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Mock Theta Functions

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1. Introduction. The mock theta functions are the subject of Ramanujan's last letter to Hardy dated January, 1920. The mathematical portions of this letter have been reproduced in [**25**, pp. 127–131] (see also [**24**, pp. 354–355] and [**29**, pp. 56–61]), and we repeat them here at the beginning to lay the groundwork for this survey.

"If we consider a ϑ-function in the transformed Eulerian form, e.g.,

$$\text{(A)}\quad 1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots,$$

$$\text{(B)}\quad 1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots,$$

and determine the nature of the singularities at the points

$$q=1, q^2=1, q^3=1, q^4=1, q^5=1, \ldots,$$

we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when $q=e^{-t}$ and $t\to 0$,

$$\text{(A)} = \sqrt{\left(\frac{t}{2\pi}\right)}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}^{*}\right)+o(1)^{\dagger},$$

$$\text{(B)} = \sqrt{\left(\frac{2}{5-\sqrt{5}}\right)}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}^{*}\right)+o(1),$$

and similar results at other singularities.

"If we take a number of functions like (A) and (B), it is only in a limited number of cases the terms close as above; but in the majority of cases they

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*It is not necessary that there should be only one term like this. There may be many terms but *the number of terms must be finite.*

†Also $o(1)$ may turn out to be $O(1)$. That is all. For instance, when $q\to 1$, the function $\{(1-q)(1-q^2)(1-q^3)\cdots\}^{-120}$ is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

never close as above. For instance, when $q = e^{-t}$ and $t \to 0$,

$$\text{(C)}\quad 1 + \frac{q}{(1-q^2)} + \frac{q^3}{(1-q^2)(1-q^2)^2} + \frac{q^6}{(1-q^2)(1-q^2)^2(1-q^3)^2} + \cdots$$

$$= \sqrt{\left(\frac{t}{2\pi\sqrt{5}}\right)} \exp\left[\frac{\pi^2}{5t}^{**} + a_1^{\dagger\dagger} t + a_2 t^2 + \cdots + O(a_k t^k)\right],$$

where $a_1 = 1/8\sqrt{5}$, and so on. The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

"Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say: Suppose there is a function in the Eulerian form and suppose that all or an infinity of points are exponential singularities, and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The questions is: Is the function taken the sum of two functions one of which is an ordinary ϑ-function and the other a (trivial) function which is $O(1)$ at all the ponts $e^{2m\pi i/n}$? The answer is *it is not necessarily so.* When it is not so, I call the function a Mock ϑ-function. I have not proved rigorously that *it is not necessarily so.* But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function. Also I have shown that if *it is necessarily so* then it leads to the following assertion—viz. it is possible to construct two power series in x, namely $\sum a_n x^n$ and $\sum b_n x^n$, both of which have *essential singularities* on the unit circle, are convergent when $|x| < 1$, and tend to *finite limits at every point* $x = e^{2r\pi i/s}$, and that at the same time the limit of $\sum a_n x^n$ at the point $x = e^{2r\pi i/s}$ is equal to the limit of $\sum b_n x^n$ at the point $x = e^{-2r\pi i/s}$.

"This assertion seems to be untrue. [H. Cohen, B. Gordon, and D. Hickerson have each pointed out to me that Ramanujan is incorrect; indeed Cohen's function $\varphi(q)$ (see (6.2) below) provides a counterexample with $\varphi(x^{24})$ and $-\varphi(x^{24})$.] Anyhow, we shall go to the examples and see how far our assertions are true.

"I have proved that, if

$$f(q) = 1 + \frac{q}{(1+q^2)} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots,$$

then

$$f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1 - 2q + 2q^4 - 2q^4 + \cdots) = O(1)$$

at all the points $q = -1$, $q^3 = -1$, $q^5 = -1$, $q^7 = -1, \cdots$; and at the same time

$$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1 - 2q + 2q^4 - 2q^9\cdots) = O(1)$$

** The coefficient $1/t$ (sic) in the index of e happens to be $\pi^2/5$ in this particular case. It may be some other transcendental numbers in other cases.

†† The coefficients of $t, t^2, \ldots$ happen to be $1/8\sqrt{5}, \ldots$ in this case. In other cases they may turn out to be some other algebraic numbers.

at all the points $q^2 = -1$, $q^4 = -1$, $q^6 = -1, \ldots$. Also, obviously, $f(q) = O(1)$ at all the points $q = 1$, $q^3 = 1$, $q^5 = 1, \ldots$. And so $f(q)$ is a Mock ϑ-function.

"When $q = -e^{-t}$ and $t \to 0$,

$$f(q) + \sqrt{\left(\frac{\pi}{t}\right)} \exp\left(\frac{\pi^2}{24t} - \frac{t}{24}\right) \to 4.$$

"The coefficient of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{\exp\left\{\pi\sqrt{\left(\frac{1}{6}n - \frac{1}{144}\right)}\right\}}{2\sqrt{\left(n - \frac{1}{24}\right)}} + O\left(\frac{\exp\left\{\frac{1}{2}\pi\sqrt{\left(\frac{1}{6}n - \frac{1}{144}\right)}\right\}}{\sqrt{\left(n - \frac{1}{24}\right)}}\right).$$

It is inconceivable that a single ϑ-function could be found to cut out the singularities of $f(q)$.

