PARTITION IDENTITIES FROM THIRD AND SIXTH ORDER MOCK THETA FUNCTIONS

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ABSTRACT. In a recent paper, the first author showed a connection between bilateral basic hypergeometric series and mock theta functions, which leads to many new identities involving mock theta functions. This paper is a sequel, and our goal is to provide partition theoretic properties of new identities involving third or sixth order mock theta functions. In his monograph, N.J. Fine gave partition theoretic interpretation for mock theta functions and derived many interesting arithmetic properties from various identities involving mock theta functions. Our theorems are inspired by Fine's work even though we have to rely on the theory of modular forms to prove some theorems.

1. INTRODUCTION

In his famous last letter to G.H. Hardy [10], S. Ramanujan introduced mock theta functions without giving an explicit definition. Ramanujan introduced 17 examples of mock theta functions in his letter. Among them, the third order mock theta functions are

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \quad \phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \\ \psi(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \quad \chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-q^m+q^{2m})}, \end{aligned} \tag{1.1}$$

where

$$(a;q)_0 := 1, \text{ and } (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \text{ for any positive integer } n.$$

Later, G. N. Watson [38] added three functions to the list of Ramanujan's third order mock theta functions. These are

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \quad \upsilon(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \quad \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{m=1}^{n+1}(1+q^{2m-1}+q^{4m-2})}. \tag{1.2}$$

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These three third order mock theta functions are actually in Ramanujan's Lost Notebook [32]. In Ramanujan's Lost Notebook [32], we are also able to find Ramanujan's sixth and tenth order mock theta functions. Among them, Ramanujan's sixth order mock theta functions are

$$\begin{aligned} \Phi(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, \quad \Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}}, \\ \rho(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}}, \quad \sigma(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}}, \\ \lambda(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}, \quad \mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n}, \quad \nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}. \end{aligned} \tag{1.3}$$

G. E. Andrews and D. Hickerson [7] established the results for sixth order mock theta functions that are similar to the mock theta conjectures. Recently, B. C. Berndt and S. H. Chan [9], and R. J. McIntosh [28] independently discovered two new sixth order mock theta functions $\phi_-(q)$ and $\psi_-(q)$ which are

$$\Phi_-(q) = \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-1}}{(q;q^2)_n} \quad \text{and} \quad \Psi_-(q) = \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n}. \tag{1.4}$$

To see the history of mock theta functions and their modern and classical developments, we recommend the survey papers [22] and [31]. In addition to their mysterious analytic properties, mock theta functions have numerous nontrivial connections to combinatorics, especially the theory of partitions [3], [6] and [19]. For example, a third order mock theta function $f(q)$ is a generating function for the number of partitions of n with even rank minus the number of partitions of n with odd rank, where the rank of a partition is defined to be its largest part minus the number of its parts.

The n-color partition and its overpartition analogue have been employed to understand q-series identities combinatorially. The n-color partition was introduced by A. K. Agarwal and G. E. Andrews [4], and its overpartition analogue was introduced by J. Lovejoy and O. Mallet [27]. The n-color partition and its overpartition analogue arise naturally, and have a connection to many other combinatorial objects [1], [2], [4] and [5]. An n-color partition of a positive integer v is a partition in which each part of size n may appear up to n different colors denoted by subscripts from 1 to n, and parts are ordered first by the size of part and then according to the color. Since we have n different copies of part n, we also call it as a partition with "n copies of n". For example, there are 6 n-color partitions of 3;

$$3_3, \quad 3_2, \quad , 3_1, \quad 2_2 1_1, \quad 2_1 1_1, \quad 1_1 1_1 1_1.$$

We define the weighted difference of two parts m_i, n_j denoted by $((m_i - n_j))$, as $m-n-i-j$ provided $m \geq n$. An n-color overpartition of a positive integer v is an n-color partition of v in which we may overline the final occurrence of each part n_j. For example, the n-color overpartitions of 2 are

$$2_2,\ \ \overline{2_2},\ \ 2_1,\ \ \overline{2_1},\ \ 1_1 1_1,\ \ 1_1\overline{1_1}.$$

We also define the weighted difference of two parts m_i, k_j in an n-color overpartition denoted also by $((m_i - k_j))$ as $m-k-i-j-\chi(m_i)-\chi(k_j)$ provided $m \geq k$, where $\chi(k_j) = 1$ if k_j is an overlined part, and 0 otherwise. We note that this definition coincides with the definition of a weighted difference of n-color partition if there is no overlined part.

In [3], Agarwal interpreted a third order mock theta function $\psi(q)$ and three fifth order mock theta functions $F_0(q)$, $\Phi_0(q)$, $\Phi_1(q)$ as generating functions of certain kinds of n-color partitions by using q-difference equations. His interpretation for $\psi(q)$, is as follows.

Theorem 1.1. *$\psi(q)$ generates n-color partitions satisfying*

(1) *the weighted difference between two consecutive parts is always 0,*
(2) *the smallest part is of the form k_k,*
(3) *even parts have even colors and odd parts have odd colors.*

In [18], the first author showed a connection between bilateral basic hypergeometric series and mock theta functions, which leads to many new identities involving mock theta functions. This paper is a sequel of [18], and the purpose of this paper is to provide partition theoretic properties of third order mock theta functions $\phi(q)$, $\psi(q)$, $\upsilon(q)$ and sixth order mock theta functions $\Psi(q)$, $\Psi_{-}(q)$, $\rho(q)$, $\lambda(q)$. Our first goal of this paper is to derive partition-theoretic interpretations for the mock theta functions above as generating functions of n-color partitions or n-color overpartitions. In particular, we will give a bijective proof of Theorem 1.1 in a constructive way, and describe similar partition-theoretic interpretation for the others. For example, the sixth order mock theta function $\Psi(q)$ can be interpreted as follows.

Theorem 1.2. *Let us define λ^1 as the largest part in the partition λ and $c(\lambda^i)$ is the color of λ^i. Then, $\Psi(q)$ generates n-color overpartitions satisfying*

(1) *the smallest part is of the form k_k and not overlined,*
(2) *the weighted differences between two consecutive parts are even and ≥ 0, where the exponent of (-1) is given by $\frac{\lambda^1 + c(\lambda^1) + \chi(\lambda^1) - 2}{2}$.*

From Theorem 1.1, we easily conclude the following corollary.

Corollary 1.3. *There is a bijection between n-color partitions described in Theorem 1.1 and partitions into odd parts without gaps. Moreover, if λ is an n-color partition corresponding to*

σ, a partition into odd parts without gaps, then $\sum_{i=1}^{\ell(\lambda)} c(\lambda^i) = \ell(\sigma)$, where $\ell(\lambda)$ is the number of parts in the partition λ. In other words, the sum of the subscripts (the colors) of each part of λ is the same as the number of parts in σ.

Even though the first part of Corollary 1.3 was first observed by Agarwal [3], a bijective proof had been unknown.

