Partitions with short sequences and mock theta functions

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P. A. MacMahon was the first to examine integer partitions in which consecutive integers were not allowed as parts. Such partitions may be described as having sequences of length 1. Recently it was shown that partitions containing no sequences of consecutive integers of length k are of interest in seemingly unrelated problems concerning threshold growth models. The object now is to develop the subject intrinsically to both provide deeper understanding of the theory and application of partitions and reveal the surprising role of Ramanujan's mock theta functions.

1. Preliminaries

In MacMahon's monumental work on combinatorial analysis (1), we find the first study of partitions without sequences. For example, there are eight partitions of 7 without sequences: $7, 6 + 1, 5 + 2, 5 + 1 + 1, 4 + 1 + 1 + 1, 3 + 3 + 1, 3 + 1 + 1 + 1 + 1$, and $1 + 1 + 1 + 1 + 1 + 1 + 1$. MacMahon was able to obtain the generating function for such partitions as a q series. He concluded his chapter on the subject by showing that the number of partitions of n without sequences and without ones equals the number of partitions of n into parts not congruent to 1 or 5 modulo 6.

Subsequently, in the 20th century, there were a few additional papers (2–4) devoted to furthering the study of such partitions.

Recently, however, Holroyd *et al.* (5) made an ingenious study of several definite integrals with widespread applications. The main integral in question is

$$-\int_0^1 \frac{\log f(x)}{x} \, dx = \frac{\pi^2}{3ab}, \tag{1.1}$$

where $0 < a < b$ and $f = f_{a,b}$ is the unique decreasing function mapping the unit interval into itself and satisfying

$$[f(x)]^a - [f(x)]^b = x^a - x^b, \quad 0 \leqq x \leqq 1. \tag{1.2}$$

This result has wide application. Holroyd *et al.* (5) prove an asymptotic theorem in probability.

Let $0 < s < 1$, and let $C_1, C_2, \ldots$ be independent events with probabilities

$$\mathbf{P}_s(C_n) = 1 - (1 - s)^n$$

under a probability measure $\mathbf{P}_s$. (We can think of C_n as the event that at least one occurs of a further set of n independent events, each of probability s.) Let k be a positive integer, and let A_k be the event

$$A_k = \bigcap_{i=1}^{\infty} (C_i \cup C_{i+1} \cup \cdots \cup C_{i+k-1})$$

that there is no sequence of k consecutive C_i values that do not occur.

Theorem 1 (5). *For every positive integer* k,

$$-\log \mathbf{P}_s(A_k) \sim \frac{\pi^2}{3k(k+1)} \frac{1}{s} \quad as \ s \to 0.$$

Holroyd *et al.* apply this theorem directly to a central probability question regarding threshold growth models. They also consider the application of Eq. **1.1** to partitions with short sequences and speculate that it might be possible to base many of their discoveries on the combinatorial aspects of the theory of partitions.

Indeed, in ref. 5, the more general partition function $p_k(n)$ is considered; $p_k(n)$ is the number of partitions of n that do not contain any sequence of consecutive integers of length k. Thus, from our previous calculation we see that $p_2(7) = 8$. Holroyd *et al.* also study the related generating function:

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$$G_k(q) = \sum_{n=0}^{\infty} p_k(n)q^n. \qquad\qquad [1.3]$$

Using the close connection of $G_k(q)$ to the probability function $\mathbf{P}_s(A_k)$, they then prove that

$$\log G_k(q) \sim \frac{\pi^2}{6}\left(1 - \frac{2}{k(k+1)}\right)\frac{1}{1-q} \quad \text{as } q \uparrow 1. \qquad\qquad [1.4]$$

From this result they deduce the asymptotic behavior of $\log p_k(n)$ as $n \to \infty$.

In the final section of ref. 5, the authors show for the case $k = 2$ how to deduce their results from MacMahon's theorem. Section 4 of ref. 5 concludes with the observation that if Eq. **1.4** could be partition-theoretically proved for general k, then it would allow a reversal of their arguments and by analytic continuation provide an independent proof of Eq. **1.1**.