Mock ϑ-functions.

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots,$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots,$$

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \cdots.$$

These are related to $f(q)$ as shown below.

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1+q)(1+q^2)(1+q^3)\cdots},$$

$$4\chi(q) - f(q) = 3\frac{(1 - 2q^3 + 2q^{12} - \cdots)^2}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

These are of the 3rd order.

Mock ϑ-functions (of 5th order).

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \cdots,$$

$$\phi(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \cdots,$$

$$\psi(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \cdots,$$

$$\chi(q) = 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)} + \cdots$$

$$= 1 + \frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots,$$

$$F(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \cdots,$$

$$\phi(-q) + \chi(q) = 2F(q),$$

$$f(-q)+2F(q^2)-2=\phi(-q^2)+\psi(-q)$$
$$=2\phi(-q^2)-f(q)=\frac{1-2q+2q^4-2q^9+\cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots},$$

$$\psi(q)-F(q^2)+1=q\frac{1+q^2+q^6+q^{12}+\cdots}{(1-q^8)(1-q^{12})(1-q^{28})\cdots}.$$

Mock ϑ-functions (of 5th order).

$$f(q)=1+\frac{q^2}{1+q}+\frac{q^6}{(1+q)(1+q^2)}+\frac{q^{12}}{(1+q)(1+q^2)(1+q^3)}+\cdots,$$
$$\phi(q)=q+q^4(1+q)+q^9(1+q)(1+q^3)+\cdots,$$
$$\psi(q)=1+q(1+q)+q^3(1+q)(1+q^2)+q^6(1+q)(1+q^2)(1+q^3)+\cdots,$$
$$\chi(q)=\frac{1}{1-q}+\frac{q}{(1-q^2)(1-q^3)}+\frac{q^2}{(1-q^3)(1-q^4)(1-q^5)}+\cdots,$$
$$F(q)=\frac{1}{1-q}+\frac{q^4}{(1-q)(1-q^3)}+\frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}+\cdots$$

have got similar relations as above.

Mock ϑ-functions (of 7th order).

$$1+\frac{q}{1-q^2}+\frac{q^4}{(1-q^3)(1-q^4)}+\frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}+\cdots,$$
$$\frac{q}{1-q}+\frac{q^4}{(1-q^2)(1-q^3)}+\frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}+\cdots,$$
$$\frac{1}{1-q}+\frac{q^2}{(1-q^2)(1-q^3)}+\frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}+\cdots.$$

These are not related to each other."

In this survey we shall try to make clear what has happened to mock theta functions since 1920 including an account of D. R. Hickerson's truly inspiring solution of the Mock Theta Conjectures **[21]**.

2. The Watson-Selberg era. G. N. Watson wrote the first papers to elucidate the mock theta functions **[29, 31]**. The first of these is Watson's Presidential Address to the London Mathematical Society in 1935. He entitled it "The Final Problem: An Account of the Mock Theta Functions." He explained the title as follows: "I doubt whether a more suitable title could be found for it than the title used by John H. Watson, M. D., for what he imagined to be his final memoir on Sherlock Holmes."

In these two papers, Watson proves most of the assertions found in the letter of Ramanujan. The first paper considers only the third-order functions.

It provides three new mock theta functions not mentioned in the letter:

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}, \tag{2.1}$$

$$\nu(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(1+q)(1+q^3)\cdots(1+q^{2n+1})}, \tag{2.2}$$

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})}. \tag{2.3}$$

The bulk of the paper is devoted to the modular transformations of these functions. For example if we let $q=e^{-\alpha}$, $\alpha\beta=\pi^2$, and $q_1=e^{-\beta}$, then (2.4)

$$q^{-1/24}f(q)=2\left(\frac{2\pi}{\alpha}\right)^{1/2}q_1^{4/3}\omega(q_1^2)+4\left(\frac{3\alpha}{2\pi}\right)^{1/2}\int_0^\infty e^{-3\alpha x^2/2}\frac{\sinh\alpha x}{\sinh(3\alpha x/2)}dx.$$

Indeed **[29]** is quite a thorough account of the third-order mock theta functions and provides a prototype for the general treatment of the subject. In **[31]**, Watson moves on to the two families of fifth-order mock theta functions. He manages to prove all of Ramanujan's assertions about these functions; however he is unable to find any results like (2.4). Consequently he is unable to establish that, in fact, these functions are indeed new functions not included under Ramanujan's ϑ-function umbrella described in the last few paragraphs of his letter.

Watson's methods were generalized in **[1, 3, 4, 5]** to prove many extensions of Ramanujan's identities. In the "Lost" Notebook, we find a number of these extensions as well as (2.1), (2.2), and (2.3) and clear indications of how to do (2.4).

The seventh-order functions were mostly neglected by Watson perhaps because Ramanujan makes no positive assertions about them. Watson does briefly and cryptically mention them **[29**, p. 80], (and they clearly are the motivation for his short paper on the dilogarithm **[30]**.) However A. Selberg **[28]** provides a full account of the behavior of the seventh-order functions near the unit circle. This requires a very adroit comparison of the seventh-order functions with q-series that Selberg **[27]** had found earlier related to the modulus 7.

