THE RAMANUJAN JOURNAL, 7, 385–400, 2003

Partitions: At the Interface of q-Series and Modular Forms

GEORGE E. ANDREWS*
The Pennsylvania State University, University Park, Pennsylvania 16802

In memory of Robert A. Rankin

Received February 10, 2003; Accepted February 20, 2003

Abstract. In this paper we explore five topics from the theory of partitions: (1) the Rademacher conjecture, (2) the Herschel-Cayley-Sylvester formulas, (3) the asymptotic expansions of E.M. Wright, (4) the asymptotics of mock theta function coefficients, (5) modular transformations of q-series.

Key words: partitions, q-series, modular forms

2000 Mathematics Subject Classification: Primary—11P82, 11P83, 11P81, 05A30, 33D15

1. Introduction

The twentieth century found two major currents in the theory of partitions. The first is characterized by the role played by modular forms. Seminal papers in this vein include the Hardy-Ramanujan paper on $p(n)$ [20], the proofs of the Ramanujan congruences by Watson [41] and Atkin [7], and the many recent contributions by Ono [28, 29] to name only a few. It should be noted that Ramanujan himself proved the congruence for general powers of 5 and likely had the corresponding proof for powers of 7 (see [12] for details).

The second has been extensive study of the application of basic hypergeometric series or q-series to partitions. This has its genesis in the work of MacMahon [25] and Schur [37]. In the 1960's a variety of results were discovered; these are described in a survey by Alder [1] (cf. [3]).

Surprisingly perhaps, there has not been a large amount of interplay between these two themes. In this paper, I hope to survey topics that lie at the interface of modular forms and q-series. Some concern important and unjustly neglected conjectures. Each should suggest a number of research possibilities.

Section 2 will be devoted to Rademacher's conjecture [32, p. 302] which concerns his partial fraction decomposition for the generating function of $p(n)$. Section 3 considers alternative representations of the restricted partition function $p(n, m)$. Section 4 looks at a sequence of papers by Wright [42–44]. The third of these is quite unlike any other in the history of partition asymptotics and foreshadows many possibilities. In Section 5, we

*Partially supported by National Science Foundation Grant DMS-9206993.

look again at problems posed by Dragonette [15] for the asymptotics of the coefficients of Ramanujan's mock theta functions. The sixth section considers the efforts by Ehrenpreis [16] and others to develop a modular type transformation theory for q-series. We conclude with brief synthesizing observations.

2. The Rademacher conjecture

Near the end of his book, Topics in Analytic Number Theory [32, p. 300], Rademacher derives the following partial fraction decomposition for the generating function of the partition function $p(n)$.

2.1. Rademacher's partial fraction decomposition

We use the following notation for forward differences

$$\Delta_\alpha f(\alpha) = f(\alpha+1) - f(\alpha),$$

and for nonnegative integers j

$$\Delta_\alpha^j f(\alpha) = \sum_{h=0}^{j} (-1)^h \binom{j}{h} f(\alpha + j - h).$$

Also

$$L_j(y) = \sum_{n=0}^{\infty} \frac{y^n}{n!\,\Gamma(j+n+1)}.$$

Then

$$F(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{h \bmod k \\ (h,k)=1}} w_{h,k} \times \sum_{j=0}^{\infty} \Delta_\alpha^j \left[L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1) \right) \right] \left(x e^{-\frac{2\pi i h}{k}} - 1 \right)^{-j-1}, \tag{2.1}$$

where $\alpha = 1/24$, w_{hk} is a root of unity [32, p. 269] and

$$F(x) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} p(n) x^n, & |x| < 1 \\ 0, & |x| > 1. \end{cases}$$

Hardy and Ramanujan had several other results of this sort, with the generating function being equal to 0 outside the unit circle. See their last published paper. In particular, see Theorem 3 on page 319 of Ramanujan's *Collected Papers*. In letters to Hardy written by

Ramanujan from nursing homes, he gives several more examples. Some of these results are recounted and proved by Berndt et al. in [11].

Immediately after proving the above expansion for $F(x)$, Rademacher [29, p. 301] observes that if $|x| < 1$

$$\lim_{N\to\infty} \frac{1}{(1-x)(1-x^2)\dots(1-x^N)} = \frac{1}{\prod_{m=1}^{\infty}(1-x^m)} = \sum_{n=0}^{\infty} p(n)x^n = F(x), \tag{2.3}$$

and if $|x| > 1$

$$\lim_{N\to\infty} \frac{1}{(1-x)(1-x^2)\dots(1-x^N)} = 0 = F(x). \tag{2.4}$$

Of course, one can perform the classical partial fraction decomposition of this finite product. Indeed, we may write

$$\frac{1}{\prod_{m=1}^{N}(1-x^m)} = \sum_{\substack{0\le h<k\le n \\ (h,k)=1}} \sum_{j=1}^{\lfloor N/k\rfloor} \frac{C_{hkj}(N)}{\left(x-e^{\frac{2\pi i h}{k}}\right)^j}. \tag{2.5}$$

As Rademacher notes, there is a natural identification suggested by these two expansions.