We shall prove the following results:

Theorem 2. *For* $|q| < 1$,

$$G_k(q) = \frac{1}{(q;q)_\infty}\sum_{s,r\geqq 0}\frac{(-1)^s q^{\binom{k+1}{2}(s+r)^2+(k+1)\binom{r+1}{2}}}{(q^k;q^k)_s(q^{k+1};q^{k+1})_r}, \qquad\qquad [1.5]$$

where

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \qquad\qquad [1.6]$$

and

$$(A;q)_\infty = \lim_{n\to\infty}(A;q)_n. \qquad\qquad [1.7]$$

To make our second result clear, we introduce further notation:

$$P(v,k;q) := (q^{k(k+1)};q^{k(k+1)})_\infty\frac{(q^{\binom{k+1}{2}+kv};q^{k(k+1)})_\infty(q^{\binom{k+1}{2}-kv};q^{k(k+1)})_\infty}{(q;q)_\infty(q^k;q^k)_\infty}, \qquad\qquad [1.8]$$

$$S(v,k;q) := (q^v;q^{k+1})_\infty\sum_{j\geqq 0}\frac{(q^{k+1-v};q^{k+1})_{kj+v-1}q^{k(k+1)j+kv}}{(q^k;q^k)_{(k+1)j+v}}. \qquad\qquad [1.9]$$

Theorem 3. *For* $|q| < 1$,

$$G_k(q) = (q^{k+1};q^{k+1})_\infty P(0,k;q) - \sum_{v=1}^{\lfloor\frac{k}{2}\rfloor}P(v,k;q)(S(v,k;q) - S(k+1-v,k;q)). \qquad\qquad [1.10]$$

Now the results in *Theorems 1* and *2* are true for all k. In light of the fact that $P(v,k;q)$ is a quotient of three theta functions, we can easily obtain asymptotic results on it of great accuracy as $q \uparrow 1$. On the other hand, we do not yet have enough information about $S(v,k;q)$ to handle its asymptotics as $q \uparrow 1$ in general. However, we shall provide a full treatment when $k = 2$ with the hope that it may point the way to understanding $G_k(q)$ fully as $q \uparrow 1$.

It is surprising that the key to the $k = 2$ case lies in one of Ramanujan's mock theta functions (6, 7):

$$\chi(q) := 1 + \sum_{n=1}^{\infty}\frac{q^{n^2}}{\prod_{j=1}^{n}(1 - q^j + q^{2j})}.$$

Theorem 4. *For* $|q| < 1$,

$$G_2(q) = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}\chi(q).$$

From here we are led directly to the full asymptotics of $\mathbf{P}_s(A_2)$. Namely:

Theorem 5. $\mathbf{P}_s(A_2) \sim \sqrt{\pi/2}\, s^{-1/2}e^{-\pi^2/18s}$ *as* $s \downarrow 0$.

Note that this is a much stronger result than the $k = 2$ instance of theorem 2 of ref. 5.

We hope that these initial successes will lead to an effort to understand the many implications of *Theorems 1* and *2*. In particular, it would be very appealing to obtain the asymptotics of $\mathbf{P}_s(A_k)$ not just for $k = 2$.

Numerical calculations of the series arising in *Theorem 2* lead us to the following:

Conjecture. *For each $k \geq 2$, there is a positive constant C_k such that*

$$\mathbf{P}_s(A_k) \sim C_k s^{-\frac{1}{2}} e^{-\frac{\pi^2}{3k(k+1)s}} \quad as \; s \downarrow 0.$$

2. Proof of *Theorem 1*

To accomplish our argument, we must refine $P_k(n)$ to $P_k(m, n)$, the number of partitions of n into m parts containing no sequences of consecutive integers of length k or longer. We then define

$$G_k(x, q) = \sum_{m,n \geq 0} p_k(m, n) x^m q^n, \tag{2.1}$$

and we note that

$$G_k(1, q) = G_k(q). \tag{2.2}$$

We begin by noting that $G_k(0, q) = 1$, and

$$G_k(x, q) = G_k(xq, q) + \sum_{j=1}^{k-1} \frac{x^j q^{\binom{j+1}{2}} G_k(xq^{j+1}, q)}{(xq; q)_j}. \tag{2.3}$$

