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Modular Transformations of Ramanujan's Fifth and Seventh Order Mock Theta Functions

BASIL GORDON bg@sonia.math.ucla.edu
Department of Mathematics, University of California, Los Angeles, California 90024

RICHARD J. MCINTOSH* mcintosh@math.uregina.ca
Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2

In memory of Robert A. Rankin

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Abstract. In his last letter to Hardy, Ramanujan defined 17 functions $F(q)$, where $|q| < 1$. He called them mock theta functions, because as q radially approaches any point $e^{2\pi ir}$ (r rational), there is a theta function $F_r(q)$ with $F(q) - F_r(q) = O(1)$. In this paper we obtain the transformations of Ramanujan's fifth and seventh order mock theta functions under the modular group generators $\tau \to \tau + 1$ and $\tau \to -1/\tau$, where $q = e^{\pi i\tau}$. The transformation formulas are more complex than those of ordinary theta functions. A definition of the order of a mock theta function is also given.

Key words: mock theta function, modular form, Mordell integral

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1. Introduction

In Ramanujan's last letter to Hardy [13, pp. 354–355; 14, pp. 127–131; 16, pp. 56–61], he observes that the asymptotic expansions of certain q-series with exponential singularities at roots of unity "close" in a striking manner. For example, let

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-3})(1-q^{5m-2})} \tag{1.1}$$

(where the last equality is the second Rogers-Ramanujan identity). If $q = e^{-t}$ and $t \to 0^+$ (so that q approaches 1 radially from inside the unit circle), then

$$H(q) = \sqrt{\frac{2}{5+\sqrt{5}}} \exp\left(\frac{\pi^2}{15t} + \frac{11t}{60}\right) + o(1). \tag{1.2}$$

In the same letter Ramanujan notes that it is only for some special q-series $f(q)$ that the exponential closes, i.e. its argument terminates with some power t^N. If $f(q)$ is not the sum

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of a theta function and a function which is $O(1)$ at all roots of unity ρ, and if for each such ρ there is an approximation of the form

$$f(q) = \sum_{\mu=1}^{M} t^{k_\mu} \exp\left(\sum_{\nu=-1}^{N} c_{\mu\nu} t^{\nu} \right) + O(1)$$

as $t \to 0^+$ with $q = \rho e^{-t}$, he calls $f(q)$ a *mock theta function*. It appears from his letter, however, that he was actually concerned with functions having the (possibly) more restrictive property that for every root of unity ρ, there are modular forms $h_j^{(\rho)}(q)$ and real numbers α_j, $1 \le j \le J(\rho)$, such that

$$f(q) = \sum_{j=1}^{J(\rho)} q^{\alpha_j} h_j^{(\rho)}(q) + O(1)$$

as q radially approaches ρ.

Ramanujan listed seventeen such functions, to which he assigned orders 3, 5 and 7. (The order is analogous to the level of a modular form.) Watson [16] found three more mock theta functions of order 3, and two more of order 5 appear in the Lost Notebook [14, p. 9]. In [16] Watson obtained the transformation laws for some of the third order functions under the action of the modular group. However, his method could not be applied to the fifth and seventh order functions, owing to the lack of suitable expansions into generalized Lambert series. In this paper we complete the transformation theory for the third order functions, and obtain the analogous formulas for some new infinite families of mock theta functions. Then, using the mock theta conjectures proved by Dean Hickerson [10], we extend the theory to Ramanujan's fifth and seventh order functions. As a corollary we show that they are not ordinary theta functions.

We use the standard notation for q-shifted factorials:

$$\begin{aligned}
(a; \pm q^k)_0 &= 1, \\
(a; q^k)_n &= (1-a)(1-aq^k)(1-aq^{2k}) \cdots \left(1 - aq^{(n-1)k}\right), \\
(a; -q^k)_n &= (1-a)(1+aq^k)(1-aq^{2k}) \cdots \left(1 - (-1)^{n-1} aq^{(n-1)k}\right), \\
(a; q^k)_\infty &= \prod_{m=0}^{\infty} (1 - aq^{mk}), \\
(a; -q^k)_\infty &= \prod_{m=0}^{\infty} (1 - (-1)^m aq^{mk}).
\end{aligned}$$

When $k = 1$ we usually write $(a)_n$ and $(a)_\infty$ instead of $(a; q)_n$ and $(a; q)_\infty$, respectively. For rational r with $0 < r < 1$, define

$$\begin{aligned}
M(r, q) &= \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^r)_n (q^{1-r})_n}, \\
M_1(r, q) &= \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(-q^r)_n (-q^{1-r})_n},
\end{aligned}$$

$$M_2(r,q)=\sum_{n=1}^{\infty}\frac{q^{n(n-1)}}{(q^r;-q)_n(-q^{1-r};-q)_n},$$
$$N(r,q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(e^{2\pi ir})_n(e^{-2\pi ir})_n}=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{\prod_{k=1}^{n}(1-2q^k\cos 2\pi r+q^{2k})},$$
$$N_1(r,q)=\sum_{n=1}^{\infty}\frac{q^{2n(n-1)}}{(e^{2\pi ir}q;q^2)_n(e^{-2\pi ir}q;q^2)_n}=\sum_{n=1}^{\infty}\frac{q^{2n(n-1)}}{\prod_{k=1}^{n}(1-2q^{2k-1}\cos 2\pi r+q^{4k-2})}.$$

Ramanujan's mock theta functions have ordinary power series expansions. For this reason, if $r=a/b$ with a and b relatively prime, we often replace q by q^b in the defintions of M, M_1 and M_2. In particular, we define

$$\mathcal{M}(r,q)=\sum_{n=1}^{\infty}\frac{q^{bn(n-1)}}{(q^a;q^b)_n(q^{b-a};q^b)_n},$$
$$\mathcal{M}_1(r,q)=\sum_{n=1}^{\infty}\frac{q^{bn(n-1)}}{(-q^a;q^b)_n(-q^{b-a};q^b)_n},$$
$$\mathcal{M}_2(r,q)=\sum_{n=1}^{\infty}\frac{q^{bn(n-1)}}{(q^a;-q^b)_n(-q^{b-a};-q^b)_n}.$$