3. Asymptotics. Watson chooses not to treat Ramanujan's assertion that: "The coefficient of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{\exp\left\{\pi\sqrt{\left(\frac{1}{6}n-\frac{1}{144}\right)}\right\}}{2\sqrt{n-\frac{1}{24}}}+O\left(\frac{\exp\left\{\frac{\pi}{2}\sqrt{\left(\frac{1}{6}n-\frac{1}{144}\right)}\right\}}{\sqrt{n-\frac{1}{24}}}\right)."$$

Watson **[29**, p. 62] states: "I have not troubled to verify this approximation; it is presumably derivable from the transformation formulae in the manner in which Hardy and Ramanujan [19] obtained the corresponding formula for $p(n)$, the number of partitions of n."

Indeed this is the case, and two of H. Rademacher's students (Dragonette [18] and Andrews [2]) carried out the full Hardy-Ramanujan-Rademacher expansion for these coefficients. The result they obtained was the following: In the series $f(q) = \sum_{n\geq 0} A(n)q^n$,

$$A(n) = \sum_{0<k\leq n^{1/2}} \frac{\lambda(k)\exp\left\{\frac{\pi}{k}\left(\frac{1}{6}n - \frac{1}{144}\right)^{1/2}\right\}}{\sqrt{k\left(n-\frac{1}{24}\right)}} + E(n).$$

Dragonette [18] showed that $\lambda(1) = (-1)^{n-1}/2$, $\lambda(k)$ is a finite exponential sum, and $E(n) = O(n^{1/2}\log n)$. Andrews [2] showed that

$$\lambda(k) = \begin{cases} \frac{1}{2}(-1)^{(k+1)/2}A_{2k}(n), & k \text{ odd,} \\ \frac{1}{2}(-1)^{k/2}A_{2k}\left(n-\frac{k}{2}\right), & k \text{ even,} \end{cases}$$

where $A_k(n)$ is the exponential sum appearing in the Hardy-Ramanujan formula for $p(n)$, and $E(n) = O(n^\varepsilon)$. Numerical computations by Dragonette [18] suggest that $E(n) \to 0$ as $n \to \infty$. Indeed $E(100) = .206$ and $E(200) = -.153$.

4. q-series and indefinite quadratic forms. Within the last few years, significant discoveries have been made that greatly extend our knowledge of the mock theta functions. However these discoveries are really just a beginning.

The basis of these discoveries lies in the method of Bailey Chains [7, p. 278; 12, Chapter 3], which relies on the following result [12, pp. 25–26].

BAILEY'S LEMMA. *If for $n \geq 0$*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{4.1}$$

then

$$\beta_n' = \sum_{r=0}^{n} \frac{\alpha_r'}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{4.2}$$

where

$$\alpha_r' = \frac{(\rho_1;q)_r(\rho_2;q)_r(aq/\rho_1\rho_2)^r\alpha_r}{(aq/\rho_1;q)_r(aq/\rho_2;q)_r} \tag{4.3}$$

and

$$\beta_r' = \sum_{j=0}^{n} \frac{(\rho_1;q)_j(\rho_2;q)_j(aq/\rho_1\rho_2;q)_{n-j}(aq/\rho_1\rho_2)^j\beta_j}{(q;q)_{n-j}(aq/\rho_1;q)_n(aq/\rho_2;q)_n} \tag{4.4}$$

where $(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$.

In practice the α_n and β_n are sequences of rational functions in a, q, and other parameters. The power of Bailey's Lemma is that it allows the construction of infinitely many pairs of sequences $(\alpha_n^{(i)}, \beta_n^{(i)})$ merely by iteration because (4.2) is precisely (4.1) with (α_n, β_n) replaced by (α_n', β_n'). Such pairs are called Bailey Pairs, and the sequence of such pairs is called a Bailey Chain.

For our purposes here we note the much simpler instance of Bailey's Lemma when $\rho_1, \rho_2, n \to \infty$ in (4.2), (4.3), and (4.4). Thus if (4.1) holds, then

$$\left(\sum_{j=0}^{\infty} a^j q^{j^2}\beta_j = \frac{1}{(aq;q)_\infty}\sum_{r=0}^{\infty} a^r q^{r^2}\alpha_r.\right) \tag{4.5}$$

Now Watson [29, p. 64] proved that

$$f(q) \equiv \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = \frac{1}{(q;q)_\infty}\left(1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right), \tag{4.6}$$

and it is not too difficult to show that his identity (which indeed Ramanujan recorded and generalized in his Lost Notebook [25, p. 202, first equation]) is an instance of (4.5) for the Bailey Pair

$$\alpha_n = \begin{cases} 1, & n = 0, \\ \dfrac{4(-1)^n q^{n(n+1)/2}}{1+q^n}, & n > 0, \end{cases} \tag{4.7}$$

$$\beta_n = \frac{1}{(-q;q)_n^2}. \tag{4.8}$$

Watson [31, p. 274] (see also Andrews [10, pp. 113–114]) expressed his doubts about finding anything comparable to (4.6) for the fifth-order mock theta functions. Indeed it was only after IBM's symbolic algebra package SCRATCHPAD was employed in a significant way that comparable results were found for most of the other mock theta functions [10, §3]. For example (using Watson's notation for the fifth-order functions)

$$\begin{aligned} f_0(q) &\equiv \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} \\ &= \frac{1}{(q;q)_\infty}\sum_{n\geq 0}\sum_{|j|\leq n}(-1)^j q^{n(5n+1)/2-j^2}(1-q^{4n+2}). \end{aligned} \tag{4.9}$$

