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Ramanujan's "Lost" Notebook VI: The Mock Theta Conjectures

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1. INTRODUCTION

In this paper we shall consider only Ramanujan's two families of fifth-order mock theta functions. These functions were briefly described in Ramanujan's last letter to G. H. Hardy [11, pp. 354–355], and G. N. Watson subsequently [12, 13] proved all the assertions about these functions contained in the letter. Subsequently the identities Watson proved were greatly generalized in [2]. Also the fifth-order mock theta functions were shown to have double series expansions involving indefinite quadratic forms [7]. However, there remains a profound mystery about these functions. Namely, no one, including Ramanujan, has ever proved that these functions are indeed mock theta functions and not just some clever combination of theta functions. This problem was described in detail by Watson [12, p. 274] and redescribed in [7, pp. 113–114]. In [5, p. 97, Eqs. (3.5) and (3.6)], a formula from the "Lost" Notebook was pointed out that would, if valid, lead to the establishment of at least some of the fifth-order mock theta functions as truly mock theta functions in the sense of Watson [12; p. 78, footnote]. There are indeed ten such identities, five for each of the two families.

We have two objects in this paper. First we wish to show that the identities in each family are equivalent (i.e., if one is true all five are true, and if one is false all five are false).

To introduce our second objective we recall some basic notions from partition theory [4, p. 142; 8; p. 84]. The rank of a partition is the largest

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part minus the number of parts, and we let $N(b, 5, n)$ denote the number of partitions of n with rank congruent to b modulo 5.

First Mock Theta Conjecture. The number of partitions of $5n$ with rank congruent to 1 modulo 5 equals the number of partitions of $5n$ with rank congruent to 0 modulo 5 plus the number of partitions of n with unique smallest part and all other parts $\leqq$ the double of the smallest part.

If we enumerate the latter described partitions by $\rho_0(n)$, then the First Mock Theta Conjecture reduces to

$$N(1, 5, 5n) = N(0, 5, 5n) + \rho_0(n). \tag{1.1}$$

EXAMPLE. $N(1, 5, 25) = 393$, $N(0, 5, 25) = 390$, and $\rho_0(5) = 3$ with the relevant partitions being $5, 3+2, 2+2+1$.

Second Mock Theta Conjecture.

$$2N(2, 5, 5n+3) = N(1, 5, 5n+3) + N(0, 5, 5n+3) + \rho_1(n) + 1, \tag{1.2}$$

where $\rho_1(n)$ is the number of partitions of n with unique smallest part and all other parts $\leqq$ one plus the double of the smallest part.

EXAMPLE. $2N(2, 5, 23) = 504$; $N(1, 5, 23) = 250$; $N(0, 5, 23) = 251$, and $\rho_0(4) = 2$ with the relevant partitions being $4, 3+1$.

The importance of these seemingly elementary assertions is that each is equivalent to the truth of the corresponding five identities for the related family of fifth-order mock theta functions. In other words, if the First Mock Theta Conjecture is true, then Ramanujan's first family of fifth-order mock theta functions does indeed consist of truly mock theta functions in the sense of Watson [12; p. 78, footnote], and similarly with the Second Mock Theta Conjecture and the second family.

In Section 2, we provide a summary of known results about the fifth-order mock theta functions. In Section 3, we present the ten assertions from the "Lost" Notebook and prove the equivalence within each set of five. In Section 4 we relate the Mock Theta Conjectures to Ramanujan's assertions, and in Section 5 we sketch how to show that the Mock Theta Conjectures imply that the corresponding functions are indeed mock theta functions.

2. Background

Most of what is known about the fifth-order mock theta functions can be found in [2, 7, 13]. Following Watson's lead with conventional notation, we present the fifth-order mock theta functions.

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n} \tag{2.1}$$

$$\phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}(-q;q^2)_n \tag{2.2}$$

$$\psi_0(q)=\sum_{n=1}^{\infty}q^{n(n+1)/2}(-q;q)_{n-1} \tag{2.3}$$

$$F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n} \tag{2.4}$$

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1};q)_n}=1+\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(q^{n+1};q)_{n+1}} \tag{2.5}$$

$$f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n} \tag{2.6}$$

$$\phi_1(q)=\sum_{n=1}^{\infty}q^{n^2}(-q;q^2)_{n-1} \tag{2.7}$$

$$\psi_1(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}(-q;q)_n \tag{2.8}$$

$$F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} \tag{2.9}$$

$$\chi_1(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1};q)_{n+1}}, \tag{2.10}$$

where

$$(A;q)_n=(1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{2.11}$$

Each set of five functions with the same subscript constitutes one of Ramanujan's families.