The second goal of this paper is to derive arithmetic properties from mock theta function identities. Every identity we examine is of the form: a linear combination of two mock theta functions is equal to a theta function. These identities yield interesting combinatorial facts about the coefficients of mock theta functions. In [19, Chapters 2 and 3], N.J. Fine gave a partition theoretic interpretation for mock theta functions, and derived many interesting properties from various identities involving mock theta functions. In particular, Fine showed that

$$f(q) = \sum_{n=0}^{\infty} \left(p(n,0,2) - p(n,1,2)\right) q^n,$$
$$\phi(q) = \sum_{n=0}^{\infty} \left(p(n,0,4) - p(n,2,4)\right) q^n,$$

and

$$\chi(q) = \sum_{n=0}^{\infty} \left(p(n,0,6) + p(n,1,6) - p(n,2,6) - p(n,3,6)\right) q^n,$$

where $f(q)$, $\phi(q)$, and $\chi(q)$ are third order mock theta functions defined by (1.1), and $p(n,d,N)$ denotes the number of partitions of n with rank $\equiv d \pmod{N}$. By using a linear relation between third order mock theta functions, he proved that

$$\sigma(2n) = p(2n,1,4) - p(2n,2,4) \tag{1.5}$$

where $\sigma(n)$ denotes the number of partitions of n into distinct odd parts without gaps. Our theorems in this paper are inspired by Fine's work in [19, Chapters 2 and 3] even though we have to rely on the theory of modular forms to prove Theorems 3.5 and 6.1.

Theorem 1.4. *We define $\beta(n) := \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)}$, where the sum runs over partitions into distinct odd parts $\leq 2\ell(\lambda) - 1$ except that 1 can be repeated and $\ell(\lambda)$ denotes the number of parts in a partition λ. Then, for all positive integers n, we have*

$$2\beta(2n) = p(2n,1,2) - 2p(2n,2,4),$$
$$2\beta(2n-1) = p(2n-1,1,2) - 2p(2n-1,0,4).$$

We also discuss Ramanujan type congruences and cranks by analyzing theta functions, which are linear sums of mock theta functions. If we say that $A_f(n)$ is the number of partitions of n with the generating function f, then we have the following congruences.

Theorem 1.5. *For all $n \geq 0$, we have*

$$A_{\Psi}(3n+3) + 2A_{\Psi_-}(3n+3) \equiv 0 \pmod 9$$

and

$$2A_{\rho}(3n+2) + A_{\lambda}(3n+2) \equiv 0 \pmod 9.$$

This paper is organized as follows. In Section 2, we introduce necessary definitions and theorems for the reminder of this paper. In Section 3, we provide combinatorial interpretations for the third order mock theta functions $\phi(q)$, $\psi(q)$, $\nu(q)$, and study their arithmetic properties. In Section 4, we study the combinatorial properties of two sixth order mock theta function identities which are proved in Section 6, and give a combinatorial interpretation for sixth order mock theta functions $\Psi(q)$, $\Psi_-(q)$, $\rho(q)$ and $\lambda(q)$ by using n-color overpartitions. In Section 5, we introduce Garvan-Kim-Stanton type crank functions for the congruences given in Section 4. In Section 6, we prove two identities involving sixth order mock functions. Finally, we conclude with a few remarks.

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2. Preliminaries

In this section, we summarize the basic definitions and theorems for partitions, q-series and modular forms.

Partitions A partition of a positive integer n is a weakly decreasing sequence of positive integers $(\lambda^1, \ldots, \lambda^r)$ such that $\lambda^1 + \cdots + \lambda^r = n$. We denote the number being partitioned by $|\lambda|$. If λ is a partition of n, then we denote that $\lambda \vdash n$. Throughout this paper, we denote $A_f(n)$ be the coefficient of q^n in the q-expansion of f. If f is a generating function for certain partitions, then we regard $A_f(n)$ as the number of such partitions of n counted by f.

p-modular Ferrers diagram. We introduce a p-modular Ferrers diagram. For a partition λ into parts λ^i congruent to r modulo p where $0 < r \leq p$, its p-modular Ferrers diagram is the diagram in which the i-th row has $\lceil \lambda^i/p \rceil$ boxes, the boxes in the last column have r, and the other boxes have p. It can easily be seen that the sum of the numbers in the boxes equals $|\lambda|$. We define the M_p-rank of partition λ as $\lceil \frac{\lambda^1}{p} \rceil$ - $\ell(\lambda)$. In other words, the M_p-rank of partition λ is the number of boxes in the largest part in the p-modular diagram minus the number of parts of λ.

2	2	2	2	1
2	2	2	1	
2	2			
1				

Figure 1. two modular diagram of a partition $\lambda = (9, 7, 4, 1)$ with M_2-rank $= 1$.

***t*-residue diagram** In the Ferrers diagram of a partition λ, we color the box at row r and column c by $c - r \pmod t$. Thus, we have t different colors, denoted by 0, 1, . . . , $t - 1$. We denote $r_j(\lambda)$ as the number of boxes with color j in the Ferrers diagram of a partition λ. For example, examine Figure 2.

0	1	2	0	1
2	0	1	2	
1	2			
0				

Figure 2. 3-residue diagram of a partition $\lambda = (9, 7, 4, 1)$ with $[r_0(\lambda), r_1(\lambda), r_2(\lambda)] = [4, 4, 4]$.

***t*-core partition**. A partition λ is said to be a t-core if there are no hook numbers that are multiples of t. For example, in Figure 3, λ is a 5-core partition. Let $a_t(n)$ be the number of

8	6	4	3	1
6	4	2	1	
3	1			
1				

Figure 3. a 5-core partition $\lambda = (5, 4, 2, 1)$ with hook numbers.

t-core partitions of n. Then, it is well-known [20] that

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}. \tag{2.1}$$

***q*-series.** We define Ramanujan's general theta function $f(a, b)$ as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Then, Jacobi's triple product identity [8, p. 10] asserts that

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{2.2}$$

where

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

We also need Jacobi's identity [8, p. 14]

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{2.3}$$

We also introduce the following space saving notations;

$$\begin{aligned}(a_1,a_2,\ldots,a_m;q)_n &:= (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,\\ (a_1,a_2,\ldots,a_m;q)_\infty &:= (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty,\\ (a)_n &:= (a;q)_n, \text{ and } (a)_\infty := (a;q)_\infty.\end{aligned}$$

Modular forms Now we give the basic properties of modular forms. For more details on this subject, consult [30], [34] and [36].

Definition. For $z \in \mathbb{H}$ and any positive integers n, m, define

$$\eta(nz) := \eta_n = q^{\frac{n}{24}}(q^n;q^n)_\infty \tag{2.4}$$

and

$$\eta_{n,m}(z) := \eta_{n,m} = q^{P_2(\frac{m}{n})\frac{n}{2}}\frac{f(-q^m,-q^{n-m})}{(q^n;q^n)_\infty}, \tag{2.5}$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second Bernoulli function, and $\{t\} := t - [t]$ is the fractional part of t.

In this paper, we only consider the cases when $m \not\equiv 0 \pmod n$ for $\eta_{n,m}$.

We define the modular group $\Gamma = SL_2(\mathbb{Z})$ and its congruence subgroups $\Gamma_0(N) := \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma : c \equiv 0 \pmod N\right\}$ and $\Gamma_1(N) := \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(N) : a \equiv b \equiv 1 \pmod N\right\}$. For a fixed real number r, a function $F(z)$, defined and meromorphic in $\mathbb{H}$, is said to be a modular form of weight r with respect to Γ, with multiplier system v, if (a) $F(z)$ satisfies $F(Mz) = v(M)(cz+d)^rF(z)$ for any $z \in \mathbb{H}$ and $M \in \Gamma$, (b) there exists a standard fundamental region R such that $F(z)$ has at most finitely many poles in $\bar{R} \cap \mathbb{H}$, and (c) $F(z)$ is meromorphic at q_j, for each cusp q_j in $\bar{R}$.