The proof of the functional Eq. **2.3** is quite straight forward. We split the partitions enumerated by $G_k(x)$ into k classes, where the jth class contains those partitions in which 1 occurs in a maximal sequence of precisely j consecutive integers. If $j = 0$ (i.e., 1 is not a part), then the relevant partitions are clearly generated by $G_k(xq, q)$. If $1 \leq j \leq k - 1$, the relevant generating function is easily seen to be

$$\frac{x^j q^{1+2+3+\cdots+j}}{(1 - xq)(1 - xq^2)\cdots(1 - xq^j)} G_k(xq^{j+1}, q), \tag{2.4}$$

and adding up the generating functions for these k exhaustive classes, we obtain Eq. **2.3**.

To prove Eq. **1.5** and consequently *Theorem 1*, we shall prove more generally

$$G_k(x, q) = \frac{1}{(xq; q)_\infty} \sum_{r,s \geq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}}}{(q^k; q^k)_s (q^{k+1}; q^{k+1})_r} := g_k(x, q). \tag{2.5}$$

Once Eq. **2.5** is proved, we may deduce *Theorem 1* by setting $x = 1$ therein and applying Eq. **2.2**.

To prove Eq. **2.5**, we proceed by denoting the right side of Eq. **2.5** by $g_k(x, q)$; we need only show that $g_k(0, q) = 1$ (which is obvious by inspection) and that $g_k(x, q)$ satisfies the functional Eq. **2.3**, which together with $g_k(0, q) = 1$ uniquely defines $G_k(x, q)$.

Apart from the right side of Eq. **2.5**, there are two other useful representations of $g_k(x, q)$. First,

$$\sum_{r,s \geq 0} \frac{x^{k(s-1)+(k+1)r}(-1)^{s-1} q^{\binom{k+1}{2}(r+s-1)^2 + (k+1)\binom{r}{2}}(1 - q^{ks+(k+1)r})}{(x; q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r}$$

$$= \sum_{r,s \geq 0} \frac{x^{ks+1+(k+1)r}(-1)^{s-1} q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}}}{(z; q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} + \sum_{r,s \geq 0} \frac{x^{ks+(k+1)r}(-1)^s q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}}}{(x; q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r}$$

$$\{\text{because } 1 - q^{ks+(k+1)r} = (1 - q(k+1)r) + q(k+1)r(1 - qks)\}$$

$$= \frac{(1 - x)}{(x; q)_\infty} \sum_{r,s \geq 0} \frac{x^{ks+(k+1)r}(-1)^s q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}}}{(q^k; q^k)_s (q^{k+1}; q^{k+1})_r} = g_k(x, q), \tag{2.6}$$

by Eq. **2.5**.

Second, we may start with the top line of Eq. **2.6**, and noting $1 - q^{ks+(k+1)r} = 1 - q^{ks} + q^{ks}(1 - q^{(k+1)r})$, we see that also

$$g_k(x, q) = \sum_{r,s \geq 0} \frac{x^{ks+(k+1)r}(-1)^s q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r}{2}}}{(x; q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} + \sum_{r,s \geq 0} \frac{x^{ks+1+(k+1)r}(-1)^{s+1} q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}+ks}}{(x; q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r}. \tag{2.7}$$

We take the representation of $g_k(x, q)$ given by Eq. **2.6** and insert it in the right side of the required functional equation

$$\sum_{j=0}^{k-1} \frac{x^j q^{\binom{j+1}{2}} g_k(xq^{j+1}, q)}{(xq; q)_j} = \left(\sum_{j=0}^{k-1} x^j q^{\binom{j+1}{2}} \sum_{r,s \geq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{(j+1)(ks+(k+1)r)+\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r}{2}}}{(xq; q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} - \sum_{j=0}^{k-1} x^j q^{\binom{j+1}{2}} \right)$$

$$\times \sum_{r,s\geqq 0} \frac{(-1)^s x^{ks+1+(k+1)r} q^{(j+1)(ks+1+(k+1)r) + \binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2} + ks}}{(xq;q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} \Bigg)$$