Rademacher's Conjecture [32, p. 302].

$$\lim_{N\to\infty} C_{hkj}(N)$$

exists and is given by

$$C_{hkj}(\infty) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \frac{w_{hk}e^{\frac{2\pi i h j}{k}}}{k^{5/2}} \Delta_{\alpha}^{j-1} L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right),$$

where $\alpha = 1/24$, and

$$L_{3/2}(-y^2) = -\frac{1}{2\sqrt{\pi}\,y}\frac{d}{dy}\left(\frac{\sin 2y}{y}\right) = -\frac{1}{2\sqrt{\pi}\,y^2}\left(2\cos 2y - \frac{\sin 2y}{y}\right).$$

Rademacher goes on to provide a small table of values (reprinted here with one correction):

N	C_{011}	C_{012}	C_{121}
1	-1	0	0
2	$-\frac{1}{4} = -0.25$	$\frac{1}{2} = 0.5$	$\frac{1}{4} = 0.25$
3	$-\frac{17}{72} = -0.23611\ldots$	$\frac{1}{4} = 0.25$	$\frac{1}{8} = 0.125$
4	$-\frac{17}{72} = -0.23611\ldots$	$\frac{59}{288} = 0.204861\ldots$	$\frac{1}{8} = 0.125$
5	$-\frac{20831}{86400} = -0.24101\ldots$	$\frac{3}{16} = 0.1875$	$\frac{13}{128} = 0.1015625$

and he lists explicitly three of the conjectured limits:

$$C_{011}(\infty) = -\frac{6}{25}\left(1 + \frac{2\sqrt{3}}{5\pi}\right) = -0.273339\ldots,$$
$$C_{012}(\infty) = \frac{24}{25 \cdot 49}\left(6 + \frac{109\sqrt{3}}{35\pi}\right) = 0.15119\ldots,$$
$$C_{121}(\infty) = -\frac{\sqrt{6}}{25}\left(\cos\frac{5\pi}{12} - \frac{12}{5\pi}\sin\frac{5\pi}{12}\right) = 0.046941.$$

Of these coefficients he says [32, p. 301]: "No explicit formula for $C_{hkl}(N)$ is known, not even for the simplest case $h = 0$, $k = 1$, $l = 1$, and variable N."

In the hope of stimulating serious consideration of this problem, I will present an explicit formula for $C_{011}(N + 1)$ below. The same procedure can obviously be applied to provide a formula for $C_{hkj}(N + 1)$ in general. However, owing to the fact that we cannot prove Rademacher's conjecture from our result, we shall not complicate matters by presenting the full formula for $C_{hkj}(N + 1)$. It should be pointed out that Rademacher's conjecture lies at the interface of the theory of modular forms and the theory of q-series. Thus progress on this problem may require contributions from two areas that have had less contact in the past than might have been expected or hoped for.

Theorem 1. *With $\rho_j = e^{2\pi i/j}$, and $H_N(x_1, \ldots, x_n)$ the N-th homogeneous symmetric function of $x_1, x_2, \ldots, x_n$, then*

$$C_{011}(N+1) = \frac{-1}{(N+1)!}\sum_{h_1=1}^{1}\sum_{h_2=1}^{2}\cdots\sum_{h_N=1}^{N}\left(\prod_{i=1}^{N}\rho_{i+1}^{-h_i}\right)H_N\left(\frac{\rho_2^{h_1}}{1-\rho_2^{h_1}}, \ldots, \frac{\rho_{N+1}^{h_N}}{1-\rho_{N+1}^{h_N}}\right).$$

Proof: Noting that (2.5) may be presented as

$$\frac{1}{(x;x)_{N+1}} = \frac{C_{01N+1}(N+1)}{(x-1)^{N+1}} + \frac{C_{01N}(N+1)}{(x-1)^{N}} + \cdots + \frac{C_{011}(N+1)}{(x-1)}$$
$$+ \sum_{\substack{0 \le h < k \le N+1 \\ (h,k)=1 \\ k>1}} \sum_{j=1}^{\lfloor n/k \rfloor} \frac{C_{hkj}(N+1)}{(x - e^{2\pi i h/k})^j},$$

we find by multiplying by $(x-1)^{N+1}$, differentiating N times and then setting $x=1$ that

$$N!\,C_{011}(N+1)=\left.\frac{d^N}{dx^N}\frac{(x-1)^{N+1}}{(x;x)_{N+1}}\right|_{x=1},$$

where

$$(A;q)_N=(1-A)(1-Aq)\dots(1-Aq^{N-1}).$$

So we clearly need the r-dimensional Leibnitz rule

$$\frac{d^N}{dx^N}f_1\,f_2\dots f_r=\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\dots+n_r=N}}\binom{N}{n_1,n_2,\dots,n_r}f_1^{(n_1)}f_2^{(n_2)}\dots f_r^{(n_r)}.$$

Now recalling that

$$\frac{1}{1-q^M}=\frac{1}{M}\sum_{j=0}^{M-1}\frac{1}{\left(1-\rho_M^j q\right)},$$

where $\rho_M=e^{2\pi i/M}$, we easily deduce

$$\frac{1-q}{1-q^M}=\frac{1}{M}\sum_{j=1}^{M-1}\frac{\left(1-\rho_M^{-j}\right)}{\left(1-\rho_M^j q\right)}.$$

Consequently

$$\frac{d^k}{dq^k}\frac{(1-q)}{(1-q^M)}=\frac{k!}{M}\sum_{j=1}^{M-1}\frac{\left(1-\rho_M^{-j}\right)\rho_M^{jk}}{\left(1-\rho_M^j q\right)^{k+1}}.$$

Next we recall the standard representation of the homogeneous symmetric function in terms of monomials:

$$H_N(X_1,X_2,\dots,X_N)=\sum_{\substack{n_1,n_2,\dots,n_N\geq 0\\ n_1+n_2+\dots+n_N=N}}X_1^{n_1}X_2^{n_2}\dots X_N^{n_N}.$$