The functions N and N_1 appear in Watson's paper [16, p. 64, 66]. The third order mock theta functions, described in Ramanujan's last letter to Hardy [13, pp. 354–355] (also see [16, p. 62]), can be expressed in terms of our new functions:

$$f(q)=N\left(\frac{1}{2},q\right)=2-2M_1(0,q)=2-2M_1(1,q),$$
$$\phi(q)=N\left(\frac{1}{4},q\right),$$
$$\psi(q)=q\mathcal{M}\left(\frac{1}{4},q\right),$$
$$\chi(q)=N\left(\frac{1}{6},q\right),$$
$$\omega(q)=N_1(0,q)=N_1\left(\frac{1}{2},-q\right)=\mathcal{M}\left(\frac{1}{2},q\right),$$
$$\upsilon(q)=N_1\left(\frac{1}{4},q^{\frac{1}{2}}\right),$$
$$\upsilon(-q)=M_2\left(\frac{1}{2},q\right),$$
$$\rho(q)=N_1\left(\frac{1}{3},q\right)=N_1\left(\frac{1}{6},-q\right).$$

The identity $f(q) = 2 - 2M_1(0, q)$ is a special case of equation (1.2) below, and the identity $\psi(q) = q\mathcal{M}(\frac{1}{4}, q)$ follows from

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \sum_{n=1}^{\infty} \left(\frac{q^{(2n-1)^2}}{(q;q^2)_{2n-1}} + \frac{q^{(2n)^2}}{(q;q^2)_{2n}} \right) = \sum_{n=1}^{\infty} \frac{q^{(2n-1)^2}}{(q;q^2)_{2n}} = q\mathcal{M}\left(\frac{1}{4}, q\right).$$

Applying the half-shift method [9, pp. 328] to a mock theta function often gives rise to another mock theta function. In view of the identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^r)_{n+1}(q^{1-r})_n} = 1 + q^r M(r, q) \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^r)_{n+1}(-q^{1-r})_n} = 1 - q^r M_1(r, q), \tag{1.2}$$

nothing new is obtained by the half-shift method. Dean Hickerson [10, pp. 648–649] prefers to work with the form

$$q(x, q) = x^{-1}\left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x)_{n+1}(q/x)_n} \right),$$

which equals $M(r, q)$ when x is replaced by q^r.

The second author proved (1.1) and (1.2) using recurrences. The following proof communicated to us by Hickerson is easier. The identity

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x)_{n+1}(q/x)_n} &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x)_n(q/x)_n} + x\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x)_{n+1}(q/x)_n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(x)_n(q/x)_n} + x\sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(x)_n(q/x)_{n-1}} \\ &= 1 + x\sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(x)_n(q/x)_n} \end{aligned}$$

with $x = q^r$ gives (1.1). Setting $x = -q^r$ gives (1.2).

To obtain transformation formulas we use the following generalized Lambert series:

$$M(r, q) = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+r}}{1 - q^{n+r}} q^{\frac{3}{2}n(n+1)}, \tag{1.3}$$

$$M_1(r, q) = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n+r}}{1 + q^{n+r}} q^{\frac{3}{2}n(n+1)}, \tag{1.4}$$

$$M_2(r, q) = \frac{1}{(-q;-q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\frac{3}{2}n(n+1)} q^{n+r}}{1 + (-1)^{n+1} q^{n+r}} q^{\frac{3}{2}n(n+1)}, \tag{1.5}$$

$$N(r,q)=\frac{1}{(q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)(2-2\cos 2\pi r)}{1-2q^n\cos 2\pi r+q^{2n}}q^{\frac{1}{2}n(3n+1)}\right], \tag{1.6}$$

$$N_1(r,q)=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(1-q^{2n+1})}{1-2q^{n+\frac{1}{2}}\cos 2\pi r+q^{2n+1}}q^{\frac{3}{2}n(n+1)}. \tag{1.7}$$

Equations (1.3) and (1.4) can be further simplified by replacing r by $1-r$ and n by $-n-1$. This gives

$$M(r,q)=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1-q^{n+r}}$$

and

$$M_1(r,q)=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1+q^{n+r}}.$$

To prove Eqs. (1.3)–(1.7) we begin with the Watson-Whipple transformation [8, p. 242, Eq. (III.17)]:

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f\\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}\end{matrix};q;\frac{a^2q^2}{bcdef}\right]\\ &=\frac{\left(aq, \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}\right)_\infty}{\left(\frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{aq}{def}\right)_\infty}{}_4\phi_3\left[\begin{matrix}\frac{aq}{bc}, d, e, f\\ \frac{aq}{b}, \frac{aq}{c}, \frac{def}{a}\end{matrix};q;q\right].\end{aligned} \tag{1.8}$$

Now

$$\frac{\left(qa^{\frac{1}{2}}\right)_n\left(-qa^{\frac{1}{2}}\right)_n}{\left(a^{\frac{1}{2}}\right)_n\left(-a^{\frac{1}{2}}\right)_n}=\frac{(1-aq^2)(1-aq^4)\cdots(1-aq^{2n})}{(1-a)(1-aq^2)\cdots(1-aq^{2n-2})}=\frac{1-aq^{2n}}{1-a}.$$

