,a}$. We denote the set of such pairs $(A|B)$ by $\mathcal{H}_{k,a}$.

We proceed to construct an involution on the set $\mathcal{H}_{k,a}$. It consists of five steps. The objective of the first step is transform a pair of partitions $(A|B)$ in $\mathcal{H}_{k,a}$ into a triple of partitions $(A|C|D)$ such that C is a partition with even parts and D is a partition with distinct odd parts. Step 2 gives an involution to cancel out certain partition pairs. In Step 3, we split the parts in C that are congruent to 2 modulo 4 but not a double of some part in D into two equal parts and move them to D. In Step 4, we apply Gordon's involution. Step 5 gives an operation on partitions that enables us to compute the generating function of the fixed points.

Step 1. Combine two equal parts in B to form an even part. Repeating this operation to generate a partition $C = (c_1, c_2, \ldots)$ with even parts. Let $D = (d_1, d_2, \ldots)$ be the partitions consisting of the remaining parts in B. Since the even parts in B appear an even number of times, D is a partition with odd parts. We denote the set of the triples $(A|C|D)$ by $\mathcal{H}'_{k,a}$.

Step 2. Now we compare the largest parts in A and C. If $c_1 > a_1$, then we move the c_1 to A. This gives a triple of partitions in $\mathcal{H}'_{k,a}$. If $c_1 \leq a_1$, we move a_1 to C to form a triple in $\mathcal{H}'_{k,a}$ except for the following cases:

(1) $a_1 = c_1 = c_2 = \cdots = c_{i-1} = c_i + 2 = \cdots = c_{\frac{k-1}{2}} + 2$, or

(2) $a_1 = c_1 = c_2 = \cdots = c_{i-1} = c_i + 2 = \cdots = c_{\frac{k-3}{2}} + 2$, and there is an odd part $d_j = a_1/2$ or $(a_1 - 2)/2$.

Step 3. For the triples of partitions $(A|C|D)$ in $\mathcal{H}'_{k,a}$ that satisfy the above exceptional conditions, we construct an involution to further cancel out certain triples. First, we need to move some parts of C to D. Consider the parts in C that are congruent to 2 modulo 4. If a part $4v - 2$ in C is twice of some part in D, then we keep it in C. Otherwise, we split the part $4v - 2$ into two equal parts and put them into D. Repeating the procedure until any part of the form $4v - 2$ is twice of some part in D. Denote the triple of partitions we have obtained this way by $(A|C'|D')$. It can be checked that such a triple satisfies the following conditions:

(1) $a_1 = c'_1 = c'_2 = \cdots = c'_{i-1} = c'_i + 2 = \cdots = c'_{\frac{k-3}{2}} + 2$;

(2) $c'_i - c'_{i + \frac{k-3}{2}} \geq 4$, and 2 appears at most $\frac{a-3}{2}$ times in C';

(3) If there is a part $4v - 2$ in C', then the part $2v - 1$ must appear in D' either once or twice. If $4v - 2$ does not appear in C' but both $4v$ and $4v - 4$ appear in C', then $2v - 1$ must appear in D'.

Step 4. For the above conditions (1), (2) and (3), we shall apply Gordon's involution to the pairs $(A|C')$ with k and a replaced by $\frac{k-1}{2}$ and $\frac{a-1}{2}$ respectively and with every part doubled. The two types of fixed points are as follows

$$(A|C'|D') = (4n, 4n - 2, \ldots, 2n + 2|(4n)^{\frac{k-a}{2}} (4n - 2)^{\frac{a-3}{2}} (4n - 4)^{\frac{k-a}{2}} \ldots |D'), \qquad (4.16)$$

and

$$(A|C'|D') = (4n - 2, \ldots, 2n|(4n - 2)^{\frac{a-3}{2}} (4n - 4)^{\frac{k-a}{2}} (4n - 6)^{\frac{a-3}{2}} \ldots |D'). \qquad (4.17)$$

Step 5. To compute the generating function for the above two types of fixed points, we need to do some transformations. Set the initial value of E to $\emptyset$. Recall that all the parts in D' are