As a result,

$$\begin{aligned}
&N!\,C_{011}(N+1)\\
&=(-1)^{N+1}\sum_{\substack{n_1,\dots,n_N\geq 0\\ n_1+\dots+n_N=N}}\binom{N}{n_1,n_2,\dots,n_N}\left(\frac{d^{n_1}}{dx^{n_1}}\frac{(1-x)}{(1-x^2)}\right)\cdots\left.\left(\frac{d^{n_N}}{dx^{n_N}}\frac{(1-x)}{(1-x^{N+1})}\right)\right|_{x=1}\\
&=(-1)^{N+1}\sum_{\substack{n_1,\dots,n_N\geq 0\\ n_1+\dots+n_N=N}}\binom{N}{n_1,n_2,\dots,n_N}\left(\frac{n_1!}{2}\sum_{j_1=1}^{2-1}\frac{\left(1-\rho_1^{-j_1}\right)\rho_2^{j_1n_1}}{\left(1-\rho_2^{j_1}q\right)^{n_1+1}}\right)
\end{aligned}$$

$$\cdots\left(\frac{n_N!}{(N+1)}\sum_{j_N=1}^{(N+1)-1}\frac{\left(1-\rho_{N+1}^{-j_N}\right)\rho_{N+1}^{j_N n_N}}{\left(1-\rho_{N+1}^{j_N}\right)^{n_N+1}}\right)\Bigg|_{x=1}$$

$$=\frac{(-1)^{N+1}}{(N+1)!}\sum_{j_1=1}^{1}\sum_{j_2=1}^{2}\cdots\sum_{j_N=1}^{N}\frac{\left(1-\rho_2^{-j_1}\right)}{\left(1-\rho_2^{j_1}\right)}\frac{\left(1-\rho_3^{-j_2}\right)}{\left(1-\rho_3^{j_2}\right)}\cdots\frac{\left(1-\rho_{N+1}^{-j_N}\right)}{\left(1-\rho_{N+1}^{j_N}\right)}$$

$$\times\sum_{\substack{n_1,\ldots,n_N\geq 0\\ n_1+\cdots+n_N=N}}\left(\frac{\rho_{N+1}^{j_N}}{1-\rho_{N+1}^{j_N}}\right)^{n_N}\cdots\left(\frac{\rho_2^{j_1}}{1-\rho_2^{j_1}}\right)^{n_1}$$

$$=\frac{-1}{(N+1)!}\sum_{j_1=1}^{1}\sum_{j_2=1}^{2}\cdots\sum_{j_N=1}^{N}\left(\prod_{i=1}^{N}\rho_{i+1}^{-j_i}\right)H_N\left(\frac{\rho_2^{j_1}}{1-\rho_2 j_1},\ldots,\frac{\rho_{N+1}^{j_N}}{1-\rho_{N+1}^{j_N}}\right),$$

as desired. □

It may be reasonably objected that Theorem 1 provides little hope of proving Rademacher's Conjecture. At the most it may suggest the value of finding better formulas for $C_{hkj}(N)$.

Ehrenpreis in [16, p. 317] mentions that his student, Jane Friedman, studied computer algorithms for this problem but he states that: "Unfortunately, the computer study proved inconclusive." While this is still true, we can add a few more values to those already computed for $C_{011}(N)$:

N	C_{011}
6	$-\frac{85823}{345600}=-0.24833$
7	$-\frac{19554517}{76204800}=-0.25660$
8	$-\frac{80858443}{304819200}=-0.26527$

3. Formulae for $p(n, m)$

We denote the number of partitions of n into at most m parts by $p(n, m)$. There is an extensive literature concerning formulae for $p(n, m)$, there is a full account of the history up to 1958 given in the Royal Society Table of Partitions [19] which is mostly devoted to extensive tables of $p(n, m)$. Many authors, among them, Paoli [30], DeMorgan [14], Herschel [21], Cayley [13], Sylvester [38], Glaisher [17] and more recently Arkin [6], have all made major contributions to the study of exact formulae for $p(n, m)$.

When one proceeds to examine the actual formulae, one is struck by a substantial difference that arises between $m \leq 4$ and $m > 4$. Namely, we see immediately that

$$p(n, 1) = 1, \tag{3.1}$$

$$p(n, 2) = \left\lfloor\frac{n+2}{2}\right\rfloor. \tag{3.2}$$

DeMorgan [14] first proved that

$$p(n,3) = \left\{\frac{(n+3)^2}{12}\right\}, \tag{3.3}$$

and Glösel proved [18, p. 138]:

$$p(n,4) = \left\{\left\lfloor\frac{n+4}{2}\right\rfloor^2\left(3\left\lfloor\frac{n+9}{2}\right\rfloor - \left\lfloor\frac{n+10}{2}\right\rfloor\right)\frac{1}{36}\right\}, \tag{3.4}$$

where $\lfloor x\rfloor$ is the greatest integer $\leq x$, and $\{x\}$ is the nearest integer to x.

Now there are several immediate observations to make about these formulae. First of all, they are all obviously real numbers, indeed positive integers. Furthermore, they are given in one line and are computed with familiar functions. We remark that (3.4) could be improved if one could eliminate some of the instances of the greatest integer function.