Let d, e and f tend to infinity. Then $(aq/d)_n$, $(aq/e)_n$ and $(aq/f)_n$ tend to 1. Also,

$$\begin{aligned}(d)_n&=(1-d)(1-dq)\cdots(1-dq^{n-1})\\ &=(-d)^n\left(-\frac{1}{d}+1\right)\left(-\frac{1}{d}+q\right)\cdots\left(-\frac{1}{d}+q^{n-1}\right)\\ &\sim(-d)^n q^{\frac{1}{2}n(n-1)}\end{aligned}$$

as $d\to\infty$. Similarly, $(e)_n\sim(-e)^n q^{\frac{1}{2}n(n-1)}$ as $e\to\infty$ and $(f)_n\sim(-f)^n q^{\frac{1}{2}n(n-1)}$ as $f\to\infty$. Hence in the limit, (1.8) becomes

$$(aq)_\infty\sum_{n=0}^{\infty}\frac{\left(\frac{aq}{bc}\right)_n}{(q)_n\left(\frac{aq}{b}\right)_n\left(\frac{aq}{c}\right)_n}a^nq^{n^2}=\sum_{n=0}^{\infty}\left(\frac{1-aq^{2n}}{1-a}\right)\frac{(a)_n(b)_n(c)_n}{(q)_n\left(\frac{aq}{b}\right)_n\left(\frac{aq}{c}\right)_n}\left(-\frac{a^2}{bc}\right)^n q^{\frac{1}{2}n(3n+1)}. \tag{1.9}$$

Now put $a = q$, $b = q^r$ and $c = q^{1-r}$ to obtain

$$(q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{1+r})_n (q^{2-r})_n} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1 - q^{2n+1}}{1-q} \right) \frac{(q^r)_n (q^{1-r})_n}{(q^{1+r})_n (q^{2-r})_n} q^{\frac{3}{2}n(n+1)}$$
$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1 - q^{2n+1}}{1-q} \right) \frac{(1-q^r)(1-q^{1-r})}{(1-q^{n+r})(1-q^{n+1-r})} q^{\frac{3}{2}n(n+1)}.$$

Thus

$$(q)_\infty M(r,q) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-q^{2n+1})}{(1-q^{n+r})(1-q^{n+1-r})} q^{\frac{3}{2}n(n+1)}.$$

Since

$$\frac{1-q^{2n+1}}{(1-q^{n+r})(1-q^{n+1-r})} = \frac{q^{n+r}}{1-q^{n+r}} + \frac{1}{1-q^{n+1-r}} = \frac{q^{n+r}}{1-q^{n+r}} - \frac{q^{-n-1+r}}{1-q^{-n-1+r}},$$

it follows that

$$\frac{(-1)^n(1-q^{2n+1})}{(1-q^{n+r})(1-q^{n+1-r})} = \frac{(-1)^n q^{n+r}}{1-q^{n+r}} + \frac{(-1)^{-n-1} q^{-n-1+r}}{1-q^{-n-1+r}}.$$

As n runs from 0 to ∞, $-n-1$ runs from -1 to $-\infty$. Moreover $q^{\frac{3}{2}n(n+1)} = q^{\frac{3}{2}(-n-1)(-n)}$. Hence

$$(q)_\infty M(r,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+r}}{1-q^{n+r}} q^{\frac{3}{2}n(n+1)},$$

which completes the proof of (1.3).

Equation (1.4) follows from (1.9) with $a = q$, $b = -q^r$ and $c = -q^{1-r}$. To prove (1.5) first replace q by $-q$ in (1.9) and then set $a = -q$, $b = q^r$ and $c = -q^{1-r}$. Equations (1.6) and (1.7) are given in [16, p. 64, 66]. If we let $a \to 1$ and put $b = e^{2\pi ir}$, $c = e^{-2\pi ir}$ in (1.9), then we obtain (1.6). Letting $a \to 1$, $b = q^{\frac{1}{2}} e^{2\pi ir}$ and $c = q^{\frac{1}{2}} e^{-2\pi ir}$ in (1.9) yields (1.7).

2. Transformation formulas

From the definitions of $\mathcal{M}(r,q)$, $\mathcal{M}_1(r,q)$ and $\mathcal{M}_2(r,q)$ we observe that when $r = a/b$,

$$\mathcal{M}(r,-q) = \begin{cases} \mathcal{M}_2(r,q) & \text{if } a \text{ is even and } b \text{ is odd,} \\ \mathcal{M}_2(1-r,q) & \text{if } a \text{ and } b \text{ are odd,} \\ \mathcal{M}_1(r,q) & \text{if } a \text{ is odd and } b \text{ is even,} \end{cases}$$

$$\mathcal{M}_1(r,-q) = \begin{cases} \mathcal{M}_2(1-r,q) & \text{if } a \text{ is even and } b \text{ is odd,} \\ \mathcal{M}_2(r,q) & \text{if } a \text{ and } b \text{ are odd,} \\ \mathcal{M}(r,q) & \text{if } a \text{ is odd and } b \text{ is even,} \end{cases}$$

$$\mathcal{M}_2(r,-q)=\begin{cases}\mathcal{M}(r,q) & \text{if } a \text{ is even and } b \text{ is odd},\\ \mathcal{M}_1(r,q) & \text{if } a \text{ and } b \text{ are odd},\\ \mathcal{M}_2(1-r,q) & \text{if } a \text{ is odd and } b \text{ is even}.\end{cases}$$

Also observe that $N_1(r,-q)=N_1(|\frac{1}{2}-r|,q)$.

Since any substitution of the modular group can be decomposed into a finite number of substitutions of the forms

$$\tau'=\tau+1 \quad \text{and} \quad \tau'=-1/\tau,$$

it is sufficient to construct the transformations that express the 10 functions $\mathcal{M}(r,\pm q)$, $\mathcal{M}_1(r,\pm q)$, $\mathcal{M}_2(r,\pm q)$, $N(r,\pm q)$ and $N_1(r,\pm q)$ in terms of similar functions of q_1 (or powers of q_1), where $q=e^{-\alpha}$ and $q_1=e^{-\beta}$ with $\alpha\beta=\pi^2$. In view of the above observations these transformations can be obtained from the following three formulas:

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}M(r,q)=\sqrt{\frac{\pi}{2\alpha}}\csc(\pi r)q_1^{-\frac{1}{6}}N\big(r,q_1^4\big)-\sqrt{\frac{3\alpha}{2\pi}}J(r,\alpha), \tag{2.1}$$

