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Note

Integrals, partitions and MacMahon's Theorem

George Andrews [a,1], Henrik Eriksson [b], Fedor Petrov [c,2], Dan Romik [d,3]

[a] *Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA*
[b] *KTH, Computer Science, SE-10044 Stockholm, Sweden*
[c] *Petersburg Department of the Russian Academy of Sciences, Russia*
[d] *Department of Statistics, 367 Evans Hall, University of California, Berkeley, CA 94720-3860, USA*

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Abstract

In two previous papers, the study of partitions with short sequences has been developed both for its intrinsic interest and for a variety of applications. The object of this paper is to extend that study in various ways. First, the relationship of partitions with no consecutive integers to a theorem of MacMahon and mock theta functions is explored independently. Secondly, we derive in a succinct manner a relevant definite integral related to the asymptotic enumeration of partitions with short sequences. Finally, we provide the generating function for partitions with no sequences of length K and part exceeding N.

Keywords: Partition identities; Partition generating functions; Partitions without consecutive parts; MacMahon's Theorem; Mock theta function

1. Introduction

In his classic two volume work, Combinatory Analysis [5], P.A. MacMahon devotes Chapter IV of volume 2 to "Partitions Without Sequences." His object in this chapter is to make a thorough study of partitions in which no consecutive integers (i.e. integers that differ by 1) occur. He concludes this chapter with what we will call MacMahon's Theorem.

E-mail addresses: andrews@math.psu.edu (G. Andrews), henrik@nada.kth.se (H. Eriksson), fedorpetrov@mail.ru (F. Petrov), romik@stat.berkeley.edu (D. Romik).

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Theorem 1.1. *The number of partitions of an integer N into parts* $\not\equiv \pm 1 \pmod 6$ *equals the number of partitions of N with no consecutive integers as summands and no ones.*

For example, for $n = 10$, the first set of partitions is 10, $8+2$, $6+4$, $6+2+2$, $4+4+2$, $4+3+3$, $4+2+2+2$, $3+3+2+2$, $2+2+2+2+2$; the second set is 10, $8+2$, $7+3$, $6+3$, $6+4$, $6+2+2$, $5+5$, $4+4+2$, $4+2+2+2$, $2+2+2+2+2$. The fact that each set of partitions has the same number of elements (in this case 9), is MacMahon's assertion.

In two previous papers [2,4], MacMahon's ideas have been generalized to the consideration of partitions in which sequences of consecutive integers have been restricted to contain fewer than k terms (MacMahon only dealt with $k = 2$).

In Section 2 of this paper we shall explore in detail various aspects of MacMahon's work in [5, vol. II, Chapter IV]. In Section 3 we discuss the generalization to partitions without k consecutive parts: First, we obtain a new and simplified proof of the Holroyd–Liggett–Romik definite integral that was used in [4] to obtain results on the asymptotic enumeration of these classes of partitions. Secondly, we strengthen the results of [2] by obtaining a double series representation of the generating function for partitions in which each part is $\leqslant N$ and sequences of consecutive integers have length less than k. Finally, Section 4 contains some remarks on a probabilistic interpretation of the mock theta function $\chi(q)$ studied by Ramanujan.

2. Investigation of MacMahon's Theorem

We begin with some definitions.

Definition 2.1. Let

$g_n =$ the number of partitions of n with no two consecutive parts,

$h_n =$ the number of partitions of n with no two consecutive parts and no 1's,

$$G_2(q) = \sum_{n=0}^{\infty} g_n q^n, \tag{2.1}$$

$$H_2(q) = \sum_{n=0}^{\infty} h_n q^n, \tag{2.2}$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n}(1 - q^j + q^{2j})}, \tag{2.3}$$

where $\chi(q)$ is one of the third-order mock theta functions studied by Ramanujan [6, p. 354].