Let $\{\Gamma, r, v\}$ denote the space of modular forms of weight r and multiplier system v on Γ, where Γ is a subgroup of $\Gamma(1)$ of finite index. When a multiplier system v is trivial, we denote $\{\Gamma, r, \upsilon\}$ as $\mathcal{M}_r(\Gamma)$. Let $ord(f;z)$ denote the invariant order of a modular form f at z. If $z \in \mathbb{H}$, then $Ord_\Gamma(f;z) := \frac{1}{\ell} ord(f;z)$, where ℓ ($\ell = 1, 2, \text{or } 3$) is the order of z as a fixed point of Γ. If z is a cusp with respect to Γ, $Ord_\Gamma(f;z) := N(\Gamma;z)\ ord(f;z)$, where $N(\Gamma;z)$ is the width of Γ at z.

Theorem 2.1. *The Dedekind eta-function $\eta(z)$ is a modular form of weight $\frac{1}{2}$ with multiplier system υ_η on $\Gamma(1)$, where the multiplier system υ_η is given by the following formula: for each* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$,

$$\upsilon_\eta(M) = \begin{cases} \left(\dfrac{d}{|c|}\right) \zeta_{24}^{bd(1-c^2)+c(a+d)-3c}, & \text{if } c \text{ is odd}, \\ \left(\dfrac{c}{|d|}\right) \zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd and either } c \geq 0 \text{ or } d \geq 0, \\ -\left(\dfrac{c}{|d|}\right) \zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd, } c < 0,\ d < 0, \end{cases}$$

where ζ_{24} is a primitive 24th root of unity.

Proof. See Theorem 2 in the page 51 of [25]. □

Theorem 2.2 (the valence formula)**.** *If $f \in \{\Gamma, r, \nu\}$ and $f \neq 0$, then*

$$\sum_{z \in R} Ord_\Gamma(f;z) = \mu r,$$

where R is any fundamental region for Γ, and $\mu := \frac{1}{12}[\Gamma(1) : \Gamma]$.

Proof. See Theorem 4.1.4 in [34]. □

Lemma 2.3. *If $m_1, m_2, \ldots, m_{2n}$ are positive integers, n is a positive integer, N is a positive even integer, and the least common multiple of $m_1, m_2, \ldots, m_{2n}$ divides N, then, for $z \in \mathbf{H}$,*

$$\eta(m_1 z)\eta(m_2 z) \cdots \eta(m_{2n} z) \in \{\Gamma_1(N), n, \upsilon\},$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, ζ_{24} is a primitive 24th root of unity, and

$$\upsilon(A) = \prod_{i=1}^{2n} \left(\frac{c/m_i}{|d|}\right) \zeta_{24}^{ac(1-d^2)/m_i + d(m_i b - c/m_i) + 3(d-1)}.$$

Proof. See Lemma 2.7. in [17]. □

Theorem 2.4. *For $z \in \mathbb{H}$, let $f(z) := \prod_{n|N,\ 0 \leq m < n} \eta_{n,m}^{r_{n,m}}(z)$, where $r_{n,m}$ are integers. If*

$$\sum_{n|N,\ 0 \leq m < n} n P_2\left(\frac{m}{n}\right) r_{n,m} \equiv 0 \pmod 2$$

and

$$\sum_{n|N,\ 0\leq m<n} \frac{N}{n} P_2(0) r_{n,m} \equiv 0 \pmod 2,$$

then $f(z) \in \{\Gamma_1(N), 0, I\}$, *where for* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, $I(M) = 1$.

Proof. See Theorem 3 in [36, p. 126]. □

Lemma 2.5. *Let* ℓ, m *and* n *be positive integers. Then, for a cusp* $k = \frac{\lambda}{\mu\epsilon}$ *for* $\Gamma_1(N)$, *where* $\epsilon \mid N$ *and* $(\lambda, N) = (\lambda, \mu) = (\mu, N) = 1$,

$$ord(\eta_{n,m}; k) + ord(\eta_n; k) \geq 0, \quad ord(\eta_{\ell n,m}; k) + \ell\ ord(\eta_n; k) \geq 0,$$

and

$$ord(\eta_{n,m}; k) + \ell\ ord(\eta_{\ell n}; k) \geq 0.$$

Proof. See Lemma 2.10 in [17]. □

For a prime p, we define the U_p-operator as follows. If $f(q)$ has a Fourier expansion $f(q) = \sum a(n)q^n$, then we define the U_p-operator by

$$U_p f(z) := \sum a(pn)q^n.$$

It is well known that $U_p f(z) \in \mathcal{M}_0(\Gamma_0(Np))$ provided $f(z) \in \mathcal{M}_0(\Gamma_0(Np^2))$. For a fixed N and integers r_i's, a function of the form

$$f(z) := \prod_{\substack{n|N \\ n>0}} \eta(nz)^{r_n}. \tag{2.6}$$

is called an η-quotient. The following theorem in [29] shows when an η-quotient becomes a modular function.

Theorem 2.6. *The* η*-quotient* (2.6) *is in* $\mathcal{M}_0(\Gamma_0(N))$ *if and only if*

(1) $\sum_{n|N} r_n = 0$,
(2) $\sum_{n|N} nr_n \equiv 0 \pmod{24}$,
(3) $\sum_{n|N} \frac{N}{n} r_n \equiv 0 \pmod{24}$,
(4) $\prod_{n|N} n^{r_n} \in \mathbf{Q}^2$.

The following theorem in [26] gives the order of the η-quotient f at the cusps c/d of $\Gamma_0(N)$ provided $f \in \mathcal{M}_0(\Gamma_0(N))$.

Theorem 2.7. *If the* η*-quotient* $f \in \mathcal{M}_0(\Gamma_0(N))$, *then its order at the cusp* c/d *of* $\Gamma_0(N)$ *is*

$$\frac{1}{24} \sum_{n|N} \frac{N(d,n)^2 r_n}{(d, N/d)dn}.$$

Recall that if $p|N$ and $f \in \mathcal{M}_0(\Gamma_0(pN))$, then $U_p f \in \mathcal{M}_0(\Gamma_0(N))$. Also, the following theorem in [21] gives bounds on the order of $U_p f$ at cusps of $\Gamma_0(N)$ in terms of the order of f at cusps of $\Gamma_0(pN)$.

Theorem 2.8. *Let p be a prime and $\pi(n)$ be the highest power of p dividing n. Suppose that $f \in \mathcal{M}_0(\Gamma_0(pN)$, where $p|N$ and $\alpha = c/d$ is a cusp of $\Gamma_0(N)$. Then,*

$$ord_\alpha U_p f \geq \begin{cases} \frac{1}{p} ord_{\alpha/p} f, & \text{if } \pi(d) \geq \frac{1}{2}\pi(N), \\ ord_{\alpha/p} f, & \text{if } 0 < \pi(d) < \frac{\pi(N)}{2}, \\ \min_{0 \leq \beta \leq p-1} ord_{(\alpha+\beta)/p} f, & \text{if } \pi(d) = 0. \end{cases}$$

3. Third order mock theta function identities

The first identity we examine is

$$\phi(q) + 2\psi(q) = \frac{(q^2;q^2)_\infty^7}{(q)_\infty^3 (q^4;q^4)_\infty^3} = (-q;q^2)_\infty \sum_{n=-\infty}^{\infty} q^{n^2} \tag{3.1}$$

where $\phi(q)$ and $\psi(q)$ are third order mock theta functions. We are able to find the equations above in [19, p. 60].

In [3], Agarwal showed that $\psi(q)$ is a generating function for certain n-color partitions by using q-difference equations. Here, we obtain the same results in a constructive way. This will give a bijective proof for Theorem 1.1.

