An Iterated Map for the Lebesgue Identity

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Abstract. We present a simple iteration for the Lebesgue identity on partitions.

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In this note, we present a simple iterated map for the Lebesgue identity on partitions. Recall that the q-shifted factorials are defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in \mathbb{Z},$$

where $|q| < 1$. The Lebesgue identity reads

$$\sum_{k=0}^{\infty} \frac{(-aq;q)_k}{(q;q)_k} q^{\binom{k+1}{2}} = (-aq^2;q^2)_\infty(-q;q)_\infty, \tag{1}$$

see, for example, Andrews [2]. There are several combinatorial proofs of the Lebesgue identity. Ramamani and Venkatachaliengar [7] found a bijection for the following generalization of (1),

$$\sum_{m=0}^{\infty} q^{m(m+1)/2} \frac{(z;q)_m}{(q;q)_m} \alpha^m = (z;q)_\infty(-\alpha q;q)_\infty \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n(-\alpha q;q)_n}.$$

Bessenrodt [3] gave a combinatorial interpretation in terms of 2-modular diagrams. Alladi and Gordon [1] provided a construction based on the standard MacMahon diagrams, see the survey of Pak [6]. Fu [4] discovered a bijective proof of the following extension of (1) by applying the insertion algorithm of Zeilberger:

$$\sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(q;q)_n} b^n q^{\binom{n+1}{2}} = (-bq;q)_\infty \sum_{k=0}^{\infty} \frac{(ab)^k q^{k(k+1)}}{(q;q)_k(-bq;q)_k}.$$

Rowell [8] presented a combinatorial proof which leads to the following finite form of (1):

$$\sum_{n=0}^{L} \begin{bmatrix} L \\ n \end{bmatrix}_q (-aq;q)_n q^{n(n+1)/2} = \sum_{k=0}^{L} \begin{bmatrix} L \\ k \end{bmatrix}_{q^2} (-q;q)_{L-k} a^k q^{k(k+1)}.$$

Recently, Little and Sellers [5] have established the relation (1) by using weighted Pell tilings.

We follow the terminology in [2]. A partition is meant to be a non-increasing finite sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The entries λ_i are called the parts of λ. The number of parts of λ is denoted by $\ell(\lambda)$, and sum of parts is denoted by $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$. The conjugate partition of λ is denoted by λ'. The partition with no parts is denoted by $\varnothing$.

Denote the left hand side of the Lebesgue identity (1) by $f(a, q)$. It is easily seen that

$$f(a,q) = \sum_{(\alpha,\beta)\in P} a^{\ell(\beta)} q^{|\alpha|+|\beta|},$$

where P denotes the set of pairs (α, β) of partitions with distinct parts such that $\ell(\alpha)$ is not less than the largest part of β. The corresponding diagram is illustrated by Figure 1.

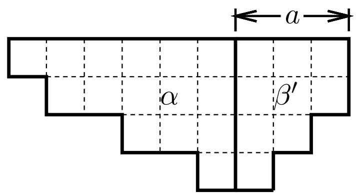

Figure 1: A pair $(\alpha, \beta) \in P$

Clearly, the right hand side of (1) has the following combinatorial interpretation

$$\sum_{(\mu,\nu)\in Q} a^{\ell(\nu)} q^{|\mu|+|\nu|},$$

where Q is the set of pairs (μ, ν) of partitions with distinct parts such that ν has only even parts.

For a triple of partitions (α, β, γ) where $(\alpha, \beta) \in P$ and $\gamma = \varnothing$ or the smallest part of γ is not less than $\ell(\beta)$, we define a map ϕ: $(\alpha, \beta, \gamma) \to (\mu, \lambda, \nu)$ as follows:

Case 1: The smallest part of β equals 1. Decrease each part of α by 1 to form a partition μ. Change the 1-part of β to a $(\ell(\alpha)+1)$-part and decrease each part of the resulting partition by 2 to generate a partition λ. Then add two $\ell(\beta)$-parts to γ to produce a partition ν. This operation can be clearly visualized as moving up the diagram on the right by two rows. See Figure 2 for an illustration, where $\alpha = (6, 5, 3, 1)$, $\beta = (4, 3, 1)$, $\gamma = (4, 4)$, $\mu = (5, 4, 2)$, $\lambda = (3, 2, 1)$, and $\nu = (4, 4, 3, 3)$.

Case 2: The smallest part of β is larger than 1. Set $\mu = \alpha$ and decrease each part of β by 2 to generate a partition λ. Then add two $\ell(\beta)$-parts to γ to form a partition ν.

It is clear that the map ϕ is reversible. Starting from $(\alpha, \beta, \varnothing)$, we can iterate the above map until λ becomes empty. This gives a pair (μ, ν') of partitions that belongs to Q, and so the combinatorial proof of the Lebesgue identity is complete.

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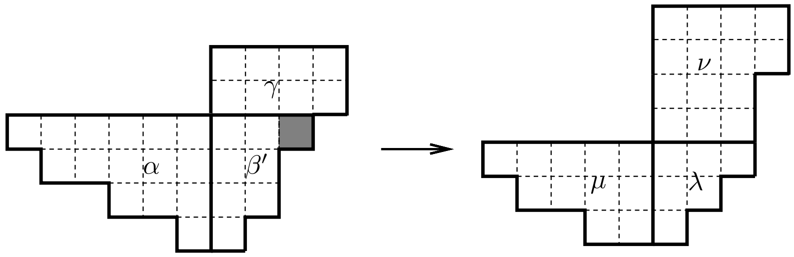

Figure 2: An example

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