Similarly for the seventh-order mock theta functions

$$\begin{aligned} \mathscr{F}_1(q) &\equiv \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n;q)_n} \\ &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2}(1-q^{2n})\sum_{j=0}^{n-1}q^{j(n-1-j)}, \end{aligned} \tag{4.10}$$

which is a succinct restatement of [10, p. 132, (7.23)] (note that the minimal exponent on q in the nth term is $\approx 7n^2/4$).

Series similar to those on the right-hand sides of (4.9) and (4.10) have arisen previously in the work of Hecke [20], Rogers [26], and Kac-Peterson [23]. The important point is the appearance of an indefinite quadratic form in the exponent of q in the sum.

As predicted in [10, p. 114] such identities have very important applications in subsequent work. In the next section we illustrate perhaps the most striking example by considering a related function also due to Ramanujan but not in his mock theta function list.

5. Partitions with distinct parts. Here we shall consider

$$\sigma(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} = \sum_{n=0}^{\infty} S(n)q^n \tag{5.1}$$
$$\equiv 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots.$$

This function appears in three identities stated in Ramanujan's Lost Notebook [25, p. 14]. These identities were proved in [9], and in [11] two conjectures were posed for $S(n)$. (The conjectures were proved in [15].)

CONJECTURE 1. $\limsup |S(n)| = +\infty$.

CONJECTURE 2. $S(n) = 0$ for infinitely many n.

Now $S(n)$ has a very simple interpretation in terms of partitions. The rank of a partition is defined as the largest part minus the number of parts. Let $\Delta_i(n)$ denote the number of partitions of n into distinct parts with rank $\equiv i \pmod 2$. Then it is easily shown [11] that

$$S(n) = \Delta_0(n) - \Delta_1(n). \tag{5.2}$$

Thus since 3 has two partitions into distinct parts, 3 and 2 + 1, and since each has even rank, we see that $S(3) = 2 - 0 = 2$.

By application of Bailey's Lemma [15, pp. 392–397], it was shown that

$$\sigma(q) = \sum_{n=0}^{\infty} \sum_{|j|\leq n} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1}), \tag{5.3}$$

a result closely resembling (4.9) and (4.10). From (5.3) it is possible to deduce the following identity [15, p. 392]:

$$\sigma(q) = \sum_{n=0}^{\infty} S(n)q^n = \sum_{n=0}^{\infty} T(24n+1)q^n, \tag{5.4}$$

where $T(n)$ is an arithmetic function defined as follows: For (positive or negative) integers $m \equiv 1 \pmod{24}$, consider Pell's equation

$$u^2 - 6v^2 = m. \tag{5.5}$$

Note that if (u, v) is a solution of this equation, then $u \equiv \pm 1 \pmod 6$ and v is even. We call two solutions (u, v) and (u', v') equivalent if

$$u' + v'\sqrt{6} = \pm(5+2\sqrt{6})^r(u+v\sqrt{6}) \tag{5.6}$$

for some integer r. By induction on $|r|$, it is easy to show that if (u, v) and (u', v') are equivalent, then $u + 3v \equiv \pm(u' + 3v') \pmod{12}$. Let $T(m)$ be the excess of the number of inequivalent solutions of (5.5) with $u + 3v \equiv \pm 1 \pmod{12}$ over the number of them with $u + 3v \equiv \pm 5 \pmod{12}$.

Once $T(m)$ is known, it is then an application of the arithmetic of $Q(\sqrt{6})$ to determine $T(m)$ fully [15, p. 401].

THEOREM 5.1. *Let $m \neq 1$ be an integer $\equiv 1 \pmod 6$. Suppose we write $m = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$, where each p_i is either a prime $\equiv 1 \pmod 6$ or the negative of a prime $\equiv 5 \pmod 6$. Then $T(m) = T(p_1^{e_1})T(p_2^{e_2})\cdots T(p_r^{e_r})$, where*

$$T(p^e) = \begin{cases} 0 & \text{if } p \not\equiv 1 \pmod{24} \text{ and } e \text{ is odd,} \\ 1 & \text{if } p \equiv 13 \text{ or } 19 \pmod{24} \text{ and } e \text{ is even,} \\ (-1)^{e/2} & \text{if } p \equiv 7 \pmod{24} \text{ and } e \text{ is even,} \\ e+1 & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = 2, \\ (-1)^e(e+1) & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = -2. \end{cases} \tag{5.7}$$

In particular, $T(m) = 0$ if and only if there is some i for which $p_i \not\equiv 1 \pmod{24}$ and e_i is odd.

From this result and (5.4) one may quickly deduce [15, p. 401]

THEOREM 5.2. *$S(n)$ is almost always 0; that is, the set of n for which $S(n) \neq 0$ has density 0. On the other hand, $S(n)$ takes on every integer value infinitely often.*

This result overwhelmingly proves the two conjectures mentioned earlier.