Proof of Theorem 1.1. In this proof, we always use 2-modular Ferrers diagrams. Recall that q^{n^2} generates partition $\tau = (1, 3, \ldots, 2n-1)$. We assign to each part color 1. Note that the weight difference between two consecutive parts is 0. Recall that $\frac{1}{(q;q^2)_n}$ generates partitions λ into odd parts $\leq 2n-1$. From the largest part of λ, we attach each part λ^i as follows. We first attach 2 from the first row to the $\frac{\lambda^i-1}{2}$-th row and attach 1 to the $\frac{\lambda^i+1}{2}$-th part. Then, we increase the color by 1 for the $\frac{\lambda^i+1}{2}$-th part of the resulting partition. For example, examine Figure 4. Note that during this process the weight difference between two consecutive parts remains the same. The second condition is clear from this construction. Since the color is increased by 1 when the parity of part is changed, the third condition holds. □

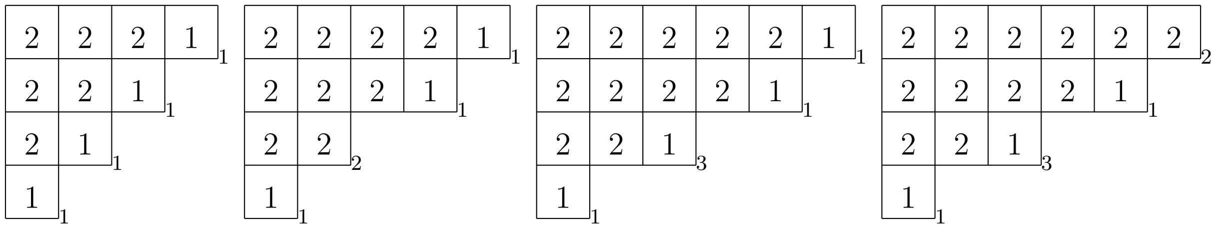

Figure 4. $\tau = (7, 5, 3, 1)$ with $\lambda = (5, 5, 1)$.

Remark. Actually, the last condition in Theorem 1.1 is not necessary. Since the weighted differences between two parts are always 0 and the smallest part is k_k, we can conclude that the parity of parts and their color should be the same.

By using the bijection above, we are now ready to prove Corollary 1.3.

Proof of Corollary 1.3. For a given n-color partition σ enumerated by $\psi(q)$, we can easily recover the partition τ and λ by reading the color of each part. By inserting parts in λ to τ in weakly decreasing order, we arrive at μ, a partition into odd parts without gaps. Since $\sum_{i=1}^{\ell(\sigma)} c(\sigma^i) = \ell(\tau) + \ell(\lambda) = \ell(\mu)$, this completes the proof. □

Example. An n-color partition $(12_2, 9_1, 5_3, 1_1)$ corresponds to the partition $(7, 5, 5, 5, 3, 1, 1)$.

Analogously, we also can obtain an n-color partition theoretic interpretation for $\phi(q)$.

Theorem 3.1. *$\phi(q)$ generates n-color partitions λ satisfying*

(1) *the smallest part is of the form $(2k-1)_k$,*
(2) *the color of λ_i is given by $\frac{\lambda_i - \lambda_{i+1}}{2}$ except the smallest part, and the exponent of (-1) is given by M_2-rank of λ.*

Remark. Since the color of each part is an integer, the conditions above imply that all parts are odd.

Proof. The constructive proof is very similar to the proof of Theorem 1.1, so we omit it. Alternatively, by splitting the partition counted by $A_\phi(m, v)$ into two classes: partitions having 1_1 as a part and the partitions without 1_1, we can see that

$$A_\phi(m, v) = A_\phi(m-1, v-2m+1) - A_\phi(m, v-2m), \tag{3.2}$$

where $A_\phi(m, v)$ is the number of n-color partitions of v with m parts. If we define

$$f(z, q) := \sum_{v,m=0}^{\infty} A_\phi(m, v) z^m q^v,$$

then, by using (3.2), we can deduce that

$$f(z, q) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(-q^2; q^2)}.$$

By setting $z = 1$, we complete the proof. □

By (3.1), it is clear that $A_\phi(\nu) + 2A_\psi(\nu) \geq 0$ for all $\nu \geq 1$. Now we show that

$$A_\phi(\nu) + A_\psi(\nu) \geq 0,$$

for all $\nu \geq 1$. To this end, we introduce a new function $\phi^*(q)$, which is defined by

$$\phi^*(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}.$$

Note that $\phi^*(q)$ generates n-color partitions described in Theorem 3.1 except that the weight is always 1. Let λ be a partition enumerated by $\phi^*(q)$. We subtract $c(\lambda^i) - 1$ from λ^i if $c(\lambda^i) > 1$, and denote the resulting partition as μ. Let r be the sum

$$\sum_{c(\lambda^i)>1} \big(c(\lambda^i) - 1\big) = \left(\sum_{1 \leq i \leq \ell(\lambda)} c(\lambda^i) \right) - \ell(\lambda).$$

We attach r to the largest part of μ, and also increase the color by r. Then, we observe that the resulting partition σ is an n-color partition counted by $\psi(q)$. Since each λ corresponds to a different σ, we have proven that

$$A_\phi(\nu) + A_\psi(\nu) \geq 0.$$

Example. An n-color partition $\lambda = (13_2, 9_1, 7_3, 1_1)$ corresponds to $\mu = (12_2, 9_1, 5_3, 1_1)$ with $r = 3$. Then, the resulting partition $\sigma = (15_5, 9_1, 5_3, 1_1)$ satisfies the conditions in Theorem 1.1 as desired.

The second identity we investigate is

$$\upsilon(q) + \upsilon_3(q, q, ; q) = 2\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} \tag{3.3}$$

where $\upsilon(q)$ is defined by (1.2) and

$$\upsilon_3(q, q; q) = \frac{1}{1+q} \sum_{n=1}^{\infty} q^n (-q^{-1};q^2)_n$$

is the function defined by Choi [18]. We easily obtain (3.3) by replacing α and z by q and q respectively in Theorem 1 of [18].

Recall that the generating function of t-core partition is (2.1). Note also that

$$\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}(-q^2;q^2)_\infty.$$

Thus, the product on the right side of (3.3) generates partition pairs (λ, σ) where λ is a 2-core partition of even parts and σ is a partition into distinct even parts.

Remark. By Gauss identity [19, p. 6],

$$\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)}.$$

Therefore, every 2-core partition consisting of even parts is of the form $(2k, 2k-2, \ldots, 2)$.

Let $b(n)$ be the number of such partition pairs. Then, we can prove the following congruence.

Theorem 3.2. *For all nonnegative integers n,*

$$b(5n+3) \equiv 0 \pmod{5}. \tag{3.4}$$

Proof. By using Jacobi identity, we arrive at

$$\begin{aligned}\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} &= \frac{(q^4;q^4)_\infty^3(q^2;q^2)_\infty^3}{(q^2;q^2)_\infty^5}\\ &\equiv \frac{\left(\sum_{m=0}^\infty(-1)^m(2m+1)q^{2m(m+1)}\right)\left(\sum_{k=0}^\infty(-1)^k(2k+1)q^{k(k+1)}\right)}{(q^{10};q^{10})_\infty} \pmod{5}.\end{aligned}$$

Since $2m(m+1)+k(k+1) \equiv 3 \pmod{5}$ holds only if $m \equiv 2 \pmod{5}$ and $k \equiv 2 \pmod{5}$, the coefficient of q^{5n+3} is divisible by 5 as desired. □

We can also find an exact formula for the generating function of $b(5n+3)$ by using modular functions.