2.1. A bijective proof of Theorem 1.1

Proof. By passing to the conjugate partition, the number of partitions of n with no 1's and no two consecutive parts is clearly seen to be equal to the number of partitions of n not containing any part *exactly once*. Here is a bijection between the set $\mathcal{C}_n$ of partitions of n not containing any part exactly once, and the set $\mathcal{B}_n$ of partitions of n into parts congruent to $0, 2, 3, 4$ mod 6: If $n = \sum_{k=1}^{\infty} k r_k$ is a partition in $\mathcal{C}_n$ (r_k is the multiplicity of k, or the number of parts equal to k in the partition), $r_k \in \{0, 2, 3, 4, \ldots\}$, then each r_k can be written uniquely as $r_k = s_k + t_k$, where $s_k \in \{0, 3\}$ and $t_k \in \{0, 2, 4, 6, 8, \ldots\}$. Define a partition $n = \sum_{j=1}^{\infty} j b_j$ by

$$\begin{aligned}
b_{6k+1} &= 0 \quad (k = 0, 1, 2, 3, \ldots),\\
b_{6k+5} &= 0,\\
b_{6k+2} &= \frac{1}{2}t_{3k+1},\\
b_{6k+4} &= \frac{1}{2}t_{3k+2},\\
b_{6k+3} &= \frac{1}{3}s_{2k+1} + t_{6k+3},\\
b_{6k+6} &= \frac{1}{3}s_{2k+2} + t_{6k+6}.
\end{aligned}$$

This partition is in $\mathcal{B}_n$, and it is not difficult to check that any partition in $\mathcal{B}_n$ is obtained in this way from a unique partition in $\mathcal{C}_n$. □

2.2. A q-series for $G_2(q)$

We give a simplified proof of the following q-series representation for $G_2(q)$, which was stated in [2, Eq. (4.2)]:

Theorem 2.2.

$$G_2(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n \prod_{j=1}^{n-1}(1 - q^j + q^{2j})}{\prod_{j=1}^{n}(1 - q^j)}. \tag{2.4}$$

Proof. Again by passing to the conjugate partition, we see that g_n is the number of partitions of n where all the parts except possibly the largest part do not appear exactly once.

Write (2.4) as

$$\begin{aligned}
G_2(q) &= 1 + \sum_{n=1}^{\infty}\left[\frac{q^n}{1-q^n} \cdot \prod_{j=1}^{n-1}\left(\frac{1 - q^j + q^{2j}}{1 - q^j}\right)\right]\\
&= 1 + \sum_{n=1}^{\infty}\left[\frac{q^n}{1-q^n} \cdot \prod_{j=1}^{n-1}\left(1 + q^{2j} + q^{3j} + q^{4j} + \cdots\right)\right].
\end{aligned}$$

The coefficient of q^N in the nth summand on the right-hand side is equal to the number of partitions of N with maximal part n, where no part except possibly the largest part appears exactly once. So the coefficient of q^N in the entire sum on the right-hand side is exactly g_N. □

2.3. The MacMahon–Fine identity

In [2], it was shown that a combination of identities due to MacMahon [5, vol. II, p. 52], and Fine [3, p. 57] shows that

$$G_2(q) = H_2(q)\chi(q). \tag{2.5}$$

This identity can be given the following combinatorial interpretation:

Theorem 2.3. *For each integer $n \geqslant 1$ and $0 \leqslant k \leqslant \sqrt{n}$, let $f_{n,k}$ be the number of partitions of $n-k^2$ in which no part which is greater than k appears exactly once. Then for each $n \geqslant 1$,*

$$g_n = \sum_{k=0}^{\lfloor \sqrt{n} \rfloor} f_{n,k}. \tag{2.6}$$

Proof. From the remark at the beginning of the proof of Theorem 2.2, we can write

$$H_2(q) = \prod \left(1 + q^{2j} + q^{3j} + q^{4j} + \cdots\right) = \prod_{n=1}^{\infty} \frac{1 - q^j + q^{2j}}{1 - q^j} \tag{2.7}$$

(this is an alternative way to prove Theorem 1.1). Now combining (2.5) and (2.7) and the definition of $\chi(q)$ gives

$$G_2(q) = \sum_{k=0}^{\infty} q^{k^2} \left(\prod_{j=1}^{k} \frac{1}{1-q^j} \right) \cdot \left(\prod_{j=k+1}^{\infty} \frac{1 - q^j + q^{2j}}{1 - q^j} \right).$$

The coefficients of q^n in the left- and right-hand side of this equation are clearly the left- and right-hand sides of (2.6), respectively. □

A natural question is whether Theorem 2.3 has a simple combinatorial explanation.