$$= \Bigg(\sum_{j=0}^{k-1} x^j q^{\binom{j+1}{2}} \sum_{r,s\geqq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{(j+1)(ks+(k+1)r) + \binom{k+1}{2}(r+s)^2 + (k+1)\binom{r}{2}}}{(xq;q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} - \sum_{j=1}^{k} x^j q^{\binom{j}{2}}$$

$$\times \sum_{r,s\geqq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{j(ks+1+(k+1)r) + \binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2} + ks}}{(xq;q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} \Bigg)$$

$$= \Bigg(\sum_{r,s\geqq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{(ks+(k+1)r) + \binom{k+1}{2}(r+s)^2 + (k+1)\binom{r}{2}}}{(xq;q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} - \sum_{r,s\geqq 0} \frac{(-1)^s x^{k(s+1)+(k+1)r}}{(xq;q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r}$$

$$\times q^{k(ks+1+(k+1)r) + \binom{k}{2} + \binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2} + ks} \Bigg)$$

{because all terms have cancelled except that $j = 0$ in the first sum and $j = k$ in the second}

$$= \Bigg(\sum_{r,s\geqq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{ks+(k+1)r + \binom{k+1}{2}(r+s)^2 + (k+1)\binom{r}{2}}}{(xq;q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r}$$

$$+ \sum_{r,s\geqq 0} \frac{(-1)^s (1-q^{ks}) x^{ks+(k+1)r}}{(xq;q)_\infty (q^k; q^k)_s (q^{k+1}; q^{k+1})_r} q^{k(k(s-1)+1+(k+1)r) + \binom{k}{2} + \binom{k+1}{2}(r+s-1)^2 + (k+1)\binom{r+1}{2} + k(s-1)} \Bigg)$$

(where s has been shifted to $s - 1$ in the second sum)

$$= \frac{1}{(xq;q)_\infty} \sum_{r,s\geqq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}}}{(q^k; q^k)_s (q^{k+1}; q^{k+1})_r}$$

$$= g_k(x, q),$$

by Eq. **2.5**, and thus we obtain the desired defining functional equation.

Hence,

$$g_k(x, q) = G_k(x, q),$$

and thus *Theorem 1* follows.

3. Proof of *Theorem 2*

The object in this second result is to obtain a representation of $G_k(q)$ that will yield the asymptotics promised in *Theorem 3*. To this end, we require several results from the literature. First is Euler's identity (see p. 490, corollary 10.2.2.b, in ref. 8),

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n} = (x;q)_\infty;$$ [3.1]

next is Jacobi's triple product identity (see p. 497, theorem 1.4.1., in ref. 8; with $q \to q^{2a}$, $x = q^{a+h}$),

$$\theta(a, h; q) := \sum_{s=-\infty}^{\infty} (-1)^s q^{as^2+hs} = (q^{2a}; q^{2a})_\infty (q^{a+h}; q^{2a})_\infty (q^{a-h}; q^{2a})_\infty.$$ [3.2]

We proceed with the proof of *Theorem 2* starting from Eq. **1.5**. Hence,

$$G_k(q) = \frac{1}{(q;q)_\infty} \sum_{s,r\geqq 0} \frac{(-1)^s q^{\binom{k+1}{2}(s+r)^2 + (k+1)\binom{r+1}{x}}}{(q^k; q^k)_s (q^{k+1}; q^{k+1})_r} = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{\binom{k+1}{2}(s+r)^2 + (k+1)\binom{r+1}{2}} (q^{k(s+1)}; q^k)_\infty}{(q;q)_\infty (q^k; q^k)_\infty (q^{k+1}; q^{k+1})_r}$$

$$= \sum_{r=0}^{\infty} \sum_{s=-\infty}^{\infty} \frac{(-1)^{s-r} q^{\binom{k+1}{2}s^2 + (k+1)\binom{r+1}{2}} (q^{k(s-r+1)}; q^k)_\infty}{(q;q)_\infty (q^k; q^k)_\infty (q^{k+1}; q^{k+1})_r} = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{s-r+j} q^{\binom{k+1}{2}s^2 + (k+1)\binom{r+1}{2} + \binom{j+1}{2}k + jk(s-r)}}{(q;q)_\infty (q^k; q^k)_\infty (q^{k+1}; q^{k+1})_r (q^k; q^k)_j}$$