For $m > 5$ (Glösel has a lengthy formula for $p(n,5)$ that is of the same type as (3.1)–(3.4)), one runs into the stark fact that one needs representations of periodic sequences. As is well-known [31, p. 76], such sequences are representable by Finite Fourier series. So, for example, an alternative to (3.4) is [26, p. 153]

$$\begin{aligned} p(n,4) = {} & \frac{1}{144}(n+5)^3 - \frac{5}{96}(n+5) + \frac{1}{32}(-1)^n(n+5) \\ & + \frac{1}{27}\left(r_3^n + r_3^{-n} - r_3^{n+1} - r_3^{-n-1}\right) \\ & + \frac{1}{16}i^n(1+(-1)^n), \end{aligned}$$

where $r_3 = e^{2\pi i/3}$.

Now this formula is rather disturbing. A little thought reveals that it is real because it is equal to its complex conjugate. However, its integrality is far from obvious.

One way to avoid this difficulty is to introduce the circulator notation for a periodic sequence of period m [19, p. xvii]

$$(A_0, A_1, \ldots, A_{m-1})crm_n = A_i \quad \text{if } n \equiv i \pmod{m}. \tag{3.6}$$

Computationally this approach avoids complex numbers, but it merely introduces notation for periodic sequences.

In this new notation we find [19, p. xxvii, Eq. (6.7)]

$$\begin{aligned} p(n,4) = {} & \frac{1}{24}\binom{n+6}{3} + \left(\frac{1}{9} - \frac{1}{16}(2,3)cr2_n\right)(n+5) \\ & + \frac{1}{9}(1,0,-1)cr3_n + \frac{1}{8}(1,0,-1,0)cr4_n. \end{aligned} \tag{3.7}$$

In this section, we shall make the simple observation that periodic sequences may also be represented by the greatest integer function. From that knowledge it is easy to derive formulas such as

$$p(n,4) = \left\{(n+5)\left(n^2+n+22+18\left\lfloor\frac{n}{2}\right\rfloor\right)\Big/144\right\}, \tag{3.8}$$

and

$$p(n,5) = \left\{(n+8)\left(n^3+n+22n^2+44n+248+180\left\lfloor\frac{n}{2}\right\rfloor\right)\Big/2880\right\}, \tag{3.9}$$

Note that these formulas are much more in the spirit of (3.1)–(3.3) than is (3.5). The formula (3.9) is strikingly simpler than any of the several given in [19]. Furthermore such formulas for $p(n, m)$ continue for larger m. For example,

$$\begin{aligned} p(n,6) = \Bigg\{(n+11)\Bigg(&(6n^4+249n^3+2071n^2-4931n+40621)/518400 \\ &+\left\lfloor\frac{n}{2}\right\rfloor(n+10)/192+\left(\left\lfloor\frac{n+1}{3}\right\rfloor+2\left\lfloor\frac{n}{3}\right\rfloor\right)\Big/54\Bigg)\Bigg\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} p(n,7) = \Bigg\{(n+14)\Bigg(&(n^5+70n^4+1785n^3-15365n^2+9702n+277032\big/3628800 \\ &+\left\lfloor\frac{n}{2}\right\rfloor(n+14)/384+\left\lfloor\frac{n}{3}\right\rfloor\Big/54\Bigg)\Bigg\}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} p(n,8) = \Bigg\{(n+18)\Bigg(&(n^6+108n^5+4503n^4+79911n^3+522148n^2-202687n \\ &+9441216)/203212800+\left\lfloor\frac{n}{2}\right\rfloor(n^2+36+231)\Big/9216 \\ &+\left(\left\lfloor\frac{n+1}{3}\right\rfloor+2\left\lfloor\frac{n}{3}\right\rfloor\right)\Big/162+\left\lfloor\frac{n}{4}\right\rfloor\Big/64\Bigg)\Bigg\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} p(n,9) = \Bigg\{(n+22)\Bigg(&(n^7+158n^6+10034n^5+327352n^4 \\ &+5419144n^3-32063602n^2+5172096n+564401888)/14631321600 \\ &+\left\lfloor\frac{n}{2}\right\rfloor(2n^2+91n+728)\Big/36864 \\ &+\left((n+20)\left\lfloor\frac{n+1}{3}\right\rfloor+2(n+23)\left\lfloor\frac{n}{3}\right\rfloor\right)\Big/2916 \\ &+\left(\left\lfloor\frac{n}{4}\right\rfloor+\left\lfloor\frac{n+2}{4}\right\rfloor\right)\Big/256\Bigg)\Bigg\}. \end{aligned} \tag{3.13}$$

In order to establish such results, we require the following:

Theorem 2. *For a fixed positive integer m, the periodic sequence in n is given by*

$$\begin{aligned}
&(A_0, A_1, \dots, A_{m-1})crm_n \\
&\quad = A_0\left\lfloor \frac{n+m}{m} \right\rfloor + \sum_{j=1}^{m-1}(A_j - A_{j-1})\left\lfloor \frac{n+m-j}{m} \right\rfloor - A_{m-1}\left\lfloor \frac{n}{m} \right\rfloor \\
&\quad = A_0 + \sum_{j=1}^{m}(A_j - A_{j-1})\left\lfloor \frac{n+m-j}{m} \right\rfloor, \qquad (3.14)
\end{aligned}$$

where we define $A_m = A_0$.