In this last letter to G. H. Hardy [11, p. 354], Ramanujan asserted relationships holding in each family. These assertions were all proved by G. N. Watson [13] and were generalized in [2]. The identities established by Watson are

$$\phi_0(-q)+\chi_0(q)=2F_0(q) \tag*{(2.12)$_{\mathrm{R}}$}$$

$$f_0(-q)+2F_0(q^2)-2=\vartheta_4(0,q)G(q) \tag*{(2.13)$_{\mathrm{R}}$}$$

$$\phi_0(-q^2)+\psi_0(-q)=\vartheta_4(0,q)G(q) \tag*{(2.14)$_{\mathrm{R}}$}$$

$$2\phi_0(-q^2)-f_0(q)=\vartheta_4(0,q)G(q) \tag*{(2.15)$_{\mathrm{R}}$}$$

$$\psi_0(q) - F_0(q^2) + 1 = q\psi(q^2)H(q^4) \tag{2.16$_R$}$$

$$\chi_1(q) - q^{-1}\phi_1(-q) = 2F_1(q) \tag{2.17$_R$}$$

$$f_1(-q) - 2qF_1(q^2) = \vartheta_4(0, q)H(q) \tag{2.18$_R$}$$

$$q^{-1}\phi_1(-q^2) + \psi_1(-q) = \vartheta_4(0, q)H(q) \tag{2.19$_R$}$$

$$2q^{-1}\phi_1(-q^2) + f_1(q) = \vartheta_4(0, q)H(q) \tag{2.20$_R$}$$

$$\psi_1(q) - qF_1(q^2) = \psi(q^2)G(q^4). \tag{2.21$_R$}$$

The auxiliary functions introduced in these identities are

$$\vartheta_4(0, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_\infty}{(-q; q)_\infty} \qquad [13, \text{p. } 276] \tag{2.22}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)}{(q; q^2)} \qquad [13, \text{p. } 276], \tag{2.23}$$

and the Rogers–Ramanujan functions [13; p. 276]

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty} \tag{2.24}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}. \tag{2.25}$$

As is our practice in this series, we put the subscript "R" on each equation that appears in the "Lost" Notebook or is equivalent to one therein.

3. The Ten Unproved Identities

In addition to the results in Section 2 (proved by Watson), the following assertions also appear in the "Lost" Notebook.

$$M_1(q) \equiv \chi_0(q) - 2 - 3\Phi(q) + A(q) = 0 \tag{3.1$_R$}$$

$$M_2(q) \equiv F_0(q) - 1 - \Phi(q) + q\psi(q^5)H(q^2) = 0 \tag{3.2$_R$}$$

$$M_3(q) \equiv \phi_0(-q) + \Phi(q) - \frac{(q^5; q^5)_\infty G(q^2)H(q)}{H(q^2)} = 0 \tag{3.3$_R$}$$

$$M_4(q) \equiv \psi_0(q) - \Phi(q^2) + qH(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+4n} = 0 \tag{3.4$_R$}$$

$$M_5(q) \equiv f_0(q) + 2\Phi(q^2) - \vartheta_4(0, q^5)G(q) = 0 \tag{3.5$_R$}$$

$$M_6(q) \equiv q\chi_1(q) - 3\Psi(q) - qD(q) = 0 \tag{3.6$_R$}$$

$$M_7(q) \equiv qF_1(q) - \Psi(q) - q\psi(q^5)G(q^2) = 0 \tag{3.7$_R$}$$

$$M_8(q) \equiv \phi_1(-q) + \Psi(q) - \frac{q(q^5;q^5)_\infty G(q)H(q^2)}{G(q^2)} = 0 \tag{3.8$_R$}$$

$$M_9(q) \equiv \psi_1(q) - \frac{1}{q}\Psi(q^2) - G(q)\sum_{n=-\infty}^{\infty}(-q)^n q^{5n^2+2n} = 0 \tag{3.9$_R$}$$

$$M_{10}(q) \equiv f_1(q) + \frac{2}{q}\Psi(q^2) - \vartheta_4(0, q^5)H(q) = 0. \tag{3.10$_R$}$$