Theorem 3.3.

$$\sum_{n=0}^\infty b(5n+3)q^n = 5q\frac{(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^2(q^{20};q^{20})_\infty}{(q^2;q^2)_\infty^4}. \tag{3.5}$$

We will follow the argument in [21] to prove (3.5).

Proof. Define $F(z)$ as

$$F(z) := \frac{\eta^3(4z)\eta^2(10z)\eta(100z)}{\eta^2(2z)\eta^4(20z)}.$$

By Theorem 2.6, we have $F(z) \in \mathcal{M}_0(\Gamma_0(100))$. Note that $U_5 f(z) \in \mathcal{M}_0(\Gamma_0(20))$. Let us define $G(z)$ as

$$G(z) := \frac{\eta^2(10z)\eta^2(20z)}{\eta^2(2z)\eta^2(4z)}.$$

Then, by Theorem 2.6, we have $G(z) \in \mathcal{M}_0(\Gamma_0(20))$. From the order at each cusp by employing Theorems 2.7 and 2.8, we see that $\frac{U_5F}{G}$ is a holomorphic modular function, namely, a constant. From this, we can easily deduce that $U_5F(z) = 5G(z)$. Recall that $U_pf(pz)g(z) = f(z)U_pg(z)$. Thus, we arrive at

$$U_5\left(\sum_{n=0}^\infty b(n)q^{n+2}\right)\frac{(q^{20};q^{20})_\infty(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^4} = 5q^2\frac{(q^{20};q^{20})_\infty^2(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^2}$$

or

$$\sum_{n=0}^\infty b(5n+3)q^n = 5q\frac{(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^2(q^{20};q^{20})_\infty}{(q^2;q^2)_\infty^4},$$

as desired. □

Remark. Theorem 3.3 and (3.3) imply that

$$\sum_{n=1}^{\infty} \left(A_{\upsilon}(5n+3) + A_{\upsilon_3}(5n+3)\right) q^n = 10q \frac{(q^4;q^4)_\infty^2 (q^{10};q^{10})_\infty^2 (q^{20};q^{20})_\infty}{(q^2;q^2)_\infty^4}.$$

Now, we will show that the left side of (3.3) generates a certain type of n-color partitions. First, we note that

$$\upsilon(q) = \upsilon_+(q) + \upsilon_-(q),$$

where

$$\upsilon_+(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_n} \text{ and } \upsilon_-(q) := \sum_{n=1}^{\infty} \frac{(-1)q^{n(n-1)}q^{2n-1}}{(-q;q^2)_n}.$$

We see that $\upsilon_+(q)$ generates n-color partitions satisfying the following properties:

(1) the smallest part is of the form $(k+1)_k$,
(2) the weighted difference between any two consecutive parts is 0, where the exponent of (-1) is $k-1$, namely the color of the smallest part minus 1.

Remark. From the condition above, we observe that odd parts have even colors, and even parts have odd colors.

Similarly, we observe that $\upsilon_-(q)$ generates n-color partitions satisfying the following properties:

(1) the smallest part is of the form k_k,
(2) the weighted difference between any two consecutive parts not containing the smallest part is 0, and 1 otherwise, where the exponent of (-1) is k, namely the color of the smallest part.

Remark. We can see that odd parts have even colors and even parts have odd colors, except for the smallest part.

In summary, $\upsilon(q)$ generates n-color partitions satisfying the following conditions.

(1) the smallest part is of the form $(k+1)_k$ or k_k.
(2) the weighted difference between any two consecutive parts is 0 except that the weighted difference involving the smallest part of the form k_k is 1,

where the exponent of (-1) is $c(\lambda^{\ell(\lambda)} - 1$ if the smallest part is $(k+1)_k$ or $c(\lambda^{\ell(\lambda)}$ if the smallest part is k_k. Now we turn to $\upsilon_3(q,q;q)$. Let us define $\upsilon^*(q) = (1+q)\upsilon_3(q,q;q)$. If we allow 0_0 as a part, then $\upsilon^*(q)$ generates n-color partitions satisfying the following properties;

(1) the smallest part is of the form 1_1 or 0_0,
(2) the weighted difference for two consecutive parts is -2 except that the weight difference involving the part 0_0 is 0.

Let denote $A_{\upsilon^*}(\nu)$ as the number of such n-color partitions of ν. Then, we have

$$\begin{aligned}\upsilon_3(q,q;q) &= \frac{1}{1+q}\sum_{\nu=0}^{\infty} A_{\upsilon^*}(\nu)q^{\nu} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}A_{\upsilon^*}(k)\right)q^n.\end{aligned}$$

Since it is clear that $b(2n+1)=0$, we have

$$A_{\upsilon}(2\nu+1) = -\sum_{k=0}^{2\nu+1}(-1)^{2\nu+1-k}A_{\upsilon^*}(k) = \sum_{k=0}^{2\nu+1}(-1)^{k}A_{\upsilon^*}(k),$$

where $A_{\upsilon}(\nu)$ is the number of n-color partitions of ν generated by $\upsilon(q)$. We easily see that $A_{\upsilon_3}(\nu)>0$, for all $\nu\geq 1$. Thus, by (3.3), $A_{\upsilon}(2\nu+1)<0$ for all nonnegative integers ν.

We turn to prove Theorem 1.4.

Proof of Theorem 1.4. Replacing q by $-q$ and setting $\alpha=-q$ and $z=q$ in the first identity of Theorem 1 in [18], we arrive at

$$2\sum_{n=1}^{\infty}(-q)^n(-q^2;q^2)_{n-1} + \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q^2;q^2)_n} = \frac{f(-q,-q)}{(-q;q)_{\infty}}. \tag{3.6}$$

Note that the first sum generates partitions into n odd parts $\leq 2n-1$ such that

(1) the only repeatable part is 1,
(2) the exponent of (-1) is the number of parts.

Let $\mathcal{O}_1$ be the set of partitions λ into distinct odd parts $\leq 2\ell(\lambda)-1$ except that 1 can be repeated. Recall that

$$\beta(n) = \sum_{\substack{\lambda\vdash n\\ \lambda\in\mathcal{O}_1}}(-1)^{\ell(\lambda)}.$$

Note that the second sum is $\phi(-q)$. Thus, from the equation (26.66) in [19], we have

$$2\phi(-q)-f(q) = f(q)+4\psi(-q) = \phi(-q)+2\sum_{n=1}^{\infty}\beta(n)q^n. \tag{3.7}$$

Recall that $\phi(q)=\sum_{n=0}^{\infty}(p(n,0,4)-p(n,2,4))q^n$. Therefore, we arrive at

$$2\beta(n) = (-1)^n p(n,0,4)-(-1)^n p(n,2,4)-p(n,0,2)+p(n,1,2).$$

Using the fact that $p(n,0,2)=p(n,0,4)+p(n,2,4)$, we deduce that

$$2\beta(2n) = p(2n,1,2)-2p(2n,2,4)$$

and

$$2\beta(2n-1) = p(2n-1,1,2) - 2p(2n-1,0,4),$$

which completes the proof of Theorem 1.4. $\square$

Let $\mathcal{O}$ be the set of partitions into odd parts without gaps. Then, $\psi(-q) = \sum_{n=0}^{\infty} \gamma(n) q^n$ where

$$\gamma(n) := \sum_{\substack{\lambda \vdash n \\ \lambda \in \mathcal{O}}} (-1)^{\ell(\lambda)}.$$

Therefore, by (3.7), we are able to derive the following theorem.