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}M_1(r,q)=-\sqrt{\frac{2\pi}{\alpha}}q_1^{\frac{4}{3}}N_1\big(r,q_1^2\big)-\sqrt{\frac{3\alpha}{2\pi}}J_1(r,\alpha), \tag{2.2}$$

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}M_2(r,q)=\sqrt{\frac{\pi}{4\alpha}}\csc\left(\frac{\pi r}{2}\right)q_1^{-\frac{1}{24}}N\left(\frac{r}{2},-q_1\right)-\sqrt{\frac{3\alpha}{2\pi}}J_2(r,\alpha), \tag{2.3}$$

where the Mordell integrals J, J_1, J_2 are defined by

$$J(r,\alpha)=\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-2)\alpha x+\cosh(3r-1)\alpha x}{\cosh\frac{3}{2}\alpha x}\,dx,$$

$$J_1(r,\alpha)=\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\sinh(3r-2)\alpha x-\sinh(3r-1)\alpha x}{\sinh\frac{3}{2}\alpha x}\,dx,$$

$$J_2(r,\alpha)=\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\left\{\cosh\left(3r-\frac{7}{2}\right)\alpha x+\cosh\left(3r-\frac{5}{2}\right)\alpha x\right.$$
$$\left.+\cosh\left(3r-\frac{1}{2}\right)\alpha x-\cosh\left(3r+\frac{1}{2}\right)\alpha x\right\}\Big/\cosh 3\alpha x\,dx.$$

The presence of the Mordell integrals proves that these functions are not ordinary theta functions.

We now prove (2.1). The proofs of (2.2) and (2.3) are similar. In (1.3) put $q=e^{-\alpha}$ with $\alpha>0$ and consider the contour integral

$$I=I_1+I_2=\frac{1}{2\pi i}\int_{-\infty-\epsilon i}^{+\infty-\epsilon i}\frac{\pi}{\sin\pi z}\frac{e^{-\alpha(z+r)}}{1-e^{-\alpha(z+r)}}e^{-\frac{3}{2}\alpha z(z+1)}dz$$
$$+\frac{1}{2\pi i}\int_{+\infty+\epsilon i}^{-\infty+\epsilon i}\frac{\pi}{\sin\pi z}\frac{e^{-\alpha(z+r)}}{1-e^{-\alpha(z+r)}}e^{-\frac{3}{2}\alpha z(z+1)}dz,$$

where $\epsilon > 0$ is sufficiently small. By Cauchy's residue theorem, I is equal to the sum of the residues of the poles of the integrand inside the contour. Now $\pi/\sin \pi z$ has a simple pole of residue $(-1)^n$ at each integer n and $1/(1-e^{-\alpha(z+r)})$ has a simple pole of residue $1/\alpha$ at $z=-r$. If ϵ is sufficiently small, there are no other poles inside the contour. Hence

$$I = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+r}}{1-q^{n+r}} q^{\frac{3}{2}n(n+1)} + \frac{\pi}{\sin(-\pi r)} \frac{q^{-\frac{3}{2}r(1-r)}}{\alpha}. \tag{2.4}$$

We now consider I_2. In the upper half plane we have

$$\frac{1}{\sin \pi z} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi i z},$$

so

$$I_2 = \sum_{n=0}^{\infty} \int_{-\infty+\epsilon i}^{+\infty+\epsilon i} \frac{e^{(2n+1)\pi i z - \alpha(z+r) - \frac{3}{2}\alpha z(z+1)}}{1-e^{-\alpha(z+r)}} dz = \sum_{n=0}^{\infty} J_n,$$

say. The integrand of J_n has poles in the upper half plane at the points z where $1-e^{-\alpha(z+r)} = 0$, that is, at the points

$$z_m = -r + \frac{2\pi i m}{\alpha}$$

for $m = 1, 2, \ldots$. The residue at z_m is

$$\begin{aligned}\lambda_{n,m} &= 2\pi i \frac{e^{(2n+1)\pi i z_m - \alpha(z_m+r) - \frac{3}{2}\alpha z_m(z_m+1)}}{\alpha} \\ &= \frac{2\pi i}{\alpha} e^{-(2n+1)\pi i r} q_1^{(2n+1)2m} q^{-\frac{3}{2}r(1-r)} e^{-\frac{3}{2}(1-2r)2\pi i m} q_1^{-6m^2},\end{aligned}$$

where $q_1 = e^{-\frac{\pi^2}{\alpha}}$. Next, we symmetrize the denominator of the integrand of J_n by using the identity

$$\frac{t}{1-t} = \frac{t^{-\frac{1}{2}} + t^{\frac{1}{2}} + t^{\frac{3}{2}}}{t^{-\frac{3}{2}} - t^{\frac{3}{2}}}.$$

Applying this with $t = e^{-\alpha(z+r)}$, we find that the integrand of J_n is

$$\frac{e^{\frac{1}{2}\alpha(z+r)} + e^{-\frac{1}{2}\alpha(z+r)} + e^{-\frac{3}{2}\alpha(z+r)}}{e^{\frac{3}{2}\alpha(z+r)} - e^{-\frac{3}{2}\alpha(z+r)}} e^{-\frac{3}{2}\alpha z} e^{(2n+1)\pi i z - \frac{3}{2}\alpha z^2}.$$

To find the saddle point, we set the derivative of the last factor equal to 0, getting $(2n+1)\pi i - 3\alpha z = 0$ or

$$z = \frac{(2n+1)\pi i}{3\alpha} = w_n,$$

say. We move the upper contour of J_n up to the horizontal line through w_n, getting J_n'. By the residue theorem,

$$J_n = J_n' + \text{sum of residues of poles of integrand between the two contours.}$$

These poles are the points $z_m = -r + 2\pi i m/\alpha$ for which $0 < 2m < (2n+1)/3$, or equivalently, $0 < m \le n/3$. Hence

$$J_n = J_n' + \sum_{0<m\le\frac{n}{3}} \lambda_{n,m}.$$

Summing over n, we obtain

$$I_2 = \sum_{n=0}^{\infty} J_n' + \sum_{m=1}^{\infty}\sum_{n=3m}^{\infty} \lambda_{n,m}.$$