H. Cohen, B. Gordon, and D. Hickerson have each pointed out that $\sigma(q)$ is not a mock theta function according to Ramanujan's description in §1. This is because [9, p. 157, (1.6)]

$$\sigma(q) = 1 + \sum_{n=0}^{\infty}(-1)^n q^{n+1}(q;q)_n; \tag{5.8}$$

consequently $\sigma(q)$ has a finite limit as $q \to e^{2\pi i m/n}$ radially. Thus $\sigma(q)$ differs from the trivial theta function 0 by a function (namely itself) which is $O(1)$ at all points $e^{2\pi i m/n}$.

6. Cohen's extensions. H. Cohen [17] has extended the results of §6 using algebraic number theory in a very substantial way. Besides the function $\sigma(q)$, the function

$$\sigma^*(q) = 2\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})} \tag{6.1}$$

was treated similarly in [15, pp. 404–405]. Cohen considers

$$\varphi(q) = q^{1/24}\sigma(q) + q^{-1/24}\sigma^*(q) = \sum_{\substack{n\in\mathbf{Z} \\ n\equiv 1 \pmod{24}}} T(n)q^{|n|/24}. \tag{6.2}$$

He then restates several results of [15] in the following:

THEOREM 6.1. *For an ideal* $a = (\alpha) \subset \mathbb{Z}[\sqrt{6}]$ *coprime to* 6, *where* $\alpha = x + y\sqrt{6}$, *define* χ_1 *by*

$$\chi_1(a) = \begin{cases} i^{yx^{-1}}\left(\frac{12}{x}\right) & \text{if } y \text{ is even,} \\ i^{yx^{-1}+1}\left(\frac{12}{x}\right) & \text{if } y \text{ is odd.} \end{cases} \tag{6.3}$$

Then χ_1 *is a well-defined character of order* 2 *and conductor* $4(3+\sqrt{6})$ *on ideals of* $\mathbb{Z}[\sqrt{6}]$, *and furthermore, setting as usual* $\chi_1(a) = 0$ *if* a *is not coprime to* 6, *we have the identity*

$$\varphi(q) = \sum_{a \subset \mathbb{Z}[\sqrt{6}]} \chi_1(a) q^{Na/24}.$$

This result is then embedded elegantly in the algebraic number theory related to the following diagram of number fields:

$$K = \mathbb{Q}(\sqrt{2}, \sqrt{3+\sqrt{3}})$$
$$|$$
$$B = \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$k_1 = \mathbb{Q}(\sqrt{6}) \quad k_2 = \mathbb{Q}(\sqrt{2}) \quad k_3 = \mathbb{Q}(\sqrt{3})$$
$$\mathbb{Q}$$

It is noted [**17**, p. 410] that K/Q is a Galois extension, that $\mathrm{Gal}(K/k_2) \cong \mathbb{Z}/4\mathbb{Z}$, that $\mathrm{Gal}(K/k_1) \cong \mathrm{Gal}(K/k_3) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and consequently that $G = \mathrm{Gal}(K/Q) \cong D_4$, the dihedral group with 8 elements.

It is now possible to go well beyond Theorem 6.1. Indeed [**17**, pp. 410–411], "the character χ_1 corresponds to a degree 1 representation of $\mathrm{Gal}(K/k_1)$. By induction to G one sees immediately that one obtains the unique irreducible representation ρ of degree 2 of G. Furthermore, Artin L-functions being preserved by induction, we have

$$L(\rho, s) = L(\chi_1, s) = \sum_{a \subset \mathbb{Z}[\sqrt{6}]} \chi_1(a)(Na)^{-s}. \tag{6.4}$$

Now ρ, being unique, is also induced by any one of the two characters χ_1, χ_2' of order 4 of $\mathrm{Gal}(K/k_2)$ and by two of the three characters of order 2 of $\mathrm{Gal}(K/k_3)$, say χ_3, χ_3'. Hence we have

$$L(\rho, s) = L(\chi_1, s) = L(\chi_2, s) = L(\chi_2', s) = L(\chi_3, s) = L(\chi_3', s)." \tag{6.5}$$

This powerful observation allows the deduction of 2 new combinatorial formulas each for $\sigma(q)$ and $\sigma^*(q)$. E.g.,

$$\sigma^*(q) = \sum_{\substack{|j| \geq |6n+1|/8 \\ j,n \in \mathbb{Z}}} (-1)^j q^{3j^2 - n(3n+1)/2}, \tag{6.6}$$

$$\sigma^*(q) = \sum_{\substack{|j| \geq |6n+1|/6 \\ j,n \in \mathbb{Z}}} (-1)^{n(n+1)/2+j} q^{2j^2 - n(3n+1)/2}. \tag{6.7}$$

Beyond the combinatorics, the related L-functions possess intriguing properties.