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odd. We shall distribute the parts of D' to E and C'. To be precise, if a part d_i' is bigger than $2n$, then we move d_i' to E. So the remaining parts in D' are less than $2n$. If there are two equal parts are left in D', we move one of them to E. Eventually, D' becomes the partition $(2n-1, 2n-3, \ldots, 3, 1)$. Now we continue to split every part of C' into two equal parts and move the remaining parts in D' to C'. It can be checked that the partition E can be any partition with distinct odd parts. So the fixed points in (4.16) and (4.17) take the following equivalent forms

$$(4n, 4n-2, \ldots, 2n+2 | (2n)^{k-a} \, (2n-1)^{a-2} \, (2n-2)^{k-a} \ldots | E), \tag{4.18}$$

and

$$(4n-2, \ldots, 2n | (2n-1)^{a-2} \, (2n-2)^{k-a} \, (2n-3)^{a-2} \ldots | E). \tag{4.19}$$

It is now easy to see that the generating function for the fixed points equals

$$(-q; q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+1)n^2 + (k+1-a)n}.$$

This completes the proof. ∎

For example, let $N = 189$, $k = a = 9$ and let $(A|B) = (16, 14, 12, 10 \mid 9, 7^8, 5^8, 3^8, 1^8)$. The first three steps are as follows:

$$(16, 14, 12, 10 \mid 9, 7^8, 5^8, 3^8, 1^8) \Rightarrow (16 \mid 14, 12, 10 \mid 14^4, 10^4, 6^4, 2^4 \mid 9)$$
$$\Rightarrow (16, 14, 12, 10 \mid 14^3, 10^3, 6^3, 2^3 \mid 9, 7, 7, 5, 5, 3, 3, 1, 1).$$

One can check that the triple of partitions $(16, 14, 12, 10 \mid 14^3, 10^3, 6^3, 2^3 \mid 9, 7, 7, 5, 5, 3, 3, 1, 1)$ is a fixed point of type (4.16) for $n = 4$. Applying the transformation in Step 5, we get a triple of partitions

$$(16, 14, 12, 10 \mid 7^7, 5^7, 3^7, 1^7 \mid 9, 7, 5, 3, 1).$$

5 Combinatorial proof of Theorem 1.4

In this section, we only give a brief description of an involution as a combinatorial proof of Theorem 1.4 since the idea is similar to the involutions for the other two identities of Andrews.

Proof of the Theorem 1.4. In view of the relation

$$\frac{1}{(-q; q^2)_\infty (q; q)_\infty} = \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty}$$

and Jacobi's triple product identity, Theorem 1.4 can be restated as follows

$$(q^2; q^2)_\infty \sum_{n=0}^{\infty} \overline{W}_{k,a}(n) q^n = (-q^2; q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+1)n^2 + (k+1-a)n}. \tag{5.20}$$

Let $(A|B)$ be a partition pair corresponding to the left hand side of (5.20), that is, $A = (a_1, a_2, \ldots)$ is a signed partition with distinct even parts and $B = (b_1, b_2, \ldots)$ is a partition

enumerated by $\overline{W}_{k,a}$. We denote the set of such partition pairs by $\mathcal{Q}_{k,a}$. We now give an involution on $\mathcal{Q}_{k,a}$ which can be described by the following five steps.

Step 1. We combine two equal parts in B to form an even part. Repeating this procedure to get an even partition $C = (c_1, c_2, \ldots)$. Suppose that the remaining parts of B form a partition $D = (d_1, d_2, \ldots)$. Since each odd part in B appears an even number of times, D is a partition with distinct even parts. We denote the set of such triples $(A|C|D)$ by $\mathcal{Q}'_{k,a}$.

Step 2. Now we compare the largest parts in A and C. If $c_1 > a_1$, we move the part c_1 to A. This leads to a triple of partitions in $\mathcal{Q}'_{k,a}$. If $a_1 \geq c_1$, we move a_1 to the partition C to form a triple in $\mathcal{Q}'_{k,a}$ unless the following two conditions are satisfied:

(1) $a_1 = c_1 = c_2 = \cdots = c_{i-1} = c_i + 2 = \cdots = c_{\frac{k-1}{2}} + 2$,

(2) $a_1 = c_1 = c_2 = \cdots = c_{i-1} = c_i + 2 = \cdots = c_{\frac{k-3}{2}} + 2$, and there is an even part d_j in D such that $d_j = a_1/2$ or $(a_1 - 2)/2$.