Now

$$\lambda_{n+1,m} = e^{2\pi i z_m}\lambda_{n,m} = e^{-2\pi i r} q_1^{4m}\lambda_{n,m}.$$

Hence

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=3m}^{\infty} \lambda_{n,m} &= \sum_{m=1}^{\infty} \frac{\lambda_{3m,m}}{1 - e^{-2\pi i r} q_1^{4m}} \\ &= \sum_{m=1}^{\infty} \frac{2\pi i}{\alpha} \frac{e^{-(6m+1)\pi i r} q_1^{(6m+1)2m} q^{-\frac{3}{2}r(1-r)} e^{-\frac{3}{2}(1-2r)2\pi i m} q_1^{-6m^2}}{1 - e^{-2\pi i r} q_1^{4m}} \\ &= \frac{2\pi i}{\alpha} q^{-\frac{3}{2}r(1-r)} e^{-\pi i r} \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{6m^2+2m}}{1 - e^{-2\pi i r} q_1^{4m}},\end{aligned}$$

so

$$I_2 = \frac{2\pi i}{\alpha} q^{-\frac{3}{2}r(1-r)} e^{-\pi i r} \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{6m^2+2m}}{1 - e^{-2\pi i r} q_1^{4m}} + \sum_{n=0}^{\infty} J_n'. \tag{2.5}$$

Before going on to evaluate the integral J_n', we remark that the integral I_1 over the lower contour can be handled similarly. This time the expansion

$$\frac{1}{\sin \pi z} = 2i \sum_{n=0}^{\infty} e^{-(2n+1)\pi i z}$$

is employed. Note that this is just the complex conjugate of the expansion used in the upper half plane. Thus

$$I_1 = \sum_{n=0}^{\infty} K_n,$$

where

$$K_n = \int_{-\infty-\epsilon i}^{+\infty-\epsilon i} \frac{e^{-(2n+1)\pi i z - \alpha(z+r) - \frac{3}{2}\alpha z(z+1)}}{1 - e^{-\alpha(z+r)}} dz.$$

The lower contour is moved down to the horizontal line through $\bar{w}_n$, giving

$$K_n = \bar{J}_n' + \sum_{0<m\le\frac{n}{3}} \bar{\lambda}_{n,m}.$$

The sum here is just the complex conjugate of the one evaluated above, so from (2.5) it follows that

$$I_1 = -\frac{2\pi i}{\alpha} q^{-\frac{3}{2}r(1-r)} e^{\pi i r} \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{6m^2+2m}}{1 - e^{2\pi i r} q_1^{4m}} + \sum_{n=0}^{\infty} \bar{J}'_n. \tag{2.6}$$

Adding (2.5) and (2.6) we obtain

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{2\pi i}{\alpha} q^{-\frac{3}{2}r(1-r)} \sum_{m=1}^{\infty} (-1)^m q_1^{6m^2+2m} \left[\frac{e^{-\pi i r}}{1 - e^{-2\pi i r} q_1^{4m}} - \frac{e^{\pi i r}}{1 - e^{2\pi i r} q_1^{4m}} \right] + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n) \\ &= \frac{4\pi}{\alpha} q^{-\frac{3}{2}r(1-r)} \sin \pi r \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{6m^2+2m} \left(1 + q_1^{4m}\right)}{1 - 2q_1^{4m} \cos 2\pi r + q_1^{8m}} + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n). \end{aligned}$$

It now follows from Eqs. (1.3) and (2.4) that

$$\begin{aligned} (q)_\infty M(r, q) &= I - \frac{\pi}{\sin(-\pi r)} \frac{q^{-\frac{3}{2}r(1-r)}}{\alpha} \\ &= \frac{\pi q^{-\frac{3}{2}r(1-r)}}{\alpha \sin \pi r} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m (2 - 2\cos 2\pi r) q_1^{6m^2+2m} \left(1 + q_1^{4m}\right)}{1 - 2q_1^{4m} \cos 2\pi r + q_1^{8m}} \right] \\ &\quad + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n). \end{aligned}$$

Using (1.6) with q replaced by q_1^4 we get

$$(q)_\infty M(r, q) = \frac{\pi q^{-\frac{3}{2}r(1-r)}}{\alpha \sin \pi r} \left(q_1^4; q_1^4\right)_\infty N\left(r, q_1^4\right) + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n). \tag{2.7}$$

We now evaluate

$$\sum_{n=0}^{\infty} (J'_n + \bar{J}'_n).$$

In the integral J'_n put $z = -r + p + x$, where $p = (2n + 1)\pi i/3\alpha$ and x is a real variable running from $-\infty$ to ∞. This gives

$$J'_n = q^{-\frac{3}{2}r} \int_{-\infty}^{\infty} ABC\, dx,$$

where

$$\begin{aligned} A &= e^{(2n+1)\pi i(-r+p+x)}, \\ B &= \frac{e^{-\alpha(p+x)} + e^{-2\alpha(p+x)} + e^{-3\alpha(p+x)}}{e^{\frac{3}{2}\alpha(p+x)} - e^{-\frac{3}{2}\alpha(p+x)}}, \\ C &= e^{-\frac{3}{2}\alpha(-r+p+x)^2}. \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} J_n' &= q^{-\frac{3}{2}r(1-r)} q_1^{\frac{(2n+1)^2}{6}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(2n+1)\pi i}{3}} e^{-\alpha x} + e^{-\frac{2(2n+1)\pi i}{3}} e^{-2\alpha x}}{2(-1)^n i \cosh \frac{3}{2}\alpha x} e^{3\alpha r x - \frac{3}{2}\alpha x^2} dx \\ &\quad - q^{-\frac{3}{2}r(1-r)} q_1^{\frac{(2n+1)^2}{6}} \int_{-\infty}^{\infty} \frac{e^{-3\alpha x} e^{3\alpha r x - \frac{3}{2}\alpha x^2}}{2(-1)^n i \cosh \frac{3}{2}\alpha x} dx \\ &= P_n + Q_n, \end{aligned}$$