For any $j = 1, 2, 3$, set

$$\Lambda(s) = (1152)^{s/2} \pi^{-s} \Gamma(s/2)^2 L(\chi_j, s). \tag{6.8}$$

Then Λ can be analytically continued to an entire function of order 1 on $\mathbb{C}$ satisfying the functional equation

$$\Lambda(1-s) = -\Lambda(s). \tag{6.9}$$

Cohen points out the importance of the factor $\Gamma(s/2)^2$ in (6.8). He notes that theta functions attached to positive definite binary quadratic forms are holomorphic modular forms of weight 1 on some congruence subgroup of the modular group due primarily to the fact that $\Gamma(s)$ itself is the Γ-factor of the associated L-function. However in Theorem 6.1, $\varphi(q)$ is a theta function attached to the indefinite form $x - 6y^2$. In the case of indefinite forms, the Γ-factor is $\Gamma(s/2)\Gamma((s+1)/2)$, $\Gamma(s/2)^2$, or $\Gamma((s+1)/2)^2$ if the infinity type of the character χ is respectively $+-$ (or $-+$), $++$, or $--$. Note that by the duplication formula

$$\Gamma(s/2)\Gamma((s+1)/2) = \sqrt{\pi} 2^{1-s} \Gamma(s),$$

so that case 1 is essentially the same situation as the positive definite case. This fact serves to explain those identities found by Hecke connecting theta-type series with indefinite quadratic forms to classical modular forms.

While we have briefly summarized Cohen's contributions to the single example of $\varphi(q)$, it is clear from his paper that the methods apply to many similarly related algebraic number fields, and Cohen describes such examples.

7. Hickerson's proof of the Mock Theta Conjectures. In Ramanujan's "Lost" Notebook [**25**, pp. 18–20], we find ten important identities for the ten fifth-order mock theta functions. Each of these identities relates a specific fifth-order mock theta function to either

$$\Phi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}, \tag{7.1}$$

or

$$\Psi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}. \tag{7.2}$$

For example [**25**, p. 19, fifth equation]

$$f_0(q) \equiv \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} = \frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} - 2\Phi(q^2). \tag{7.3}$$

In [**16**], F. Garvan and I show that these ten identities split into two sets of five each and that in each class the five are either true or false together.

Furthermore one identity in each class has an especially simple formulation in terms of partitions. To state these conjectures we require the function $N(b, 5, n)$, the number of partitions of n with rank $\equiv b \pmod 5$.

First Mock Theta Conjecture. $N(1,5,5n) - N(0,5,5n)$ equals the number of partitions of n with unique smallest part and no parts exceeding the double of the smallest part.

Second Mock Theta Conjecture. $2N(2,5,5n+3) - N(1,5,5n+3) - N(0,5,5n+3) - 1$ equals the number of partitions of n with unique smallest part and all other parts at most one larger than the double of the smallest part.

In [13], the constant term method (see [12, Chapter 4] for background on constant term problems) was first applied to the fifth-order mock theta functions with the following wish [13, p. 48]. "It was our initial hope that by exhibiting the fifth-order mock theta functions as constant terms we could make some progress on the Mock Theta Conjectures described in [16]. So far the Mock Theta Conjectures remain unresolved."

Recently, D. R. Hickerson [21] proved much more explicit and powerful constant term identities than those in [13]. From his new method and discoveries he was able to prove the Mock Theta Conjectures.

Hickerson's proof rests on two different dissections of the function

(7.4)
$$B(z) = \frac{z^2(-z, -q/z, q; q)_\infty (z, q^3/z, q^3; q^3)_\infty}{(z, q^2/z; q^2)_\infty},$$

where

$$(A_1, A_2, \ldots, A_r; q)_\infty = (A_1; q)_\infty (A_2; q)_\infty \cdots (A_r; q)_\infty.$$

Greatly extending the sorts of expansions considered in §4, he shows that

(7.5)
$$\begin{aligned} B(z) = q f_0(q) &\left[\sum_{\lambda=-\infty}^{\infty} (-1)^\lambda z^{5\lambda+1} q^{15\lambda^2-9\lambda} + \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda z^{5\lambda+4} q^{15\lambda^2+9\lambda} \right] \\ + f_1(q) &\left[\sum_{\lambda=-\infty}^{\infty} (-1)^\lambda z^{5\lambda+2} q^{15\lambda^2-3\lambda} + \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda z^{5\lambda+3} q^{15\lambda^2+3\lambda} \right] \\ + 2 &\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3} z^{5r+5}}{1 - q^{6r+2} z} + 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3} z^{-5r}}{1 - q^{6r+2} z^{-1}}, \end{aligned}$$

where

(7.6)
$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n}$$

is a second fifth-order mock theta function.

From the form of (7.5) it is clear that one wants

(7.7)
$$B(z) = \sum_{i=0}^{4} z^i B_i(z^5).$$

In particular, it is a straightforward deduction from (7.5) to see that $f_0(q)$ will be directly involved in the constant term for $B_1(z^5)$ that $f_1(z)$ will arise similarly in $B_2(z^5)$. For example, Hickerson derives

(7.8)
$$\begin{aligned} B_1(z^5) = q f_0(q) &\sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{15\lambda^2-9\lambda} z^{5\lambda} \\ &+ 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+21r+5} z^{5r+5}}{1 - q^{30r+10} z^5} \\ &+ 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+39r+11} z^{-5r-5}}{1 - q^{30r+10} z^{-5}}. \end{aligned}$$

On the other hand, Hickerson uses (7.4) to find a pure theta function expansion for $B_1(z^5)$:

(7.9)
$$\begin{aligned} B_1(z^5) = &\frac{q(q^5, q^5, q^{10}; q^{10})_\infty (q^2, q^3, q^5; q^5)_\infty \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{15\lambda^2-9\lambda} z^{5\lambda}}{(q;q)_\infty} \\ &- \frac{2q^3 (q^{10}; q^{10})_\infty^2 \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{5\lambda^2-3\lambda} z^{5\lambda} \sum_{\mu=-\infty}^{\infty} (-1)^\mu q^{15\mu^2-15\mu} z^{5\mu}}{(q^2, q^8, q^{10}; q^{10})_\infty \sum_{\nu=-\infty}^{\infty} (-1)^\nu q^{5\nu^2-5\nu} z^{5\nu}}. \end{aligned}$$

In order to pick out the constant term in comparing (7.8) and (7.9), substantial work remains. Indeed Hickerson accomplishes this with a partial fractions type decomposition of the second term in (7.9). It is only then that he is able to read off (7.3), a result equivalent to the first Mock Theta Conjecture. Similar treatment of $B_2(z^5)$ yields the second conjecture as well.

In a second paper [22], Hickerson applies these methods to the seventh-order mock theta functions. Again his approach is totally successful, and he derives analogs of (7.3) for each of the seventh-order mock theta functions.

As was pointed out in [16, §5], the proof of these conjectures and their seventh-order counterparts establishes formulae that will clearly yield the behavior of the fifth- and seventh-order mock theta functions near the unit circle. Furthermore the asymptotic behavior of the resulting Mordell integrals (see (2.4) as an example, also [6]) should clearly establish that these functions just like the third-order functions are truly mock theta functions [29, p. 78, footnote] in the sense described by Watson. Dean Hickerson has noted a discrepancy between Watson's assertion and Ramanujan's original definition. He notes that for a function to be a mock theta function it must be of the form (θ-function)+$O(1)$ at each root of unity, but there must not be a single theta function that works for all roots of unity. Watson only proves that the third-order functions are not equal to θ-functions; that is, the $O(1)$ terms cannot be identically zero.

8. Combinatorics. The mock theta functions are closely allied with generating functions for certain polynomials that have arisen in the study of partitions [9, 14]. This relationship provides possible combinatorial applications for whatever we learn subsequently about the mock theta functions. In order to present this in a self-contained manner, we shall restrict ourselves to three examples:

$$M\theta_3(q,t) \equiv \sum_{n=0}^{\infty} \frac{t^{2n}q^{n^2}}{(t;q)_{n+1}(tq)_n} = 1 + \sum_{N=1}^{\infty} t^N \sum_{m=0}^{N-1} q^m \begin{bmatrix} N-1 \\ m \end{bmatrix}; \tag{8.1}$$

$$M\theta_5(q,t) \equiv \sum_{n=0}^{\infty} \frac{t^{2n}q^{n^2}}{(t;q)_{n+1}} = \sum_{N=0}^{\infty} t^N \sum_{m=-N}^{N} (-1)^\lambda q^{\lambda(5\lambda+1)/2} \begin{bmatrix} N \\ [\frac{N-5\lambda}{2}] \end{bmatrix}, \tag{8.2}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{(1-q^A)(1-q^{A-1})\cdots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\cdots(1-q)}, \tag{8.3}$$

and

$$[x] = \text{the largest integer not exceeding } x; \tag{8.4}$$

$$M\theta_7(q,t) \equiv \sum_{n=0}^{\infty} \frac{t^{2n}q^{n^2}}{(t;q)_{n+1}(t^2q;q^2)_n} = \sum_{N=0}^{\infty} t^N \sum_{m=-N}^{N} (-1)^m q^{m(7m+1)} \begin{bmatrix} N \\ [\frac{N-7m}{2}] \end{bmatrix}. \tag{8.5}$$

Each of the polynomials appearing in (8.1), (8.2), and (8.5) as coefficients of t^N has an interpretation as a generating function for a certain class of partitions.

In particular, $\sum_{m\geq 0} q^m \left[\begin{smallmatrix} N-1 \\ m \end{smallmatrix}\right]$ is the generating function for all partitions wherein the largest part plus the number of parts is at most N. The polynomial in (8.2) is the generating function for partitions with largest part $\leq N$ and difference at least 2 between parts. The polynomial in (8.5) is subject to a somewhat more complicated interpretation [8, p. 14, (5.17)].

We remark that for $i = 3, 5, 7$

$$\lim_{t\to 1}(1-t)M\theta_i(q,t) = \begin{cases} \left(\sum_{\lambda=-\infty}^{\infty}(-1)^\lambda q^{\lambda(3\lambda-1)/2}\right)^{-1}, & i=3, \\ \frac{\sum_{\lambda=-\infty}^{\infty}(-1)^\lambda q^{\lambda(5\lambda-1)/2}}{\prod_{n=1}^{\infty}(1-q^n)}, & i=5, \\ \frac{\sum_{\lambda=-\infty}^{\infty}(-1)^\lambda q^{\lambda(7\lambda-1)}}{\prod_{n=1}^{\infty}(1-q^n)}, & i=7, \end{cases} \tag{8.6}$$

while

$$2M\theta_i(q,-1) = \begin{cases} f(q) \text{ in (4.6)}, & i=3, \\ f_0(q) \text{ in (4.9)}, & i=5, \\ \text{the first of the 7th-order mock } \vartheta\text{-functions given in §1}, & i=7, \end{cases} \tag{8.7}$$

Thus we see a close tie among mock theta functions in (8.7), classical modular functions in (8.6), and certain polynomial generating functions exemplified by (8.1), (8.2), and (8.5). Most of the mock theta functions can be placed in this sort of three-way relationship [8].