3. Partitions without *k* consecutive parts

3.1. The Holroyd–Liggett–Romik integral

In [4], the following result concerning the asymptotic enumeration of partitions without k consecutive parts was proved:

Theorem 3.1. (Holroyd, Liggett and Romik [4]) *Let $p_k(n)$ denote the number of partitions of n not containing k consecutive parts. Then for each fixed $k > 1$, we have as $n \to \infty$*

$$p_k(n) = e^{(1+o(1))c_k\sqrt{n}},$$

where

$$c_k = \pi \sqrt{\frac{2}{3}\left(1 - \frac{2}{k(k+1)}\right)}.$$

The proof of this result relies on a special case of the following family of definite integrals, also proved in [4]: For every $0 < a < b$, a decreasing function $f_{a,b} : [0,1] \to [0,1]$ can be defined by $f_{a,b}(0) = 1$, $f_{a,b}(1) = 0$ and $f_{a,b}(x)^a - f_{a,b}(x)^b = x^a - x^b$ in between. In the simplest case $f_{1,2} - f_{1,2}^2 = x - x^2$, we have $f_{1,2}(x) = 1 - x$. Then we have:

Theorem 3.2. (Holroyd, Liggett and Romik [4])

$$\int_0^1 \frac{-\log f_{a,b}(x)}{x}\, dx = \frac{\pi^2}{3ab}.$$

We give here a new and shorter proof of this result. We remark that the proof given in [4], while considerably more complicated, seems to contain more interesting information, see [7].

Proof. The integral in the theorem can be interpreted as a double integral:

$$I_{a,b} := \int_0^1 \frac{-\log f_{a,b}(x)}{x}\,dx = \int_0^1 \frac{dx}{x} \int_{f_{a,b}(x)}^1 \frac{dy}{y} = \iint_D \frac{dx\,dy}{xy},$$

where D is a symmetric domain bounded below by $y^a - y^b = x^a - x^b$, above by $y = 1$, and to the right by $x = 1$. Bisect it along its symmetry axis $y = x$ and substitute $y = xt$, $dy = x\,dt$ to get

$$I_{a,b} = 2\iint_{D'} \frac{dx\,dt}{xt},$$

where D' is bounded below by $x^{b-a} = (1-t^a)/(1-t^b)$, above by $t = 1$, and to the right by $x = 1$. Integrating x we get

$$I_{a,b} = \frac{2}{b-a}\int_0^1 \log\left(\frac{1-t^b}{1-t^a}\right)\frac{dt}{t}.$$

Finally, if we split the logarithm in two and substitute $x = t^b$ in the first integral and $x = t^a$ in the second, the desired result is obtained:

$$I_{a,b} = \frac{2}{b-a}\left(-\frac{1}{b} + \frac{1}{a}\right)\int_0^1 \frac{\log(1-x)}{x}\,dx = \frac{\pi^2}{3ab}. \qquad \square$$

3.2. The restricted generating function

We must now substantially extend the definitions that appear at the beginning of Section 2. Let

$g_{m,n}(k,N) =$ the number of partitions of n into m parts in which each part is $\leqslant N$ and there is no string of parts forming a sequence of consecutive integers of length k,

$$G_k(N; x, q) = \sum_{m,n=0}^{\infty} g_{m,n}(k,N)x^m q^n.$$

We note in passing that with regard to the definitions in Section 2,

$$g_n = \sum_{m\geqslant 0} g_{m,n}(2,\infty),$$

and

$$G_2(q) = G_2(\infty; 1, q).$$

In [2, Eq. (2.5)], it was proven that

$$G_k(\infty; x, q) = \frac{1}{(xq;q)_\infty} \sum_{r,s \geqslant 0} \frac{(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}}}{(q^k;q^k)_s (q^{k+1};q^{k+1})_r}, \tag{3.1}$$

where

$$(A;q)_t = (1-A)(1-Aq)\cdots\left(1-Aq^{t-1}\right), \qquad (A;q)_0 = 1.$$

Our object here is to prove the following result for $G_k(N; x, q)$ which reduces to (3.1) when $N \to \infty$.