say. Since Q_n is purely imaginary, we have $J_n' + \bar{J}_n' = P_n + \bar{P}_n$. Making use of the fact that

$$\sin \frac{(2n+1)\pi i}{3} = \sin \frac{2(2n+1)\pi i}{3}$$

for all integers n, we find that

$$\begin{aligned} P_n + \bar{P}_n &= (-1)^{n+1} \sin \frac{(2n+1)\pi}{3} q^{-\frac{3}{2}r(1-r)} q_1^{\frac{(2n+1)^2}{6}} \\ &\quad \times \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh(3r-2)\alpha x + \cosh(3r-1)\alpha x}{\cosh \frac{3}{2}\alpha x} \, dx \\ &= 2(-1)^{n+1} \sin \frac{(2n+1)\pi}{3} q^{-\frac{3}{2}r(1-r)} q_1^{\frac{(2n+1)^2}{6}} J(r,\alpha), \end{aligned}$$

where $J(r, \alpha)$ is the Mordell integral used in formula (2.1). In [18, p. 464] we see that

$$\begin{aligned} 2\sum_{n=0}^{\infty} (-1)^n \sin \frac{(2n+1)\pi}{3} q_1^{\frac{(2n+1)^2}{6}} &= \theta_1\left(\frac{\pi}{3}, q_1^{\frac{2}{3}}\right) \\ &= 2q_1^{\frac{1}{6}} \sin \frac{\pi}{3} \prod_{m=1}^{\infty} \left(1 - q_1^{\frac{4m}{3}}\right)\left(1 - 2q_1^{\frac{4m}{3}} \cos \frac{2\pi}{3} + q_1^{\frac{8m}{3}}\right) \\ &= \sqrt{3} q_1^{\frac{1}{6}} \prod_{m=1}^{\infty} \left(1 - q_1^{\frac{4m}{3}}\right)\left(1 + q_1^{\frac{4m}{3}} + q_1^{\frac{8m}{3}}\right) \\ &= \sqrt{3} q_1^{\frac{1}{6}} \left(q_1^4; q_1^4\right)_\infty, \end{aligned}$$

so

$$\sum_{n=0}^{\infty} (J_n' + \bar{J}_n') = \sum_{n=0}^{\infty} (P_n' + \bar{P}_n') = -\sqrt{3} q^{-\frac{3}{2}r(1-r)} q_1^{\frac{1}{6}} \left(q_1^4; q_1^4\right)_\infty J(r, \alpha).$$

Substituting this into Eq. (2.7) yields

$$(q)_\infty M(r,q) = \frac{\pi q^{-\frac{3}{2}r(1-r)}}{\alpha \sin \pi r} \left(q_1^4; q_1^4\right)_\infty N\left(r, q_1^4\right) - \sqrt{3} q^{-\frac{3}{2}r(1-r)} q_1^{\frac{1}{6}} \left(q_1^4; q_1^4\right)_\infty J(r, \alpha). \quad (2.8)$$

By the functional equation for the Dedekind η-function (see, for example [5, p. 48]) we have

$$(q)_\infty = \sqrt{\frac{2\pi}{\alpha}}\, q^{-\frac{1}{24}} q_1^{\frac{1}{6}} \left(q_1^4; q_1^4\right)_\infty. \tag{2.9}$$

Using this to eliminate $(q)_\infty$ from (2.8) we obtain

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}} M(r,q) = \sqrt{\frac{\pi}{2\alpha}} \csc(\pi r) q_1^{-\frac{1}{6}} N\left(r, q_1^4\right) - \sqrt{\frac{3\alpha}{2\pi}} J(r,\alpha),$$

which completes the proof of (2.1).

In [16] Watson does not give transformation formulas for the mock theta functions χ and ρ. These transformations involve the new mock theta functions $\xi(q) = 1 + 2q\mathcal{M}(\frac{1}{6}, q)$ and $\sigma(q) = \mathcal{M}_1(\frac{1}{3}, q)$. It follows from (2.1)–(2.3) that

$$\begin{aligned}
q^{-\frac{1}{24}}\chi(q) &= \sqrt{\frac{\pi}{2\alpha}}\,\xi\left(q_1^{\frac{2}{3}}\right) + \sqrt{\frac{3\alpha}{2\pi}} W_1(\alpha),\\
q^{-\frac{1}{24}}\chi(-q) &= \sqrt{\frac{\pi}{\alpha}}\, q_1^{\frac{7}{24}} \sigma\left(-q_1^{\frac{1}{3}}\right) + \sqrt{\frac{3\alpha}{2\pi}} W(\alpha),\\
q^{\frac{2}{3}}\rho(q) &= -\sqrt{\frac{\pi}{\alpha}}\, q_1^{\frac{7}{12}} \sigma\left(q_1^{\frac{2}{3}}\right) + \sqrt{\frac{3\alpha}{4\pi}} W_2\left(\frac{\alpha}{2}\right),\\
q^{\frac{2}{3}}\rho(-q) &= \sqrt{\frac{\pi}{4\alpha}}\,\xi\left(-q_1^{\frac{1}{3}}\right) - \sqrt{\frac{3\alpha}{\pi}} W_3(\alpha),\\
\xi(q) &= \sqrt{\frac{4\pi}{3\alpha}}\, q_1^{-\frac{1}{36}} \chi\left(q_1^{\frac{2}{3}}\right) - \sqrt{\frac{9\alpha}{\pi}} W_2\left(\frac{3\alpha}{2}\right),\\
\xi(-q) &= \sqrt{\frac{4\pi}{3\alpha}}\, q_1^{\frac{2}{9}} \rho\left(-q_1^{\frac{1}{3}}\right) + \sqrt{\frac{36\alpha}{\pi}} W_3(3\alpha),\\
q^{\frac{7}{8}}\sigma(q) &= -\sqrt{\frac{2\pi}{3\alpha}}\, q_1^{\frac{4}{9}} \rho\left(q_1^{\frac{2}{3}}\right) + \sqrt{\frac{9\alpha}{2\pi}} W_1(3\alpha),\\
q^{\frac{7}{8}}\sigma(-q) &= \sqrt{\frac{\pi}{3\alpha}}\, q_1^{-\frac{1}{72}} \chi\left(-q_1^{\frac{1}{3}}\right) - \sqrt{\frac{9\alpha}{2\pi}} W(3\alpha),
\end{aligned}$$