Theorem 3.4. *For all positive integers n, $\gamma(n) = \beta(n)$.*

We provide a bijective proof.

Proof. Let λ be a partition in $\mathcal{O}$. Let σ be a partition consisting of parts $\lambda^i - 1$ for all $1 \le i \le \ell(\lambda)$. Let σ' be a partition obtained by conjugating 2-modular diagram of σ. We attach 1 from the first part to $\ell(\lambda)$-th part of σ'. Then, the resulting partition μ is in $\mathcal{O}_1$. Since the number of parts of λ and that of μ is the same, this completes the proof. $\square$

4. Sixth order mock theta function identities

In this section, we discuss the following two identities involving sixth order mock theta functions

$$\Psi(q) + 2\Psi_{-}(q) = 3\frac{q(q^6;q^6)_\infty^3}{(q)_\infty (q^2;q^2)_\infty} \tag{4.1}$$

and

$$2\rho(q) + \lambda(q) = 3\frac{(q^3;q^3)_\infty^3}{(q)_\infty (q^2;q^2)_\infty}. \tag{4.2}$$

We will prove these identities in Section 6.

First, note that the right sides of (4.1) and (4.2) generate partitions analogous to the partitions defined by

$$\frac{1}{(q)_\infty (q^2;q^2)_\infty},$$

which have been studied by H.C. Chan [11], [12], [13]. This partition function satisfies many congruences [14], [15], and [37], and a crank function for this partition and its overpartition analogue are studied by the second author [23] and [24].

Here, we study two analogous partition functions defined by

$$\sum_{n=1}^{\infty} c(n)q^n = \frac{q(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty} \tag{4.3}$$

and

$$\sum_{n=0}^{\infty} d(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}. \tag{4.4}$$

Remark. From the generating function for t-core partition 2.1, we can regard these partitions as 3-core partition analogues of H.-C. Chan's partitions.

We can easily prove that these two partition functions satisfy the following congruences.

Theorem 4.1.

$$c(3n) \equiv 0 \pmod 3, \tag{4.5}$$

$$d(3n+2) \equiv 0 \pmod 3. \tag{4.6}$$

Now we obtain exact generating functions for these arithmetic progressions. Since $\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}$ is a newform in $\mathcal{M}_2(\Gamma_0(6))$, we see that

$$U_3\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)} = 3\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}. \tag{4.7}$$

Remark. A classical proof of (4.7) can be found in Fine's book [19, (33.124)].

Proof of Theorem 4.1. By (4.3), (4.4) and (4.7), we see that

$$\left(\sum_{n=1}^{\infty} c(3n)q^n\right)(q)_\infty^3 = 3q\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}$$

and

$$\left(\sum_{n=1}^{\infty} d(3n-1)q^n\right)(q^2;q^2)_\infty^3 = 3q\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty},$$

which implies that

$$\sum_{n=1}^{\infty} c(3n)q^n = 3q\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty^4(q^2;q^2)_\infty}$$

and

$$\sum_{n=0}^{\infty} d(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty^4}.$$

□

Now, we will give a combinatorial interpretation for the sixth order mock theta functions $\Psi(q)$, $\Psi_-(q)$, $\rho(q)$ and $\lambda(q)$ by using n-color overpartitions.

Proof of Theorem 1.2. We rewrite $\Psi(q)$ as

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q^2)_{n+1}(-q^2;q^2)_n}.$$

Recall that $(n+1)^2$ generates partition into odd parts from 1 to $2n+1$. We assign the color 1 to each part. Then, the weight difference of two consecutive parts is 0. We attach each part λ^i in λ generated by $\frac{1}{(q^2;q^2)_n}$ as follows. We attach 2 from the first row to the $\frac{\lambda^i}{2}$-th row. Then, we can see that the weighted difference between the $\frac{\lambda^i}{2}$-th part and the $\frac{\lambda^i}{2}+1$-th part of the resulting partition increases by 2. We also attach each part σ^j in σ generated by $\frac{1}{(q;q^2)_{n+1}}$ as follows. We attach 2 from the first row to the $\frac{\sigma^j-1}{2}$-th row and attach 1 to the $\frac{\sigma^j+1}{2}$-th row. Then, we increase the color of the $\frac{\sigma^j+1}{2}$-th part of the resulting partition by 1. We can observe that this does not affect the weight difference. Finally, we attach each part of μ^k in μ generated by $(q;q^2)_n$ as we did for σ^j, and overline the $\frac{\mu^k+1}{2}$-th part of the resulting partition. We see that this also does not affect the weight difference. By tracking the exponent of (-1), we complete the proof. $\square$

By employing a similar argument, we can prove the following theorem.

Theorem 4.2. $\Psi_-(q)$ *generates n-color overpartitions satisfying*

(1) *the smallest part is of the form k_k, which cannot be overlined,*
(2) *the weighted difference between two consecutive parts is* 0 *or* -2.

$\rho(q)$ *generates n-color overpartitions satisfying that*

(1) *the smallest part is of the form k_k or $\overline{(k+1)_k}$,*
(2) *the weighted difference of two consecutive part is -2 if the smaller part is overlined and 0 or -1 if it involves the unoverlined smallest part and -1, otherwise.*

And $\lambda(q)$ generates n-color overpartitions λ satisfying

(1) *the smallest part is of the form k_k or $\overline{(k+1)_k}$,*
(2) *the weighted difference of two consecutive parts forms a non-decreasing sequence of which sum equals $-2(\ell(\lambda)-1)$, where the exponent of (-1) is the sum of colors plus the number of overlined parts.*

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Combining Theorems 1.2, 4.1, and 4.2, we can derive the following congruences. For all $n \geq 0$, we have

$$A_{\Psi}(3n+3) + 2A_{\Psi_-}(3n+3) \equiv 0 \pmod 9,$$
$$2A_{\rho}(3n+2) + A_{\lambda}(3n+2) \equiv 0 \pmod 9.$$

We have completed the proof of Theorem 1.5. □

5. Crank analogue for $c(n)$ and $d(n)$

Recall that $c(n)$ and $d(n)$ are partition functions defined by (4.3) and (4.4), respectively. We find a Garvan-Kim-Stanton type crank [20] for $c(n)$ and $d(n)$ by modifying a crank given in Z. Reti's thesis [35]. Since Reti's result has not been published and is not well-known, we give details from his thesis, and show how this crank can be extended to $c(n)$ and $d(n)$. Interested readers should consult [20] and [35]. The following lemma enables us to extend a crank for t-core partitions to a crank for ordinary partitions. Here and in the sequel, $\mathcal{P}$ denotes the set of ordinary partitions and $\mathcal{P}_t^*$ is the set of t-core partitions.

Lemma 5.1 (Bijection 1 of [20])**.** *There is a bijection between $\pi \in \mathcal{P}$ and $[\pi_0, \dots, \pi_{t-1}, \pi^*] \in \mathcal{P} \times \cdots \times \mathcal{P} \times \mathcal{P}_t$, which satisfies*

$$|\pi| = t\sum_{j=0}^{t-1} |\pi_j| + |\pi^*|.$$

Let us define the set

$$S^*(n) := \{[\pi(1), \pi(2)] \in \mathcal{P}_3^* \times \mathcal{P}_3^* : |\pi(1)| + 2|\pi(2)| = n\}.$$

Recall that $r_j(\pi)$ is the number of dots colored j in the 3-residue diagram of π. We define a coordinate system $\underline{a}$ by

$$\underline{a} := [r_0(\pi(1)) - r_1(\pi(1)), r_0(\pi(1)) - r_2(\pi(1)), r_0(\pi(2)) - r_1(\pi(2)), r_0(\pi(2)) - r_2(\pi(2))],$$

where $[\pi(1), \pi(2)] \in S^*(n)$. We understand $\#S^*(n, A)$ as the number of elements in the set S^* satisfying the property A. Now we are ready to give cranks for $S^*(3n+2)$.