Proof: Let us define $f(n)$ to be the right-hand side of (3.14).

$$\begin{aligned}
f(n+m) - f(n) &= \sum_{j=1}^{m}(A_j - A_{j-1})\left(\left\lfloor \frac{n+2m-j}{m} \right\rfloor - \left\lfloor \frac{n+m-j}{m} \right\rfloor\right) \\
&= \sum_{j=1}^{m}(A_j - A_{j-1}) = A_m - A_0 = 0.
\end{aligned}$$

Hence $f(n)$ has period m. Furthermore for $0 \le i < m$,

$$\begin{aligned}
f(i) &= A_0 + \sum_{j=1}^{m}(A_j - A_{j-1})\left\lfloor \frac{i+m-j}{m} \right\rfloor \\
&= A_0 + \sum_{j=1}^{i}(A_j - A_{j-1}) \\
&= A_i.
\end{aligned}$$

Therefore $f(n) = (A_0, A_1, \dots, A_{m-1})crm_n$ as desired. □

Corollary.

$$\begin{aligned}
&(A_0, A_1, \dots, A_{m-1})crm_n \\
&\quad = A_0 + (A_1 - A_0)n + \sum_{j=2}^{m}(A_j - A_{j-1} - A_1 + A_0)\left\lfloor \frac{n+m-j}{m} \right\rfloor.
\end{aligned}$$

Proof: This follows immediately from Theorem 1 once we recall that

$$\left\lfloor \frac{n+m-1}{m} \right\rfloor = n - \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{n+1}{m} \right\rfloor - \cdots - \left\lfloor \frac{n+m-2}{m} \right\rfloor.$$

□

Formulas (3.8)–(3.13) are now easily derived from the literature, in particular, from the formulas [19, pp. xxvii–xxviii, Eq. (6.7)]. Since the process is routine, we illustrate with the cases $m = 4$ and 5.

Note that in (3.7) we may drop the $\frac{1}{9}(1, 0, -1)cr3_n + \frac{1}{8}(1, 0, -1, 0)cr4_n$ because this expression is in absolute value $<\frac{1}{9} + \frac{1}{8} < \frac{1}{2}$. Hence immediately

$$
\begin{aligned}
p(n, 4) &= \left\{\binom{n+6}{3}\frac{1}{24} + \left(\frac{1}{9} - \frac{1}{16}\left(2 + n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)\right)(n+5)\right\} \\
&= \left\{(n+5)\left(n^2 + n + 22 + 18\left\lfloor \frac{n}{2} \right\rfloor\right) \Big/ 144\right\}, \qquad (3.16)
\end{aligned}
$$

as desired.

Similarly [6, p. xxvii, Eq. (6.7)]

$$
\begin{aligned}
p(n, 5) &= \frac{1}{120}\binom{n+9}{4} - \frac{5}{288}\binom{n+8}{2} + \frac{1}{64}(n+8)(1, -1)cr2 \\
&\quad + \frac{1}{9}(1, 0, 0)cr3_n + \frac{1}{64}(0, 1, -8, -7)cr4_n + \frac{1}{5}(1, 0, 0, 0, 0)cr5_n \\
&= \left\{\frac{1}{120}\binom{n+9}{4} - \frac{5}{288}\binom{n+8}{2} + \frac{1}{64}(n+8)\left(1 - 2n + 4\left\lfloor \frac{n}{2} \right\rfloor\right)\right\} \\
&= \left\{(n+8)\left(n^3 + 22n^2 + 44n + 248 + 180\left\lfloor \frac{n}{2} \right\rfloor\right) \Big/ 2880\right\} \qquad (3.17)
\end{aligned}
$$

as desired.

Formulas (3.10)–(3.13) are handled in exactly this way.

4. Asymptotics of non-modular generating functions

The topic of this section has had extensive development since 1941 when Ingham's seminal paper appeared [22]. Among the authors obtaining general theorems on this topic are Bateman and Erdös [8], Nicolas and Sárközy [27], Richmond [35], Roth and Szekeres [36, 39], and McIntosh [24]. This list is undoubtedly incomplete and is actually only tangential to the real object of this section. Of special interest are Ramanujan's observations as explicated by Berndt [10, p. 269, Entry 7].

Prior to Ingham's work, E.M. Wright wrote three papers on the asymptotics of particular generating functions. The first of these [42] was devoted to the asymptotics of the plane partition function. Its method was essentially the saddlepoint method that forms the foundation of many of the succeeding works mentioned above. Wright's second paper [43] in this series was based on a more elementary than analytic account but was not a surprising development.

The point I wish to make in the section is that Wright's third paper [44] on partitions into powers IS UNIQUE in the history of this subject. Its starting point and fundamental philosophy are different from anything that has come before or since.

Wright's focus in [44] is

$$f(x)=\prod_{\ell=1}^{\infty}\left(1-x^{\ell^k}\right)^{-1}=1+\sum_{n=1}^{\infty}p_k(n)x^n, \tag{4.1}$$

where $p_k(n)$ is the number of partitions of n into k-th powers.

His results are intricate to say the least. However, after stating his central results we shall examine the import of his work.