Besides the auxiliary functions $\vartheta_4(0, q)$, $\psi(q)$, $G(q)$, and $H(q)$ defined in (2.22)–(2.25), we also have

$$\Phi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}, \tag{3.11}$$

$$\Psi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n} \tag{3.12}$$

$$A(q) = \frac{G(q)^2(q^5;q^5)_\infty}{H(q)} \tag{3.13}$$

$$D(q) = \frac{H(q)^2(q^5;q^5)_\infty}{G(q)}. \tag{3.14}$$

There are several ways to treat these identities. We shall use the method of generalized Lambert series developed in Section 3 of [5]. This method allows us to reduce all our computations to straightforward manipulations of these series. The reductions we require are the following:

$$\begin{aligned}
&\vartheta_4(0, q)G(q)\\
&= \frac{(q^5;q^5)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty}{(-q;q)_\infty}\\
&= \frac{1}{(q^{10};q^{10})_\infty}\frac{(q^5;q^5)^2_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty(-q^2;q^5)_\infty(-q^3;q^5)_\infty}\\
&= \frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q^2)^n}{1+q^{5n+1}} \qquad \text{(by [5, Eq. (6.1)]).}
\end{aligned} \tag{3.15}$$

Similarly

$$\vartheta_4(0, q)H(q) = \frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}q^{2n+1}}{1+q^{5n+4}} \tag{3.16}$$

$$\psi(q^2)H(q^4)$$
$$=\frac{1}{(q^{10};q^{10})_\infty}\frac{(q^4;q^{20})_\infty(q^{16};q^{20})_\infty(q^{20};q^{20})_\infty^2}{(q^2;q^{20})_\infty(q^6;q^{20})_\infty(q^{14};q^{20})_\infty(q^{18};q^{20})_\infty}$$
$$=\frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{20n+14}} \quad \text{(by [5, Eq. (6.1)])}. \tag{3.17}$$

Similarly

$$\psi(q^2)G(q^4)=\frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{20n+6}} \tag{3.18}$$

$$A(q)=\frac{1}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{5n+1}} \quad \text{(by [5, Eq. (3.7)}_{\text{R}}\text{])} \tag{3.19}$$

$$D(q)=\frac{1}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{5n+2}} \quad \text{(by [5, Eq. (3.8)}_{\text{R}}\text{])} \tag{3.20}$$

$$\psi(q^{10})H(q^4)=\frac{-q^6(q^{20};q^{20})_\infty^2(q^{-6};q^{20})_\infty(q^{26};q^{20})_\infty}{(q^8;q^{20})_\infty(q^{12};q^{20})_\infty(q^6;q^{20})_\infty(q^{14};q^{20})_\infty}\cdot\frac{1}{(q^{10};q^{10})_\infty}$$
$$=\frac{-1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{14n+6}}{1-q^{20n+12}} \quad \text{(by [5, Eq. (6.1)])}. \tag{3.21}$$

Similarly

$$\psi(q^{10})G(q^4)$$
$$=\frac{-1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{12n+8}}{1-q^{20n+16}} \tag{3.22}$$

$$\frac{(q^5;q^5)_\infty G(q^2)H(q)}{H(q^2)}$$
$$=\frac{1}{(q^5;q^5)_\infty}\frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty(q^5;q^5)_\infty^2}{(q;q^5)_\infty(q^4;q^5)_\infty(-q;q^5)_\infty(-q^4;q^5)_\infty}$$
$$=\frac{1}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q)^n}{1-q^{5n+1}} \quad \text{(by [5, Eq. (6.1)])}. \tag{3.23}$$

Similarly

$$\frac{(q^5;q^5)_\infty H(q^2)G(q)}{G(q^2)}=\frac{1}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q^2)^n}{1-q^{5n+2}} \tag{3.24}$$

$$H(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+4n}$$
$$= \frac{1}{(q^{10}; q^{10})_\infty} \frac{(q; q^{10})_\infty (q^9; q^{10})_\infty (q^{10}; q^{10})_\infty^2}{(q^2; q^{10})_\infty (q^8; q^{10})_\infty (q^3; q^{10})_\infty (q^7; q^{10})_\infty}$$
$$= \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{7n}}{1-q^{10n+2}} \qquad \text{(by [5, Eq. (6.1)]).} \tag{3.25}$$