Step 3. We wish to construct an involution for further cancelation of triples $(A|C|D)$ in $\mathcal{Q}'_{k,a}$ that satisfy the above exceptional conditions. If a part $4v$ in C is twice of some part in D, then we keep it in C. Otherwise, we split this part $4v$ into two equal parts and put them in D. After this operation, we denote the resulting triple of partitions by $(A|C'|D')$. It can be verified that such a triple $(A|C'|D')$ satisfies the following conditions:

(1) $a_1 = c'_1 = c'_2 = \cdots = c'_{i-1} = c'_i + 2 = \cdots = c'_{\frac{k-3}{2}} + 2$;

(2) $c'_i - c'_{i+\frac{k-3}{2}} \geq 4$, and 2 appears at most $\frac{a-2}{2}$ times in C';

(3) If there is a part $4v$ in C', then the part $2v$ must appear in D' either once or twice. If $4v$ does not appear in C' but both $4v - 2$ and $4v + 2$ appear in C', then $2v$ must appear in D'.

Step 4. For a partition triple $(A|C'|D')$ satisfying the above conditions (1), (2) and (3), we apply Gordon's involution on the pair $(A|C')$ with k and a replaced by $\frac{k-1}{2}$ and $\frac{a}{2}$ respectively and with every part doubled. The two types of fixed points are as follows

$$(A|C'|D') = (4n, 4n - 2, \ldots, 2n + 2 | (4n)^{\frac{k-a-1}{2}} (4n - 2)^{\frac{a-2}{2}}, (4n - 4)^{\frac{k-a-1}{2}} \ldots | D'), \quad (5.21)$$

and

$$(A|C'|D') = (4n - 2, \ldots, 2n | (4n - 2)^{\frac{a-2}{2}}, (4n - 4)^{\frac{k-a-1}{2}} (4n - 6)^{\frac{a-2}{2}} \ldots | D'). \quad (5.22)$$

Step 5. We shall make some transformations on the above fixed points in order to compute their generating function. Set the initial value of E to $\emptyset$. Since all the parts in D' are even, we may distribute the parts of D' to E and C'. For a fixed point of type (5.21), if a part d'_i is bigger than $2n + 1$, then we move d'_i to E. After the completion of this process, the remaining parts in D' are less than $2n + 1$ in the fixed point of type (5.21). And for a fixed point of type (5.22), if a part d'_i is bigger than $2n - 1$, then we move d'_i to E. Then the remaining parts in D' are less than $2n - 1$ in the fixed point of type (5.22).

If there are two equal parts that are left in D', we move one of them to E. Then there are some remaining parts in D'. For a fixed point of type (5.21), the remaining parts in D' are $(2n, 2n - 2, \ldots, 4, 2)$. While for a fixed point of type (5.22), the remaining parts in D' are $(2n - 2, 2n - 4, \ldots, 4, 2)$.

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Now we continue to split every part of C' into two equal parts and move the remaining parts in D' to C'. It can be seen that the partition E can be an arbitrary partition with distinct odd parts. Therefore, the fixed points can be represented by the following types of partitions

$$(4n, 4n-2, \ldots, 2n+2|(2n)^{k-a} \ (2n-1)^{a-2} \ (2n-2)^{k-a} \ldots |E), \qquad (5.23)$$

and

$$(4n-2, \ldots, 2n|(2n-1)^{a-2} \ (2n-2)^{k-a} \ (2n-3)^{a-2} \ldots |E). \qquad (5.24)$$

Hence we get the generating function

$$(-q^2; q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+1)n^2+(k+1-a)n}.$$

This completes the proof. $\blacksquare$

For example, let $k = 7$, $a = 6$ and $N = 40$ and let $(A|B) = (10, 2 \mid 4, 4, 4, 4, 4, 2, 2, 2, 1, 1)$. The first step makes the transformation

$$(10, 2 \mid 4, 4, 4, 4, 4, 2, 2, 2, 1, 1) \Rightarrow (10, 2 \mid 8, 8, 4, 2 \mid 4, 2),$$

One can check that the partition $(10, 2 \mid 8, 8, 4, 2 \mid 4, 2)$ belongs U_1^1 as defined in Section 2 when assuming that every part for Gordon's involution is doubled. Using the map β, also with every part doubled, we get $(10 \mid 10, 8, 4, 2 \mid 4, 2)$ from the partition pair $(10 \mid 5, 5, 4, 4, 4, 2, 2, 2, 1, 1)$ via the following steps:

$$(10 \mid 5, 5, 4, 4, 4, 2, 2, 2, 1, 1) \Rightarrow (10 \mid 10, 8, 4, 2 \mid 4, 2).$$

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