Theorem 3.3.

$$G_k(N; x, q) = \frac{1}{(xq;q)_N} \sum_{r,s \geqslant 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2 + (k+1)\binom{r+1}{2}} \times \begin{bmatrix} N-kr-ks-r+1 \\ s \end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1}, \tag{3.2}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_t = \begin{cases} 0 & \text{if } B < 0 \text{ or } B > A, \\ \frac{(q^t;q^t)_A}{(q^t;q^t)_B (q^t;q^t)_{A-B}} & \text{for } 0 \leqslant B \leqslant A. \end{cases}$$

Proof. We begin by noting that there is a defining recurrence for $G_k(N; x, q)$. Namely,

$$G_k(N; x, q) = \begin{cases} \frac{1}{(xq;q)_N}, & \text{if } 0 \leqslant N < k, \\ G_k(N-1; x, q) + \sum_{i=1}^{k-1} \frac{x^i q^{N+(N-1)+\cdots+(N-i+1)} G_k(N-i-1; x, q)}{(1-xq^N)(1-xq^{N-1})\cdots(1-xq^{N-i+1})}. \end{cases} \tag{3.3}$$

This last assertion is easily verified as follows. If $N < k$, then there can be no sequences of k consecutive integers among the parts. Hence for $N < k$, all partitions with parts $\leqslant N$ must be included and the generating function in this case is

$$\frac{1}{(xq;q)_N}$$

as asserted.

To establish the bottom line of (3.3), we note that among the partitions generated by $G_k(N; x, q)$ there are some in which N does not appear as a part. These are generated by $G_k(N-1; x, q)$. If N does appear as a part, it then lies in a sequence of consecutive integers of maximal length i where $1 \leqslant i < k$. The portion of such partitions containing only parts in $[N-i+1, N]$ is generated by

$$\frac{x^i q^{N+(N-1)+\cdots+(N-i+1)}}{(1-xq^N)(1-xq^{N-1})\cdots(1-xq^{N-i+1})},$$

and all other parts must be $< N-i$, and consequently are generated by $G_k(N-i-1; x, q)$. Hence the right-hand side of (3.3) generates precisely those partitions generated by $G_k(N; x, q)$ thus establishing the second line of (3.3).

We now define

$$S(k, N) = (xq;q)_N G_k(N; x, q). \tag{3.4}$$

Consequently $S(k, N)$ is uniquely determined by the recurrence

$$S(k,N)=\begin{cases}1, & \text{if } 0\leqslant N<k,\\ \sum_{i=1}^{k-1} x^i q^{N+(N-1)+\cdots+(N-i+1)}(1-xq^{N-i})S(k,N-i-1).\end{cases} \tag{3.5}$$

We now define

$$\sigma(k,N)=\sum_{r,s\geqslant 0}(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}} \times \begin{bmatrix} N-kr-ks-r+1 \\ s\end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r\end{bmatrix}_{k+1}. \tag{3.6}$$

Note that one may equivalently take $-\infty<r,s<\infty$ as the range of summation in (3.6), since the terms with negative r,s will be 0. We wish to show that $S(k,N)=\sigma(k,N)$ in order to complete the proof of this theorem. To do this we need only show that $\sigma(k,N)$ also satisfies the defining recurrence (3.5).

Immediately we see that if $N<k$, then the only non-vanishing term of the double sum in (3.6) occurs for $s=r=0$. Hence

$$\sigma(k,N)=1 \quad \text{if } 0\leqslant N<k.$$

We shall prove the following equivariant recurrence for $\sigma(k,N)$ when $N\geqslant k$:

$$\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}}\big(\sigma(k,N-i)-\sigma(k,N-i-1)\big) + x^k q^{kN-\binom{k}{2}}\sigma(k,N-k)=0. \tag{3.7}$$