Similarly

$$G(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+2n}$$
$$= \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{10n+6}} \tag{3.26}$$
$$\vartheta_4(0, q^5) G(q)$$
$$= \frac{1}{(q^{10}; q^{10})_\infty} \frac{(q^5; q^5)_\infty^2 (-q^2; q^5)_\infty (-q^3; q^5)_\infty}{(q; q^5)_\infty (q^4; q^5)_\infty (-q^2; q^5)_\infty (-q^3; q^5)_\infty}$$
$$= \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{5n+1}} \qquad \text{(by [5, Eq. (6.1)]),} \tag{3.27}$$

and finally

$$\vartheta_4(0, q^5) H(q) = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1-q^{5n+3}} \qquad \text{(by [5, Eq. (6.1)]).} \tag{3.28}$$

THEOREM 1. *Identities* $(3.1)_R$–$(3.5)_R$ *are equivalent, and identities* $(3.6)_R$–$(3.10)_R$ *are equivalent.*

Proof. We begin with

$$M_5(q) - 2M_3(q^2)$$
$$= f_0(q) - 2\phi_0(-q^2) - \vartheta_4(0, q^5) G(q) + \frac{2(q^{10}; q^{10})_\infty G(q^4) H(q^2)}{H(q^4)}$$
$$= -\vartheta_4(0, q) G(q) - \vartheta_4(0, q^5) G(q)_\infty + \frac{2}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{20n+2}} \qquad \text{(by } (2.15)_R \text{ and (3.23))}$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \left(- \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1+q^{5n+1}} \right.$$

$$\left. + 2 \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{10n+2}} \right) - \vartheta_4(0, q^5) G(q) \qquad \text{(by (3.15))}$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \left(- \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n (1-q^{5n+1})}{1-q^{10n+2}} \right.$$

$$\left. + 2 \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{10n+2}} \right) - \vartheta_4(0, q^5) G(q)$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n (1+q^{5n+1})}{1-q^{10n+2}} - \vartheta_4(0, q^5) G(q)$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{5n+1}} - \vartheta_4(0, q^5) G(q)$$

$$= 0 \qquad \text{(by (3.27))}. \tag{3.29}$$

Next

$$M_5(q) + 2M_2(q^2)$$

$$= f_0(q) + 2F_0(q^2) - 2 + 2q^2 \psi(q^{10}) H(q^4) - \vartheta_4(0, q^5) G(q)$$

$$= \vartheta_4(0, -q) G(-q) + 2q^2 \psi(q^{10}) H(q^4)$$

$$- \vartheta_4(0, q^5) G(q) \qquad \text{(by (2.13))}$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \left(\sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-(-1)^n q^{5n+1}} \right.$$

$$\left. - \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{5n+1}} \right) + 2q^2 \psi(q^{10}) H(q^4) \qquad \text{(by (3.15))}$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \left(\sum_{n=-\infty}^{\infty} \frac{(-q^2)^{2n+1}}{1+q^{10n+6}} \right.$$

$$\left. - \sum_{n=-\infty}^{\infty} \frac{(-q^2)^{2n+1}}{1-q^{10n+6}} \right) + 2q^2 \psi(q^{10}) H(q^4)$$

$$= \frac{2}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{14n+8}}{1-q^{20+12}} + 2q^2 \psi(q^{10}) H(q^4)$$