Write

$$a=\frac{1}{k},\quad b=\frac{1}{k+1},\quad j=j(k)=0\quad (k \text{ even}); \tag{4.2}$$

$$j=j(k)=\frac{(-1)^{(k+1)/2}}{(2\pi)^{k+1}}\Gamma(k+1)\zeta(k+1)\quad (k \text{ odd}). \tag{4.3}$$

Define a generalized Bessel function as follows:

$$\varphi(z)=\sum_{\ell=0}^{\infty}\frac{z^\ell}{\Gamma(\ell+1)\Gamma\left(\ell a-\frac{1}{2}\right)}. \tag{4.4}$$

Wright then produces the following asymptotic formula for $p_k(n)$ which is obtained by examining in detail the singularity of $f(x)$ at $x=1$:

$$p_k(n)=(n+j)^{-\frac{3}{2}}(2\pi)^{-\frac{k}{2}}\varphi(\Gamma(1+a)\zeta(1+a)(n+j)^a)+O\left(e^{(\Delta-\alpha)n^b}\right), \tag{4.5}$$

where

$$\Delta=(k+1)(a\Gamma(1+a)\zeta(1+a))^{1-b},\quad \alpha=\alpha(k)>0, \tag{4.6}$$

and $\zeta(s)$ is the Riemann zeta function.

Now the important point here is that the error term in (4.5) is of *exponentially* lower order of magnitude than the main term.

Wright accomplishes this by proving a complete generalization of the modular transformation formulae for Dedekind's eta function.

Let p, q, h, s and l be integers with $1\le p<q$, $(p,q)=1$, except that, when $q=1$, $p=0$,

$$1\le h\le q,\quad 1\le s\le k,\quad l\ge 0.$$

Define d_h by

$$ph^k\equiv d_h\ (\mathrm{mod}\ q),\ 0\le d_h<q.$$

If $d_h\ne 0$, we write

$$\mu_{h,s}=\frac{d_h}{q}(s \text{ odd}),\quad \mu_{h,s}=\frac{q-d_h}{q}\ (s \text{ even}). \tag{4.7}$$

If $d_h=0$, we take $\mu_{h,s}=1$. Hence always $q\mu_{h,s}\ge 1$.

In connection with any particular values of p and q we write

$$X = xe_q(-p) = e^{-y}, \quad Y = q^k y, \tag{4.8}$$

and we take y real and positive when X is real and $0 < X < 1$. We write also

$$t_s = \left(\frac{2\pi}{Y}\right)^a \exp\left\{a\,\pi\,i\left(s - \frac{1}{2}\right)\right\}, \tag{4.9}$$

where Y^a is that k-th root of Y which is real and positive when Y is real and positive,

$$\begin{aligned} g(h, l, s) &= \exp\left\{2\,\pi\,i(l + \mu_{h,s})^a t_s\right\} e_q(-h) \\ &= \exp\left\{\frac{(2\pi)^{1+a}(l + \mu_{h,s})^a e^{\frac{1}{2}\pi ai(2s+k-1)}}{qy^a} - \frac{2h\,\pi\,i}{q}\right\}, \end{aligned} \tag{4.10}$$

and

$$P_{p,q} = \prod_{h=1}^{q}\prod_{s=1}^{k}\prod_{l=0}^{\infty}\{1 - g(h, l, s)\}^{-1}. \tag{4.11}$$

Then we have:

Wright's Transformation Theorem [*44, p. 149, Theorem 4*]. *If $\Re\,(y) > 0$, then $P_{p,q}$ is convergent and*

$$f(x) = f(e^{-y}e_q(p)) = C_{p,q}\,y^{\frac{1}{2}}e^{jy}\exp\left(\frac{\Lambda_{p,q}}{y^a}\right)P_{p,q}, \tag{4.12}$$

where $C_{p,q}$ and $\Lambda_{p,q}$ are explicitly given constants depending on p and q.

In the case $k = 1$, (4.12) reduces to the modular function transformation used by Hardy and Ramanujan [20].

In addition, various instances of $k = 2$ appear in [20] and have been extended by Baxter [9] in his proof of the Doochel Kim conjectures [23].

There are a couple of things to say about Wright's theorems. First, how large is the class of partition generating functions for which there are analogs of Wright's Transformation Theorem? Second, how good is the full analog of the Hardy-Ramanujan-Rademacher formula for $p(n)$ that Wright gives for $p_k(n)$ (Theorem 3 of [44] but not restated here). Wright suggests that each of his main terms has order

$$n^{c_1}e^{c_2 n^{\frac{1}{k+1}}}.$$

He goes on to compare his error term to the famous $O(n^{-\frac{1}{4}})$ that appears in the Hardy-Ramanujan formula for $p(n)$ [20]. "If we attempt to make a similar improvement for $p_k(n)$, we find that we can choose $q_0 = q_k(k, n)$ so that the error term is $O(e^{n^d})$, where $d < b$.

This is not so good as the result for $k = 1$, and, in view of the heavy analysis required for this further step, I am content to prove $[O(e^{\epsilon n^b})]$."

One would hope that the advances on closely related work on modular forms would allow a complete analysis of this error term. Does it tend to 0 as $n \to \infty$? If not, is it bounded by polynomial growth? If not, what then?