Lemma 5.2 (Theorem 5 of [35])**.** *The following two vectors are cranks for $S^*(3n+2)$*

$$\underline{f}(1) := [-1, 1, -1, 1] \text{ and } \underline{f}(2) := [-1, 1, 1, -1],$$

in the sense of

$$\#S^*(3n+2, \underline{f} \cdot \underline{a} \equiv k \pmod 3) = \frac{\#S^*(3n+2)}{3},$$

for all $0 \leq k \leq 2$, where $\#(S)$ is the number of element in the set S.

Even though the two cranks above are defined only on the set of $S^*(n)$, we can extend these cranks to $S_1(n)$ (resp. $S_2(n)$) by using Lemma 5.1, where $S_1(n)$ (resp. $S_2(n)$) is the set of partitions enumerated by $c(n)$ (resp. $d(n)$). In the next proposition, we give such an extension in the spirit of [20, Proposition 1].

Proposition 5.3. *Let $[\pi(1), \pi(2)]$ be a partition in $S_1(n)$ or $S_2(n)$ and $r_j(\pi)$ be the number of j-colored boxes in the 3-residue diagram of π. Then, the following two linear combinations*

$$r_1(\pi(1)) - r_2(\pi(1)) + r_1(\pi(2)) - r_2(\pi(2)) \text{ and } r_1(\pi(1)) - r_2(\pi(1)) - r_1(\pi(2)) + r_2(\pi(2)),$$

are crank statistics for $S_1(n)$ and $S_2(n)$.

The proof of the above proposition is analogous to that of Proposition 1 in [20]. The key idea is that the above statistics are invariant under the removal of 3-rim hooks. By using Proposition 5.3, we can deduce the crank statistics, which can be calculated from the Ferrers diagram in the spirit of Theorem 3 in [20].

Theorem 5.4. *For all partitions $[\pi(1), \pi(2)] \in S_1(n)$ (or $S_2(n)$), we can define a crank from $f(1)$ by*

$$\sum_{j=1}^{2} \sum_{i=1}^{\ell(\pi(j))} \left(\delta(\pi(j)_i - i) - \delta(-i)\right),$$

where $\delta(x) = 1$ for $x \equiv 1 \pmod 3$ and 0, otherwise, and $\ell(\pi)$ is the number of parts in π. We can also define a crank from $f(2)$ by

$$\sum_{i=1}^{\ell(\pi(1))} \left(\delta(\pi(1)_i - i) - \delta(-i)\right) - \sum_{i=1}^{\ell(\pi(2))} \left(\delta(\pi(2)_i - i) - \delta(-i)\right).$$

The proof of Theorem 5.4 is easily obtained by calculating the contribution of each row to the crank from Proposition 5.3, so we omit it.

6. Proof of two sixth order mock theta function identities

In this section, we prove the following two identities which played an important role in Section 4.

Theorem 6.1. *For $|q| < 1$,*

$$\Psi(q) + 2\Psi_-(q) = 3\frac{q(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}, \tag{6.1}$$

$$2\rho(q) + \lambda(q) = 3\frac{(q^3;q^3)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}, \tag{6.2}$$

where $\Psi(q)$, $\Psi_-(q)$, $\rho(q)$, and $\lambda(q)$ are the sixth order mock theta functions defined by (1.3).

Before proving these identities, we need to prove the following two eta function identities. Throughout the proof, we let E_N be a complete set of inequivalent cusps for $\Gamma_1(N)$.

Theorem 6.2. *For $z \in \mathbb{H}$,*

$$-\eta_2^4\eta_4^2\eta_6^6\eta_{12}^2\eta_{4,2}^2\eta_{6,2}^6\eta_{12,2}^2 + 4\eta_1^2\eta_3^2\eta_4^8\eta_6^2\eta_{3,1}^2\eta_{6,1}^2 = 3\eta_1^8\eta_4^2\eta_6^2\eta_{12}^2. \tag{6.3}$$

Proof. For $1 \le i \le 3$, let f_i^1 be the product of eta-functions in each of the 3 products in (6.3), and for $1 \le i \le 2$, g_i^1 be the product of the generalized eta-functions in each of the 2 products in (6.3). Each f_i^1 is the product of 14 eta-functions, and by Lemma 2.3 and a straightforward calculation, each f_i^1 is a modular form of weight 7 on $\Gamma_1(72)$ with the multiplier system υ_1, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(72)$, $\upsilon_1(A) = \zeta_{24}^{4bd}$. By Theorem 2.4 and a straightforward calculation, each g_i^1 is a modular form of weight 0 on $\Gamma_1(72)$ with the multiplier system I. Therefore, $f_1^1g_1^1$, $f_2^1g_2^1$, and f_3^1 are modular forms of weight 7 on $\Gamma_1(72)$ with multiplier system υ_1.

Recall that $[\Gamma(1) : \Gamma_1(72)] = 3456$. Let F_1 denote the difference of the left and right sides of (6.3). By applying the three equations in Lemma 2.5 to F_1 and a straightforward calculation, we find that for each $k \in E_{72}$, $k \neq \infty$,

$$ord(F_1; k) \ge 0. \tag{6.4}$$

Applying Theorem 2.2 for a fundamental region R for $\Gamma_1(72)$, and using (6.4), we deduce that, for F_1,

$$\sum_{z\in R} Ord_{\Gamma_1(72)}(F_1; z) = \frac{7 \cdot 3456}{12} = 2016 \ge ord(F_1; \infty), \tag{6.5}$$

since both sides of (6.3) are analytic on R. Using *Maple*, we calculated the Taylor series of F_1 about $q = 0$ (or about the cusp $z = \infty$) and found that $F_1 = O(q^{2017})$. Unless F_1 is a constant, we have a contradiction to (6.5). We have thus completed the proof of Theorem 6.2. □

Theorem 6.3. *For $z \in \mathbb{H}$,*

$$-\eta_1^{16}\eta_4^4\eta_6^4\eta_{12}^4 + \eta_2^{16}\eta_3^8\eta_{12}^4\eta_{3,1}^4\eta_{12,2}^2 = 12\eta_1^{10}\eta_2^2\eta_3^2\eta_4^6\eta_6^2\eta_{12}^6. \tag{6.6}$$

Proof. For $1 \le i \le 3$, let f_i^2 be the product of eta-functions in each of the 3 products in (6.6), and $g^2 := \eta_{3,1}^4\eta_{12,2}^2$ be the product of the generalized eta-functions in the second term in (6.6). Each f_i^2 is the product of 28 eta-functions, and by Lemma 2.3 and a straightforward calculation, each f_i^2 is a modular form of weight 14 on $\Gamma_1(24)$ with the multiplier system υ_2, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(24)$, $\upsilon_2(A) = \zeta_{24}^{8bd}$. By Theorem 2.4 and a straightforward calculation, g^2 is a modular form of weight 0 on $\Gamma_1(24)$ with the multiplier system I. Therefore, f_1^2, $f_2^2 g^2$, f_3^2 are modular forms of weight 14 on $\Gamma_1(24)$ with multiplier system υ_2.