$$= 0 \qquad \text{(by (3.21))}. \tag{3.30}$$

We now consider

$$\begin{aligned}
&M_4(-q)+M_3(q^2)\\
&\quad=\psi_0(-q)+\phi_0(-q^2)-\frac{(q^{10};q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}\\
&\qquad-qH(-q)\sum_{n=-\infty}^{\infty} q^{5n^2+4n}\\
&\quad=\vartheta_4(0,q)G(q)-\frac{(q^{10};q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}\\
&\qquad-qH(-q)\sum_{n=-\infty}^{\infty} q^{5n^2+4n}\\
&\quad=\frac{1}{(q^{10};q^{10})_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-q^2)^n}{1-q^{5n+1}}-\sum_{n=-\infty}^{\infty}\frac{(-q^2)^n}{1-q^{10n+2}}\right)\\
&\qquad-qH(-q)\sum_{n=-\infty}^{\infty} q^{5n^2+4n}\qquad\text{(by (3.15) and (3.23))}\\
&\quad=\frac{1}{(q^{10};q^{10})_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-q^2)^n(1+q^{5n+1})}{1-q^{10n+2}}\right.\\
&\qquad\left.-\sum_{n=-\infty}^{\infty}\frac{(-q^2)^n}{1-q^{10n+2}}\right)-qH(-q)\sum_{n=-\infty}^{\infty} q^{5n^2+4n}\\
&\quad=\frac{q}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q^7)^n}{1-q^{10n+2}}\\
&\qquad-qH(-q)\sum_{n=-\infty}^{\infty} q^{5n^2+4n}\\
&\quad=0\qquad\text{(by (3.25)).}
\end{aligned}\tag{3.31}$$

Finally

$$\begin{aligned}
&M_1(q^2)+3M_3(q^2)\\
&\quad=\chi_0(q^2)+3\phi_0(-q^2)-2+A(q^2)\\
&\qquad-\frac{3(q^{10};q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}\\
&\quad=2F_0(q^2)+2\phi_0(-q^2)-2+A(q^2)\\
&\qquad-\frac{3(q^{10};q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}\qquad\text{(by (2.12))}
\end{aligned}$$

$$= \vartheta_4(0, -q)G(-q) + \vartheta_4(0, q)G(q) + A(q^2) - \frac{3(q^{10}; q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}$$

$$= \frac{1}{(q^{10}; q^{10})_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-(-1)^n q^{5n+1}} + \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1+q^{5n+1}} + \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{10n+2}} - 3\sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{10n+2}}\right) \quad \text{(by (3.15), (3.19), and (3.23))}$$

$$= \frac{1}{(q^{10}; q^{10})_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{10n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1+q^{10n+6}} + \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1+q^{10n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1+q^{10n+6}} - 2\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{20n+2}} + 4\sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1-q^{20n+12}}\right)$$

$$= \frac{1}{(q^{10}; q^{10})_\infty}\left(2\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{20n+2}} - 2\sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1+q^{10n+6}} - 2\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{20n+2}} + 4\sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1-q^{20n+12}}\right)$$

$$= \frac{1}{(q^{10}; q^{10})_\infty}\left(-2\sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1+q^{10n+6}} + 2\sum_{n=-\infty}^{\infty} q^{4n+2}\left(\frac{1}{1+q^{10n+6}} + \frac{1}{1-q^{10n+6}}\right)\right)$$

$$= \frac{2}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1-q^{10n+6}}$$

$$= \frac{2q^2}{(q^{10}; q^{10})_\infty} \frac{(q^{10}; q^{10})_\infty^2 (1; q^{10})_\infty (q^{10}; q^{10})_\infty}{(q^6; q^{10})_\infty^2 (q^4; q^{10})_\infty^2} \quad \text{(by [5, Eq. (6.1)])}$$

$$= 0 \tag{3.32}$$

as desired.

Now we remark that if any of the $M_i(q) \equiv 0$ then (3.29)–(3.32) imply that all are zero. Conversely if any $M_i(q) \not\equiv 0$ then none is identically zero.

Exactly parallel arguments establish the identities

$$M_{10}(q) - \frac{2}{q} M_8(q^2) = 0 \tag{3.33}$$

$$M_{10}(q) + \frac{2}{q} M_7(q^2) = 0 \tag{3.34}$$

$$M_9(q) + \frac{1}{q} M_8(q) = 0 \tag{3.35}$$

and

$$M_6(q) + 3M_8(q) = 0. \tag{3.36}$$

Hence Theorem 1 is established.