We do not know what more general relationships there are between these combinatorial observations and the work in §§4–7; however the fact that most of the mock theta functions arise as specializations of polynomial generating functions suggests that the study of such relationships may be fruitful.

9. Conclusion. I wish to thank H. Cohen, B. Gordon, and D. Hickerson for helpful conversations and letters that greatly assisted me in the preparation of this paper.

References

1. R. P. Agarwal, *Certain basic hypergeometric identities associated with mock theta functions*, Quart J. Math. Oxford Ser. (2) **20** (1969), 121–128.
2. G. E. Andrews, *On the theorems of Watson and Dragonette for Ramanujan's mock theta functions*, Amer. J. Math. **88** (1966), 454–490.
3. ——, *On basic hypergeometric series, mock theta functions, and partitions*. I, Quart. J. Math. Oxford Ser. (2) **17** (1966), 64–80.
4. ——, *On basic hypergeometric series, mock theta functions, and partitions*. II, Quart J. Math. Oxford Ser. (2) **17** (1966), 132–143.
5. ——, *q-Identities of Auluck, Carlitz and Rogers*, Duke Math. J. **33** (1966), 575–582.
6. ——, *The Mordell integrals and Ramanujan's "lost" notebook*, Lecture Notes in Math., vol. 899, Springer-Verlag, 1981, pp. 10–48.
7. ——, *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. **114** (1984), 267–283.
8. ——, *Combinatorics and Ramanujan's "lost" notebook*, Surveys in Combinatorics 1985, I. Anderson, editor, London Math. Soc. Lecture Note Series, No. 103, Cambridge Univ. Press, 1985, pp. 1–23.
9. ——, *Ramanujan's "lost" notebook*. V: *Euler's partition identity*, Adv. in Math. **61** (1986), 156–164.
10. ——, *The fifth and seventh order mock theta functions*, Trans. Amer. Math. Soc. **293** (1986), 113–134.
11. ——, *Questions and conjectures in partition theory*, Amer. Math. Monthly **93** (1986), 708–711.
12. ——, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra*, CBMS Regional Conf. Ser. in Math., no. 66, Amer. Math. Soc., Providence, R.I., 1986.
13. ——, *Ramanujan's fifth order mock theta functions as constant terms*, Ramanujan Revisited, G. E. Andrews et al., editors, Academic Press, 1988, pp. 47–56.
14. G. E. Andrews, R. J. Baxter, D. Bressoud, W. H. Burge, P. J. Forrester, and G. Viennot, *Partitions with prescribed hook differences*, European J. Combin. **8** (1987), 341–350.
15. G. E. Andrews, F. J. Dyson, and D. R. Hickerson, *Partitions and indefinite quadratic forms*, Invent. Math. **91** (1988), 391–407.

16. G. E. Andrews and F. G. Garvan, *Ramanujan's "lost" notebook. VI: The mock theta conjectures*, Adv. in Math. (to appear).

17. H. Cohen, *q-Identities for Maass waveforms*, Invent. Math. **91** (1988), 409–422.

18. L. Dragonette, *Some asymptotic formulae for the mock theta series of Ramanujan*, Trans. Amer. Math. Soc. **72** (1952), 474–500.

19. G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115; also in [**23**, pp. 276–309].

20. E. Hecke, *Über das Verhalten von* $\sum e^{2i\pi\tau|m^2-2n^2|/8}$ *und ähnlichen Funktionen bei Modulsubstitutionen*, Math. Werke, no. 25, Vandenhoeck und Ruprecht, Göttingen, 1970, pp. 487–498.

21. D. R. Hickerson, *A proof of the mock theta conjectures*, Invent. Math. (to appear).

22. ——, *On the seventh order mock theta functions*, Invent. Math. (to appear).

23. V. G. Kac and D. H. Peterson, *Affine Lie algebras and Hecke modular forms*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), 1057–1061.

24. S. Ramanujan, *Collected papers*, Cambridge Univ. Press, 1927; reprinted by Chelsea, New York, 1962.

25. ——, *The Lost Notebook and other unpublished papers*, Narosa, New Delhi, 1987.

26. L. J. Rogers, *On the expansion of some infinite products*, Proc. London Math. Soc. **24** (1893), 337–352.

27. A Selberg, *Über einige arithmetische Identitäten*, Avhl. Norske Vid. **8** (1936), 23 pp.

28. ——, *Über die Mock-Thetafunktionen siebenter Ordnung*, Arch. Math. og Naturvidenskab **41** (1938), 3–15.

29. G. N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc. **11** (1936), 55–80.

30. ——, *A note on Spence's logarithmic transcendant*, Quart. J. Math. Oxford Ser. **8** (1937), 39–42.

31. ——, *The mock theta functions* (2), Proc. London Math. Soc. (2) **42** (1937), 274–304.

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