5. The power series coefficients of Mock theta functions

In his last letter to Hardy [34, pp. 127–131], Ramanujan first presents his mock theta functions. Hardy [33, p. 354] summarizes the idea of a mock theta function: It is a function, "defined by a q-series convergent for $|q| < 1$, for which we can calculate asymptotic formulae, when q tends to a "rational point $e^{2\pi ir/s}$, of the same degree of precision as those furnished, for the ordinary ϑ-functions, by the theory of linear transformation." Subsequently, starting with G.N. Watson's famous L.M.S. Presidential address [40], there have been a number of studies of mock theta functions. Much of this work was summarized in a survey article [5]. Ramanujan focuses upon one mock theta function in particular:

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\ldots(1+q^n)^2}. \tag{5.1}$$

He asserts that: "The coefficient of q^n in $f(q)$ is

$$\frac{(-1)^{n-1}e^{\pi\left(\frac{n}{6}-\frac{1}{144}\right)^{\frac{1}{2}}}}{2\left(n-\frac{1}{24}\right)^{\frac{1}{2}}} + O\left(\frac{e^{\frac{\pi}{2}\left(\frac{n}{6}-\frac{1}{144}\right)^{\frac{1}{2}}}}{\left(n-\frac{1}{24}\right)^{\frac{1}{2}}}\right). \tag{5.2}$$

In 1964, building on the work of Watson [40] and Dragonette [15], I proved [2] a theorem equivalent to the assertion that if

$$f(q) = \sum_{n=0}^{\infty} \alpha(n)q^n, \tag{5.3}$$

then

$$\alpha(n) = \sum_{k=1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}(n - k(1+(-1)^k)/4) \sinh\left(\frac{\pi}{k}\left(\frac{n}{6} - \frac{1}{144}\right)^{\frac{1}{2}}\right)}{k^{\frac{1}{2}}\left(n - \frac{1}{24}\right)^{\frac{1}{2}}} + O(n^{\epsilon}) \tag{5.4}$$

[2, p. 455, Theorem 5.1]. The $A_k(n)$ is the exponential sum that appears in the formula for $p(n)$ [4, p. 70].

However, earlier Dragonette [15, p. 494], [2, p. 456] had suggested that in fact the actual error might indeed be smaller than $\frac{1}{2}$ in absolute value. If so, then the main sum rounded to the nearest integer would give $\alpha(n)$ exactly. Dragonette supported this possibility with calculations at $n = 100$ and 200.

Indeed it is possible that the main sum *converges* when extended to infinity and, if so, may actually equal $\alpha(n)$ [2, p. 456]. The following table provides further evidence:

n	$\alpha(n)$	The main sum in (5.4)	The main sum in (5.4) extended to n terms
100	−18520	−18520.18	−18520.01
200	−2660008	−2660008.01	−2660008.05
300	−128045286	−128045285.83	−128045285.99
400	−3447212602	−3447212601.86	−34472122602.05
500	−63676485905	−63676485905.01	−63676485905.06
600	−897840541970	−897840541969.91	−897840541969.98
700	−10302538222405	−10302538222405.14	−10302538222405.02
800	−100343014357869	−100343014357869.11	−100343014357869.01
900	−854282301584078	−854282301584078.03	−854282301584078.02
1000	−649583638177105	−649583638177105.004	−649583638177105.105

While three of the errors of $\sqrt{n}$ terms are smaller than the corresponding errors for n terms, it is nonetheless the case that the average error in column 2 is 0.0884 and in column 4 it is 0.0355 which offers some support to our conjecture that the series in (5.4) actually converges to $\alpha(n)$.

Indeed, in order to prove convergence of the series, it would suffice to prove that

$$\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}(n - k(1 + (-1)^k)/4)}{k^{\frac{3}{2}}}$$

converges. This in turn asks for an analysis of Dirichlet series such as

$$A(n, s) = \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}(n - k(1 + (-1)^k)/4)}{k^s} \tag{5.6}$$

which can easily be shown to converge absolutely for Re $s > \frac{3}{2}$.

6. Modular transformations and q-hypergeometric series

Let us state an example of the type of problem that concerns us here.

Problem. Prove directly that if

$$P(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \dots (1-q^n)^2}, \tag{6.1}$$

then for $\Im(\tau) > 0$

$$P(e^{2\pi i\tau}) = e^{\frac{-\pi i}{4}} \tau^{\frac{1}{2}} q^{\frac{\pi i}{12}(\tau+\frac{1}{\tau})} P\left(e^{-\frac{2\pi i}{\tau}}\right). \tag{6.2}$$

The catch in this problem is the single word "directly." Otherwise the problem is merely a reformulation of the transformation of Dedekind's η-function because [4, p. 70]

$$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{6.3}$$

where $q = e^{2\pi i\tau}$.

None of the methods which have been used to prove (6.3) seems to adapt to a direct treatment of the series in (6.1).

The only work I know of on this topic is by L. Ehrenpreis. His observations are presented in [16]; see especially Section 3 thereof. Of his work, he says: "This method sheds light on the question of why the generating function for Rogers-Ramanujan is an automorphic function and why automorphicity is difficult to prove."

7. Conclusion

The five topics chosen for this paper are tightly related. The problems presented do not fall naturally into the home turf of either modular forms or q-series. Consequently they have received little attention. However, they are calling out for an attack by a joint effort of these two areas.