[3.3]

$$= \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{\theta\left(\dfrac{k+1}{2}, j; q^k\right)(-1)^{r+j} q^{(k+1)\binom{r+1}{2} + k\binom{j+1}{2} - jkr}}{(q; q)_\infty (q^k; q^k)_\infty (q^{k+1}; q^{k+1})_r (q^k; q^k)_j}$$

$$= \sum_{j=0}^{\infty} \frac{\theta\left(\dfrac{k+1}{2}, j; q^k\right)(-1)^j q^{k\binom{j+1}{2}}}{(q; q)_\infty (q^k; q^k)_\infty (q^k; q^k)_j} \, (q^{k+1-jk}; q^{k+1})_\infty,$$

by Eq. **3.1**.

Now we note several things about this last expression. First, if j is a positive multiple of $(k + 1)$, then the infinite product in the summand vanishes. Next, we observe that $\theta(a, -h; q) = \theta(a, h; q)$; also,

$$\theta(a, h + 2a; q) = \sum_{s=-\infty}^{\infty} (-1)^s q^{a(s+1)^2 + h(s+1) - a - h} = -q^{-a-h} \theta(a, h; q), \qquad [3.4]$$

and thus,

$$\theta(a, a; q) = -\theta(a, -a; q) = -\theta(a, a; q) = 0.$$

This last identity means that if $j \equiv k + 1/2 \pmod{k + 1}$, the sum in Eq. **3.4** also vanishes. Finally, we note that the $j = 0$ term in Eq. **3.3** is in fact $(q^{k+1}; q^{k+1})_\infty P(0, k; q)$ by Eqs. **1.8** and **3.2**.

Hence, separating the terms of the sum in Eq. **3.2** according to the residue of j modulo $k + 1$, we see that

$$G_k(q) = (q^{k+1}; q^{k+1})_\infty P(0, k; q) + \sum_{v=1}^{\lfloor \frac{k}{2} \rfloor} \left\{ \sum_{j=0}^{\infty} \frac{\theta\left(\dfrac{k+1}{2}, (k+1)j + v; q^k\right)(-1)^{(k+1)j+v} (q^{k+1-((k+1)j+v)k}; q^{k+1})_\infty}{(q; q)_\infty (q^k; q^k)_\infty (q^k; q^k)_{(k+1)j+v}} \, q^{k\binom{(k+1)j+v+1}{2}} \right.$$

$$\left. + \sum_{j=0}^{\infty} \frac{\theta\left(\dfrac{k+1}{2}, (k+1)j + k + 1 - v; q^k\right)(-1)^{(k+1)j+k+1-v} (q^{k+1-((k+1)j+k+1-v)k}; q^{k+1})_\infty}{(q; q)_\infty (q^k; q^k)_\infty (q^k; q^k)_{(k+1)j+k+1-v}} \, q^{k\binom{(k+1)j+k+1-v+1}{2}} \right\}.$$

This last expression simplifies term by term into the formulation in *Theorem 2*; in the simplification one must invoke Eq. **3.2** and the fact that

$$(Aq^{-N}; q)_\infty = (-1)^N q^{-\binom{N+1}{2}} A^N (q/A; q)_N (A; q)_\infty. \qquad [3.5]$$

4. Proof of *Theorem 3*

It is surprising that half the proof of *Theorem 3* relies on an identity of MacMahon (1), and half relies on an identity of Fine (7). Thus, we shall refer to the result as the MacMahon–Fine identity. We begin with MacMahon, (ref. 1, p. 52) who proved

$$G_2(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n (q^6; q^6)_{n-1}}{(1 - q^n)(q^2; q^2)_{n-1}(q^3; q^3)_{n-1}}, \qquad [4.1]$$

which may be algebraically simplified term by term to

$$G_2(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n \prod_{j=1}^{n-1}(1 - q^j + q^{2j})}{(q; q)_n}. \qquad [4.2]$$

Next, in his study of Ramanujan's mock theta functions, Fine (7) showed that

$$\chi(q) = \frac{(q^2 q^2)_\infty}{(-q^3; q^3)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{q^n \prod_{j=1}^{n-1}(1 - q^j + q^{2j})}{(q; q)_n} \right). \qquad [4.3]$$

Combining Eqs. **4.2** and **4.3**, we obtain the assertion of *Theorem 3*.