We now simplify the left-hand side of (3.7):

$$\begin{aligned}
&\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}}\big(\sigma(k,N-i)-\sigma(k,N-i-1)\big)\\
&\quad=\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}} \sum_{-\infty<r,s<\infty}(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}}\\
&\qquad\times\Bigg\{q^{k(N-i-kr-ks-r+1-s)}\begin{bmatrix} N-kr-ks-r-i \\ s-1\end{bmatrix}_k \begin{bmatrix} N-kr-ks-i \\ r\end{bmatrix}_{k+1}\\
&\qquad+q^{(k+1)(N-i-kr-ks-r)}\begin{bmatrix} N-kr-ks-r-i \\ s\end{bmatrix}_k \begin{bmatrix} N-kr-ks-1-i \\ r-1\end{bmatrix}_{k+1}\Bigg\}\\
&\quad=\sum_{i=0}^{k-1} q^{Ni-\binom{i}{2}} \sum_{-\infty<r,s<\infty}(-1)^{s+i} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-i+1}{2}}\\
&\qquad\times\Bigg\{q^{k(N-i-k(r+s)-r+i+1-s-i)}\begin{bmatrix} N-kr-ks-r \\ s+i-1\end{bmatrix}_k \begin{bmatrix} N-kr-ks-i \\ r-i\end{bmatrix}_{k+1}\\
&\qquad+q^{(k+1)(N-i-k(r+s)-r+i)}\begin{bmatrix} N-kr-ks-r \\ s+i\end{bmatrix}_k \begin{bmatrix} N-kr-ks-i-1 \\ r-i-1\end{bmatrix}_{k+1}\Bigg\}
\end{aligned}$$

(having replaced s by $s+i$ and r by $r-i$)

$$= \sum_{i=0}^{k-1} q^{Ni-\binom{i}{2}} \sum_{-\infty<r,s<\infty} (-1)^{s+i} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-i+1}{2}}$$
$$\times q^{k(N-i-k(r+s)-r-s+1)} \begin{bmatrix} N-kr-ks-r \\ s+i-1 \end{bmatrix}_k \begin{bmatrix} N-i-kr-ks \\ r-i \end{bmatrix}_{k+1}$$
$$+ \sum_{i=1}^{k} q^{N(i-1)-\binom{i-1}{2}} \sum_{-\infty<r,s<\infty} (-1)^{s+i-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-i+2}{2}}$$
$$\times q^{(k+1)(N-kr-ks-r)} \begin{bmatrix} N-kr-ks-r \\ s+i-1 \end{bmatrix}_k \begin{bmatrix} N-i-kr-ks \\ r-i \end{bmatrix}_{k+1}$$

(having replaced i by $i-1$ in the second sum).

Now examination of the exponents on x and q reveals that each term in the second sum for $1 \leqslant i \leqslant k-1$ is the negative of each term in the first sum. Hence all that remains after cancellation is the term $i=0$ in the first sum and the term $i=k$ in the second.

Hence

$$\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}} \big(\sigma(k, N-i) - \sigma(k, N-i-1)\big)$$
$$= \sum_{-\infty<r,s<\infty} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}+k(N-(k+1)(r+s)+1)}$$
$$\times \begin{bmatrix} N-kr-ks-r \\ s-1 \end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1}$$
$$+ q^{N(k-1)-\binom{k-1}{2}} \sum_{-\infty<r,s<\infty} (-1)^{s+k-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-k+2}{2}}$$
$$\times q^{(k+1)(N-kr-ks-r)} \begin{bmatrix} N-kr-ks-r \\ s+k-1 \end{bmatrix}_k \begin{bmatrix} N-k-kr-ks \\ r-k \end{bmatrix}_{k+1}$$
$$:= S_1 + S_2. \tag{3.8}$$

Let us now define

$$S_3 := x^k q^{N+(N-1)+\cdots+(N-k+1)} \sigma(k, N-k) \tag{3.9}$$
$$= x^k q^{kN-\binom{k}{2}} \sum_{-\infty<r,s<\infty} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}}$$
$$\times \begin{bmatrix} N-k-kr-ks-r+1 \\ s \end{bmatrix}_k \begin{bmatrix} N-k-kr-ks \\ r \end{bmatrix}_{k+1}$$
$$= q^{kN-\binom{k}{2}} \sum_{-\infty<r,s<\infty} (-1)^{s-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s-1)^2+(k+1)\binom{r+1}{2}}$$
$$\times \begin{bmatrix} N-kr-ks-r+1 \\ s-1 \end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1}$$