References

1. H.L. Alder, "Partition identities—from Euler to the present," *Amer. Math. Monthly* **76** (1969), 733–746.
2. G.E. Andrews, "On the theorems of Watson and Dragonette for Ramanujan's mock theta functions," *Amer. J. Math* **88** (1966), 454–490.
3. G.E. Andrews, "Partition identities," *Advances in Math.* **9** (1972), 10–51.
4. G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading 1976; reprinted, Cambridge University Press, Cambridge, 1984, 1998.
5. G.E. Andrews, "Mock theta functions," *Proc. Symp. in Pure Math* **49** (1989), 283–298.
6. J. Arkin, "Researches on partitions," *Duke Math. J.* **38** (1970), 403–409.
7. A.O.L. Atkin, "Proof of a conjecture of Ramanujan," *Glasgow Math. J.* **8** (1967), 14–32.
8. P. Bateman and P. Erdös, "Monotonicity of partition functions," *Mathematika* **3** (1956), 1–14.
9. R.J. Baxter, "A direct proof of Kim's identities," *J. Phys. A: Math. Gen.* **31** (1998), 1105–1108.
10. B.C. Berndt, *Ramanujan's notebooks*, Part IV, Springer Verlag, New York, 1994.
11. B.C. Berndt, P.B. Bialek, and A.J. Yee, "Formulas of Ramanujan for the power series coefficients of certain quotients of eisenstein series," *Inter. Math. Res. Not.* **21** (2002), 1077–1109.
12. B.C. Berndt and K. Ono, "Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary," (Andrews Festschrift, D. Foata, and G.-N. Han, eds.), Springer Verlag, Berlin, 2001, pp. 39–110.
13. A. Cayley, "Researches on the partition of numbers," *Phil. Trans. Royal Soc.* **146** (1856), 127–140, (Reprinted: *Coll. Math. Papers*, **2** (1889), 235–249).

14. A. DeMorgan, "On a new form of difference equation," *Cambridge Math. J.* **4** (1843), 87–90.
15. L. Dragonette, "Some asymptotic formulae for the mock theta series of Ramanujan," *Trans. Amer. Math. Soc.* **72** (1952), 474–500.
16. L. Ehrenpreis, "Function theory for Rogers-Ramanujan-like partition identities," *Contemporary Math* **143** (1993), 259–320.
17. J.W.L. Glaisher, "On the number of partitions of a number of partitions of a number into a given number of parts," *Quart. J. Pure and Appl. Math.* **40** (1908), 57–143.
18. K. Glösel, "Über die Zerlegung der ganzen Zahlen," *Monatschefte Math. Phys*. **7** (1896), 133–141.
19. H. Gupta, E.E. Gwyther, and J.C.P. Miller, *Tables of Partitions*, Royal Soc. Math. Tables, Cambridge University Press, Cambridge, 1958, Vol. 4.
20. G.H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis," *Proc. London Math. Soc.* **17**(2) (1918), 75–115.
21. J.F.W. Herschel, "On circulating functions and on the integration of a class of equations of finite differences into which they enter as coefficients," *Phil. Trans. Royal Soc. London* **108** (1818), 144–168.
22. A.E. Ingham, "A Tauberian theorem for partitions," *Annals of Math.* **42** (1941), 1075–1090.
23. D. Kim, "Asymmetric XXZ chain at the antiferromagnetic transition: Spectra and partition functions," *J. Phys. A: Math. Gen.* **30** (1996), 3817–3836.
24. R. McIntosh, "Some asymptotic formulae for q-hypergeometric series," *J. London Math. Soc.* **51**(2) (1995), 120–136.
25. P.A. MacMahon, *Combinatory Analysis*, Cambridge University Press, London, 1916, Vol. 2. reprinted Chelsea, New York, 1960.
26. E. Netto, *Lehrbuch der Combinatorik* 2nd edn., Teubner, Berlin, 1927; reprinted: Chelsea, New York, 1958.
27. J.L. Nicolas and A. Sárközy, "On the asymptotic behavior of general partition functions," *Ramanujan Journal* **4** (2000), 29–39.
28. K. Ono, "On the parity of the partition function in arithmetic progressions," *J. für die r. und a. Math.* **472** (1996), 1–16.
29. K. Ono, "Distribution of the partition function modulo m," *Annals of Math.* **151** (2000), 293–307.
30. P. Paoli, *Opuscula Analytica*, Liburni, 1780, Opusc. II (Meditations Arith.), Section 1.
31. H. Rademacher, *Lectures on Elementary Number Theory*, Blaisdell, New York, 1964.
32. H. Rademacher, *Topics in Analytic Number Theory*, Springer, New York, 1973.
33. S. Ramanujan, *Collected Papers*, Cambridge University Press, London, 1927; reprinted: A.M.S. Chelsea, 2000 with new preface and extensive commentary by B. Berndt.
34. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
35. L.B. Richmond, "A general asymptotic result for partitions," *Canadian J. Math.* **27** (1975) 1083–1091.
36. K.F. Roth and G. Szekeres, "Some asymptotic formulae in the theory of partitions," *Quant. J. Math. Oxford Series* **5**(2) (1954), 241–259.
37. I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss., Phys.-Math. Kl., 1926, 488–495.
38. J.J. Sylvester, "On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order. With an excursus on rational fractions and partitions," *Amer. J. Math.* **5** (1882), 79–136.
39. G. Szekeres, "An asymptotic formula in the theory of partitions, II," *Quart. J. Math. Oxford Series* **2**(2) (1951), 85–108.
40. G.N. Watson, "The final problem: An account of the mock theta functions," *J. London Math. Soc.* **11** (1936), 55–80.
41. G.N. Watson, "Ramanujans Vermutung über Zerfälllungsanzahlen," *J. reine und angew. Math.* **179** (1938), 97–128.
42. E.M. Wright, "Asymptotic partition formulae, I. plane partitions," *Quart. J. Math., Oxford Series* **2** (1931), 177–189.
43. E.M. Wright, "Asymptotic partition formulae, II. Weighted partitions," *Proc. London Math. Soc.* **36**(2) (1932), 117–141.
44. E.M. Wright, "Asymptotic partitions formulae, III. Partitions into k-th powers," *Acta Math.* **63** (1934), 143–191.