where $q = e^{-\alpha}$, $q_1 = e^{-\beta}$ with $\alpha\beta = \pi^2$ and the Watson-Mordell integrals W, W_1, W_2, W_3 are defined in [16, pp. 77–79] by

$$\begin{aligned}
W(\alpha) &= \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x} dx,\\
W_1(\alpha) &= \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\sinh \alpha x}{\sinh\frac{3}{2}\alpha x} dx,
\end{aligned}$$

$$W_2(\alpha) = \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh \alpha x}{\cosh 3\alpha x} dx,$$
$$W_3(\alpha) = \int_0^\infty e^{-3\alpha x^2} \frac{\sinh \alpha x}{\sinh 3\alpha x} dx.$$

These are related to our Mordell integrals by

$$W(\alpha) = 2J\left(\frac{1}{4}, 4\alpha\right) = J_2\left(\frac{1}{3}, \alpha\right) = \frac{1}{2}(J_2(0,\alpha) + J_2(1,\alpha)) = \frac{1}{2}J_2(1,\alpha) + \sqrt{\frac{\pi}{6\alpha}} e^{\frac{1}{24}\alpha},$$
$$W_1(\alpha) = -2J_1\left(\frac{1}{4}, 4\alpha\right) = -J_1\left(\frac{1}{3}, \alpha\right),$$
$$W_2(\alpha) = J\left(\frac{1}{2}, 4\alpha\right) = 2J\left(\frac{1}{6}, 4\alpha\right) - \sqrt{\frac{\pi}{6\alpha}},$$
$$W_3(\alpha) = -\frac{1}{2}J_1\left(\frac{1}{2}, 2\alpha\right) = J_1\left(\frac{1}{6}, 2\alpha\right) + \sqrt{\frac{\pi}{12\alpha}},$$

where we used the fact that

$$J_2(0,\alpha) = 2\int_0^\infty e^{-\frac{3}{2}\alpha x^2} \cosh\left(\frac{1}{2}\alpha x\right) dx = \sqrt{\frac{2\pi}{3\alpha}} e^{\frac{1}{24}\alpha}.$$

Watson's asymptotic expansion of $W_3(\alpha)$ is incorrect. It should read

$$W_3(\alpha) = \frac{1}{6}\sqrt{\frac{\pi}{3\alpha}}\left(1 - \frac{2}{9}\alpha + \frac{1}{9}\alpha^2 - \cdots\right)$$

for small values of α.

The function $\gamma(q) = N(\frac{1}{3}, q)$ is defined in Ramanujan's "lost" notebook [14, p. 17] (also see [4, p. 62]), but the related function $\mathcal{M}(\frac{1}{3}, q)$ is not mentioned there.

3. The fifth and seventh order mock theta functions

Ramanujan's fifth order mock theta functions are [13, pp. 354–355; 14, pp. 127–131; 17, pp. 277–278]:

$$f_0(q) = \sum_{n=0}^\infty \frac{q^{n^2}}{(-q)_n}, \quad f_1(q) = \sum_{n=0}^\infty \frac{q^{n(n+1)}}{(-q)_n},$$
$$F_0(q) = \sum_{n=0}^\infty \frac{q^{2n^2}}{(q;q^2)_n}, \quad F_1(q) = \sum_{n=0}^\infty \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}},$$
$$\phi_0(q) = \sum_{n=0}^\infty q^{n^2}(-q;q^2)_n, \quad \phi_1(q) = \sum_{n=0}^\infty q^{(n+1)^2}(-q;q^2)_n,$$

$$\psi_0(q)=\sum_{n=0}^{\infty}q^{\frac12(n+1)(n+2)}(-q)_n,\quad \psi_1(q)=\sum_{n=0}^{\infty}q^{\frac12 n(n+1)}(-q)_n,$$

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1})_n},\quad \chi_1(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1})_{n+1}}.$$

These functions are connected to the functions $M(\frac15,q)$ and $M(\frac25,q)$ by the mock theta conjectures [3, pp. 245–246] (see also [10]):

$$\begin{aligned}
f_0(q)&=-2q^2M\left(\frac15,q^{10}\right)+\theta_4(0,q^5)G(q),\\
f_1(q)&=-2q^3M\left(\frac25,q^{10}\right)+\theta_4(0,q^5)H(q),\\
F_0(q)-1&=qM\left(\frac15,q^5\right)-q\psi(q^5)H(q^2),\\
F_1(q)&=qM\left(\frac25,q^5\right)-\psi(q^5)G(q^2),\\
\phi_0(-q)&=-qM\left(\frac15,q^5\right)+\frac{(q^5;q^5)_\infty G(q^2)H(q)}{H(q^2)},\\
\phi_1(-q)&=q^2M\left(\frac25,q^5\right)-\frac{q(q^5;q^5)_\infty G(q)H(q^2)}{G(q^2)},\\
\psi_0(q)&=q^2M\left(\frac15,q^{10}\right)+q(q;q^{10})_\infty(q^9;q^{10})_\infty(q^{10};q^{10})_\infty H(q),\\
\psi_1(q)&=q^3M\left(\frac25,q^{10}\right)+(q^3;q^{10})_\infty(q^7;q^{10})_\infty(q^{10};q^{10})_\infty G(q),\\
\chi_0(q)-2&=3qM\left(\frac15,q^5\right)-\frac{(q^5;q^5)_\infty G(q)^2}{H(q)},\\
\chi_1(q)&=3qM\left(\frac25,q^5\right)+\frac{(q^5;q^5)_\infty H(q)^2}{G(q)},
\end{aligned}$$

where

$$\begin{aligned}
\theta_4(0,q)&=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}=\frac{(q)_\infty}{(-q)_\infty}=\frac{(q)_\infty^2}{(q^2;q^2)_\infty},\\
\psi(q)&=\sum_{n=0}^{\infty}q^{\frac12 n(n+1)}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\frac{(q^2;q^2)_\infty^2}{(q)_\infty},\\
G(q)&=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty},\\
H(q)&=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.
\end{aligned}$$

Note that Eqs. $(3.8)_{\mathrm{R}}$ and $(3.9)_{\mathrm{R}}$ in [3, pp. 245–246] are stated incorrectly.