(where we have replaced s by $s-1$). (3.10)

In order to complete the proof of the recurrence (3.7) for $\sigma(k, n)$ we need only show that

$$S_1 + S_2 = -S_3.$$

Now

$$\begin{aligned}
&S_1 + S_3 \\
&= \sum_{-\infty<r,s<\infty} (-1)^s x^{ks+(k+1)r} \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1} \\
&\quad \times \left\{ q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}+k(N-(k+1)(r+s)+1)} \begin{bmatrix} N-kr-ks-r \\ s-1 \end{bmatrix}_k \right. \\
&\quad \left. - q^{kN-\binom{k}{2}+\binom{k+1}{2}(r+s-1)^2+(k+1)\binom{r+1}{2}} \begin{bmatrix} N-kr-ks-r+1 \\ s-1 \end{bmatrix}_k \right\} \\
&= -\sum_{-\infty<r,s<\infty} (-1)^s x^{ks+(k+1)r} \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1} q^{kN-\binom{k}{2}+\binom{k+1}{2}(r+s-1)^2+(k+1)\binom{r+1}{2}} \\
&\quad \times q^{k(N-kr-ks-r-s+2)} \begin{bmatrix} N-kr-ks-r \\ s-2 \end{bmatrix}_k \\
&\quad \text{(by [1, Eq. (3.3.3), p. 35])} \\
&= -\sum_{-\infty<r,s<\infty} (-1)^{s+k+1} x^{ks+(k+1)r} \begin{bmatrix} N-k-kr-ks \\ r-k \end{bmatrix}_{k+1} \\
&\quad \times q^{kN-\binom{k}{2}+\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-k+1}{2}} \\
&\quad \times q^{k(N-k(r+s+1)-(r+s+1)+2)} \begin{bmatrix} N-kr-ks-r \\ s+k-1 \end{bmatrix}_k \\
&= -S_2.
\end{aligned}$$

Thus $S_1 + S_2 = -S_3$; so the desired recurrence is established for $\sigma(k,n)$. Consequently $S(k,n) = \sigma(k,n)$ for all $k \geqslant 1, n \geqslant 0$ which is the result to be proved. □

4. Further remarks

4.1. A probabilistic interpretation of $\chi(q)$

The mock theta function $\chi(q)$ has an interpretation in terms of conditional probabilities in some probability space. Let $0 < q < 1$, and let $C_1, C_2, \ldots$ be a sequence of independent events with probabilities

$$P(C_n) = 1 - q^n, \quad n = 1, 2, 3, \ldots.$$

Define events A and B by

$$A = \bigcap_{n=1}^{\infty} (C_n \cup C_{n+1}),$$

$$B = \bigcap_{n=2}^{\infty} (C_n \cup C_{n+1}).$$

Theorem 4.1. *The following relations hold*:

$$\mathbf{P}(A|B) = (1-q)\chi(q),$$
$$\mathbf{P}(C_1|A) = 1/\chi(q).$$

Proof. Let

$$F(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Holroyd, Liggett and Romik [4] proved that

$$\mathbf{P}(A) = \frac{G_2(q)}{F(q)},$$

and by a similar argument it follows that

$$\mathbf{P}(B) = \frac{H_2(q)}{(1-q)F(q)}.$$

Then, using (2.5):

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(1-q)G_2(q)}{H_2(q)} = (1-q)\chi(q),$$
$$\mathbf{P}(C_1|A) = \frac{\mathbf{P}(C_1\cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(C_1\cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(C_1)\mathbf{P}(B)}{\mathbf{P}(A)}$$
$$= \frac{(1-q)H_2(q)/(1-q)F(q)}{G_2(q)/F(q)} = 1/\chi(q).$$

Incidentally, since probabilities are between 0 and 1, we get that for $0 < q < 1$,

$$\chi(q) < \frac{1}{1-q}. \qquad \square$$

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