4. The Mock Theta Conjectures

The importance of the Mock Theta Conjectures is imbedded in the following:

THEOREM 2. *The First Mock Theta Conjecture is equivalent to*

$$M_1(q) = 0, \tag{4.1}$$

and the second is equivalent to

$$M_6(q) = 0. \tag{4.2}$$

Proof. In [9], F. G. Garvan showed that some of the main identities used by Atkin and Swinnerton-Dyer [8] in their proofs of the Dyson conjectures actually occur in Ramanujan's "Lost" Notebook. Defining

$$R_{b,c}(d) = \sum_{n=0}^{\infty} (N(b, 5, 5n+d) - N(c, 5, 5n+d))\, q^n, \tag{4.3}$$

we have

$$R_{1,2}(0) = \Phi(q) \qquad \text{(Garvan [9, (2.7.39)])} \tag{4.4}$$

and

$$R_{2,0}(3) = \frac{1}{q} \Psi(q) \qquad \text{(Garvan [9, (2.7.40)])}. \tag{4.5}$$

Furthermore Atkin and Swinnerton-Dyer showed

$$R_{0,2}(0)+2R_{1,2}(0)=A(q)-1 \qquad [8, \text{p. } 101, \text{Eq. } (6.12)] \tag{4.6}$$

$$R_{0,1}(3)+R_{0,2}(3)=D(q)-1 \qquad [8, \text{p. } 101, \text{Eq. } (6.18)]. \tag{4.7}$$

Now if we rewrite the First Mock Theta Conjecture in generating function form (see [3, Lemma 2] for $\chi_0(q)$), we have equivalently

$$\chi_0(q)-1=R_{1,0}(0). \tag{4.8}$$

But

$$\begin{aligned} M_1(q) &= \chi_0(q)-2-3\Phi(q)+A(q) \\ &= \chi_0(q)-1+R_{0,2}(0)+2R_{1,2}(0)-3R_{1,2}(0) \\ &= \chi_0(q)-1-R_{1,0}(0). \end{aligned} \tag{4.9}$$

Hence (4.8), the First Mock Theta Conjecture, is equivalent to $M_1(q)=0$ by (4.9).

In the same manner, we note that the Second Mock Theta Conjecture is equivalent to

$$\chi_1(q)-1=R_{2,1}(3)+R_{2,0}(3). \tag{4.10}$$

However,

$$\begin{aligned} \frac{1}{q}M_6(q) &= \chi_1(q)-\frac{3}{q}\Phi(q)-D(q) \\ &= \chi_1(q)-3R_{2,0}(3)-R_{0,1}(3)-R_{0,2}(3)-1 \\ &= \chi_1(q)-1-R_{2,0}(3)-R_{2,1}(3). \end{aligned} \tag{4.11}$$

Hence (4.10), the Second Mock Theta Conjecture, is equivalent to $M_6(q)=0$ by (4.11).

Therefore Theorem 2 is established.

5. $\Phi(q)$ AND $\Psi(q)$ UNDER MODULAR TRANSFORMATIONS

We shall present a brief sketch of how to obtain the modular transformations of $\Phi(q)$ and $\Psi(q)$.

We note that F. G. Garvan [8, (2.7.36)] has shown (see also [5, (3.3), (3.5), and (3.6)])

$$q^{-1}\Phi(q)=\frac{1}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{15n(n+1)/2}}{1-q^{5n+1}}. \tag{5.1}$$

Hence if $q=e^{-\alpha}$, then

$$q^{1/15}\Phi(q^{1/5})(q;q)_\infty=\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(3/2)(n+1/3)^2}}{\sinh((\alpha/2)(n+1/5))}. \tag{5.2}$$

The behavior of $\Phi(q)$ under modular transformations is now determined in exactly the same way it was for

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(3/2)n^2}}{\cosh((\alpha/2)n)}$$

in Theorem 2.1 of [1] using the Poisson summation formula. Comparable transformations were done by Kothmann in [10] for the generating functions associated with $N(b, 5, n)$.

If the mock theta conjectures are proved, it may well be worthwhile to flesh out the sketch given in this section. However, the first order of business now is to prove the Mock Theta Conjectures.

6. Conclusion

It is typical of Ramanujan to discover truly surprising results. It should be emphasized that while the Mock Theta conjectures are simple assertions in arithmetic, their implications are quite profound. Indeed they imply the quite subtle behavior of the fifth-order mock theta functions in the neighborhood of the unit circle.

Note added in proof. D. R. Hickerson has proved the Mock Theta Conjectures. His work will appear shortly in Inventiones Mathematicae.

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