Recall that $[\Gamma(1) : \Gamma_1(24)] = 384$. Let F_2 denote the difference of the left and right sides of (6.6). By applying the three equations in Lemma 2.5 to F_2 and a straightforward calculation, we have that for each $k \in E_{24}$, $k \neq \infty$,

$$ord(F_2; k) \geq 0. \tag{6.7}$$

By using the similar argument in the proof of Theorem 6.2, we can see that $F_2 = 0$ by checking the first 449 terms, which was done by *Maple*. □

We now derive two theta function identities from the previous eta function identities.

Theorem 6.4. *For* $|\, q \,| < 1$,

$$- f(q, q^5)^6 f(q^3, q^3)^2 + f(q^2, q^4)^6 f(1, q^6)^2 = 3 \frac{(q;q)_\infty^2 (q^3;q^3)_\infty^2 (q^6;q^6)_\infty^8}{(q^2;q^2)_\infty^4}.$$

Proof. By the Jacobi triple product identity,

$$f(q, q^5) = (-q;q^6)_\infty (-q^5;q^6)_\infty (q^6;q^6)_\infty = \frac{(q^2;q^{12})_\infty (q^{10};q^{12})_\infty}{(q;q^6)_\infty (q^5;q^6)_\infty} (q^6;q^6)_\infty, \tag{6.8}$$

$$f(q^3, q^3) = (-q^3;q^6)_\infty^2 (q^6;q^6)_\infty = \frac{(q^6;q^{12})_\infty^2}{(q^3;q^6)_\infty^2} (q^6;q^6)_\infty, \tag{6.9}$$

$$f(1, q^6) = 2(-q^6;q^6)_\infty^2 (q^6;q^6)_\infty = 2 \frac{(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty}, \tag{6.10}$$

and by Euler's identity,

$$f(q^2, q^4) = (-q^2;q^6)_\infty (-q^4;q^6)_\infty (q^6;q^6)_\infty = \frac{(q^4;q^{12})_\infty (q^8;q^{12})_\infty}{(q^2;q^6)_\infty (q^4;q^6)_\infty} (q^6;q^6)_\infty. \tag{6.11}$$

Dividing both sides of (6.3) by $q^2 \frac{\eta_2^4 \eta_4^2 \eta_{12}^2 \eta_{6,1}^6 \eta_{6,2}^6 \eta_{6,3}^2}{\eta_6^2}$, using $\eta_{2\ell,\ell} \eta_{2\ell}^2 = \eta_\ell^2$ and $\eta_{3\ell,\ell} \eta_{3\ell} = \eta_\ell$ frequently, and employing (6.8)–(6.11), we get the identity in Theorem 6.4. □

Theorem 6.5. *For* $|\, q \,| < 1$,

$$- (q;q^2)_\infty^6 f(q^3, q^3) + (-q;q^2)_\infty^6 f(-q^3, -q^3) = 12q \frac{(q^6;q^6)_\infty (q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^2 f(-q^2, -q^{10})^2}.$$

Proof. By Jacobi triple product identity, we can derive that

$$f(q^3,q^3)=(-q^3;q^6)_\infty^2(q^6;q^6)_\infty=\frac{(q^6;q^{12})_\infty}{(q^3;q^6)_\infty}(q^6;q^6)_\infty \tag{6.12}$$

and

$$(-q;q^2)_\infty=\frac{(q^2;q^4)_\infty}{(q;q^2)_\infty}. \tag{6.13}$$

Now, dividing both sides of (6.6) by $q^{-\frac{1}{4}}\eta_1^4\eta_2^{10}\eta_4^6\eta_6^3\eta_{12}^4\eta_{2,1}^3\eta_{6,3}\eta_{12,2}^2$, using $\eta_{2\ell,\ell}\eta_{2\ell}^2=\eta_\ell^2$ and $\eta_{3\ell,\ell}\eta_{3\ell}=\eta_\ell$ frequently, and employing (6.12) and (6.13), we derive the identity in Theorem 6.5. □

Finally, we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. First, we prove (6.1). Replacing z by q in Theorem 4 [18] and using Theorem 6.4, we deduce that

$$\begin{aligned}
&\frac{1+q}{q}\left(\psi(q)+2\psi_-(q;q)\right)\\
&=-\frac{q^2}{2}\frac{(-q^{-1},-q^{-1},-q^3,-q,-q;q^2)_\infty}{(q,-q^2,q,q,q;q^2)_\infty}f(1,q^6)+\frac{1}{2}\frac{(-1,-1,-q^2,-q^2,-1;q^2)_\infty}{(q,-q^3,q,q,q;q^2)_\infty}f(q^3,q^3)\\
&=-\frac{(1+q)(-1;q)_\infty^3}{128}\left(8(-q;q^2)_\infty^6f(1,q^6)-(-1;q^2)_\infty^6f(q^3,q^3)\right)\\
&=-\frac{(1+q)(-1;q)_\infty^3}{128}\frac{16(-q^3;q^3)_\infty^2}{(q^6;q^6)_\infty^7}\left\{f(q,q^5)^6f(q^3,q^3)^2-f(1,q^6)^2f(q^2,q^4)^6\right\}\\
&=3\frac{(1+q)(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}.
\end{aligned}$$

Multiplying both sides of the above equation by $\frac{q}{1+q}$, we conclude

$$\psi(q)+2\psi_-(q;q)=3\frac{q(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}.$$

Now we turn to (6.2). Using Theorem 6.5 with q replaced by $q^{1/2}$, we obtain

$$\begin{aligned}
2\rho(q)+\lambda(q;q)&=-\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty^6f\left(q^{\frac{3}{2}},q^{\frac{3}{2}}\right)-(-q^{\frac{1}{2}};-q^{\frac{1}{2}})_\infty^6f\left(-q^{\frac{3}{2}},-q^{\frac{3}{2}}\right)}{4q^{\frac{1}{2}}(q;q)_\infty^3(q^2;q^2)_\infty^3}\\
&=-\frac{(q^{\frac{1}{2}};q)_\infty^6f(q^{\frac{3}{2}},q^{\frac{3}{2}})-(-q^{\frac{1}{2}};q)_\infty^6f(-q^{\frac{3}{2}},-q^{\frac{3}{2}})}{4q^{\frac{1}{2}}(-q;q)_\infty^3}\\
&=3\frac{(q;q)_\infty(q^3;q^3)_\infty(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^3f(-q,-q^5)^2}\\
&=3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}.
\end{aligned}$$

□

7. Concluding Remarks

The following questions naturally arise from our work. First, it is desirable to find a proof for Theorem 6.2 by using Ramanujan's modular equations. From the fact that the generating function for n-color partitions (resp. n-color overpartitions) and the generating function for plane partitions (resp. plane overpartitions) are the same, it would be interesting if we can describe mock theta functions as generating functions for certain class of plane partitions or plane overpartitions. Finally, we define the function $M(m,n)$ by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n) z^m q^n = \frac{(q^3;q^3)_\infty^3}{(q;q^2)_\infty (zq^2;q^2)_\infty (z^{-1}q^2;q^2)_\infty}. \tag{7.1}$$

Note that $M(0,3,3n+2) + M(1,3,3n+2) + M(2,3,3n+2) = d(3n+2)$ and $M(0,3,3n+2) = M(1,3,3n+2) = M(2,3,3n+2)$ for any nonnegative integer n, where $M(i,3,n) = \sum_{m\equiv i \pmod 3} M(m,n)$. Since the function $M(m,n)$ explains the congruence $d(3n+2) \equiv 0 \pmod 3$, we can call the function $M(m,n)$ as a crank function for $d(n)$. Therefore, it is natural to ask what $M(m,n)$ counts.

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