5. Proof of *Theorem 4*

Here we rely principally on one of Ramanujan's mock theta function identifies (6), which we write as

$$\chi(q) = \frac{1}{4} f(q) + \frac{3}{4} \frac{(q^3; q^3)_\infty^4}{(q)_\infty (q^6; q^6)_\infty^2},$$ [5.1]

where

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}.$$ [5.2]

In light of the fact that $q/(1 + q)^2$ is increasing for $0 \leqq q \leqq 1$, one may easily deduce that

$$f(q) \rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{4}{3} \quad \text{as} \quad q \uparrow 1.$$ [5.3]

Finally, we require (see theorem 1.12.8 in ref. 8)

$$\frac{1}{(e^{-s}; e^{-s})_\infty} \sim \frac{1}{\sqrt{2\pi}} s^{\frac{1}{2}} e^{\frac{\pi^2}{6s}} \quad \text{as} \quad s \downarrow 0.$$ [5.4]

Combining these results with *Theorem 3*, we see that as $s \downarrow 0$ and $q = e^{-s}$,

$$G_2(q) = \frac{(-q^3; q^3)_\infty}{(q^2; q^2)_\infty} \chi(q) = \frac{(q^6; q^6)_\infty}{(q^2; q^2)_\infty (q^3; q^3)_\infty} \left(\frac{1}{4} f(q) + \frac{3}{4} \frac{(q^3; q^3)_\infty^4}{(q)_\infty (q^6; q^6)_\infty^2}\right) \sim \frac{1}{6\pi} s^{\frac{1}{2}} e^{\frac{\pi^2}{9s}} + \frac{1}{2} e^{\frac{\pi^2}{9s}} \sim \frac{1}{2} e^{\frac{\pi^2}{9s}}.$$ [5.5]

Finally, then, from the identity of Holroyd *et al.* (5),

$$P_s(A_2) = \frac{G_2(q)}{G(q)} \sim \frac{1}{2} e^{\frac{\pi^2}{9s}} \sqrt{2\pi s}^{-\frac{1}{2}} e^{\frac{-\pi^2}{6s}} \sim \sqrt{\frac{\pi}{2}} s^{-\frac{1}{2}} e^{\frac{-\pi^2}{18s}} \quad \text{as} \quad s \downarrow 0,$$ [5.6]

which is the assertion in *Theorem 4*.

6. Conclusions

Several themes have been brought together in this article. First, the partition function, $G_k(q)$, of Holroyd *et al.* (5) has been represented in explicit q series. In *Theorem 1*, $G_k(q)$ is represented as an infinite product times a double series, and *Theorem 2*, $G_k(q)$ is represented by less than k single-fold series multiplied by infinite products.

Empirical studies of the functions appearing on the right side in *Theorem 2* suggest that $(q^{k+1}; q^{k+1})_\infty P(0, k; q)$ is the dominant term. This observation is what led to *Conjecture*.

We hope that *Theorem 3* provides a hint of further discoveries for $G_k(q)$ in general. The fact that $G_2(q)$ is an infinite product (in fact, a modular form) times the mock theta function, $\chi(q)$, suggests that the analytic nature of $G_k(q)$ for $k > 2$ may be very interesting. Indeed, additional discoveries may well lead to the extension of *Theorem 4* to results such as those stated in *Conjecture*.

The striking applications of such results pioneered by Holroyd *et al.* (5) make clear the potential significance of additional discoveries.

1. MacMahon, P. A. (1916) *Combinatory Analysis* (Cambridge Univ. Press, Cambridge, U.K.), Vol. II, pp. 49–58.
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