To prove the transformation formulas for the fifth order mock theta functions we need the following transformations obtained from (2.9):

$$\begin{aligned} q^{\frac{1}{24}}(q)_\infty &= \sqrt{\frac{2\pi}{\alpha}}q_1^{\frac{1}{6}}\left(q_1^4;q_1^4\right)_\infty, \\ \theta_4(0,q) &= \sqrt{\frac{4\pi}{\alpha}}q_1^{\frac{1}{4}}\psi\left(q_1^2\right), \\ q^{\frac{1}{8}}\psi(q) &= \sqrt{\frac{\pi}{2\alpha}}\theta_4\left(0,q_1^2\right), \end{aligned} \tag{3.1}$$

where $q = e^{-\alpha}$ and $q_1 = e^{-\beta}$ with $\alpha\beta = \pi^2$. The Rogers-Ramanujan functions G and H satisfy [15] (also see [12])

$$q^{-\frac{1}{60}}G(q) = \sqrt{\frac{5+\sqrt{5}}{10}}q_1^{-\frac{1}{15}}G\left(q_1^4\right) + \sqrt{\frac{5-\sqrt{5}}{10}}q_1^{\frac{11}{15}}H\left(q_1^4\right) \tag{3.2}$$

and

$$q^{\frac{11}{60}}H(q) = \sqrt{\frac{5-\sqrt{5}}{10}}q_1^{-\frac{1}{15}}G\left(q_1^4\right) - \sqrt{\frac{5+\sqrt{5}}{10}}q_1^{\frac{11}{15}}H\left(q_1^4\right).$$

The transformation formulas for Ramanujan's fifth order mock theta functions are:

$$\begin{aligned} q^{-\frac{1}{60}}f_0(q) &= \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}}q_1^{-\frac{1}{60}}\left(F_0\left(q_1^2\right)-1\right) + \sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}}q_1^{\frac{71}{60}}F_1\left(q_1^2\right) \\ &\quad + \sqrt{\frac{60\alpha}{\pi}}J\left(\frac{1}{5},10\alpha\right), \\ q^{\frac{11}{60}}f_1(q) &= \sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}}q_1^{-\frac{1}{60}}\left(F_0\left(q_1^2\right)-1\right) - \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}}q_1^{\frac{71}{60}}F_1\left(q_1^2\right) \\ &\quad + \sqrt{\frac{60\alpha}{\pi}}J\left(\frac{2}{5},10\alpha\right), \\ q^{-\frac{1}{60}}f_0(-q) &= -\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{60}}f_0(-q_1) + \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{\frac{11}{60}}f_1(-q_1) \\ &\quad + \sqrt{\frac{60\alpha}{\pi}}J\left(\frac{1}{5},10\alpha\right), \\ q^{\frac{11}{60}}f_1(-q) &= \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{60}}f_0(-q_1) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{\frac{11}{60}}f_1(-q_1) \\ &\quad - \sqrt{\frac{60\alpha}{\pi}}J\left(\frac{2}{5},10\alpha\right), \end{aligned}$$

$$q^{-\frac{1}{120}}(F_0(q)-1) = \sqrt{\frac{\pi(5-\sqrt{5})}{20\alpha}}\,q_1^{-\frac{1}{30}} f_0\big(q_1^2\big) + \sqrt{\frac{\pi(5+\sqrt{5})}{20\alpha}}\,q_1^{\frac{11}{30}} f_1\big(q_1^2\big) - \sqrt{\frac{15\alpha}{2\pi}}\,J\left(\frac{1}{5}, 5\alpha\right),$$

$$q^{\frac{71}{120}} F_1(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{20\alpha}}\,q_1^{-\frac{1}{30}} f_0\big(q_1^2\big) - \sqrt{\frac{\pi(5-\sqrt{5})}{20\alpha}}\,q_1^{\frac{11}{30}} f_1\big(q_1^2\big) - \sqrt{\frac{15\alpha}{2\pi}}\,J\left(\frac{2}{5}, 5\alpha\right),$$

$$q^{-\frac{1}{120}}(F_0(-q)-1) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}(F_0(-q_1)-1) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{\frac{71}{120}} F_1(-q_1) + \sqrt{\frac{15\alpha}{2\pi}}\,J_2\left(\frac{4}{5}, 5\alpha\right),$$

$$q^{\frac{71}{120}} F_1(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}(F_0(-q_1)-1) + \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{\frac{71}{120}} F_1(-q_1) + \sqrt{\frac{15\alpha}{2\pi}}\,J_2\left(\frac{2}{5}, 5\alpha\right),$$

$$q^{-\frac{1}{120}}\phi_0(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}\phi_0(q_1) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{49}{120}}\phi_1(q_1) - \sqrt{\frac{15\alpha}{2\pi}}\,J_2\left(\frac{4}{5}, 5\alpha\right),$$

$$q^{-\frac{49}{120}}\phi_1(q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}\phi_0(q_1) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{49}{120}}\phi_1(q_1) - \sqrt{\frac{15\alpha}{2\pi}}\,J_2\left(\frac{2}{5}, 5\alpha\right),$$

$$q^{-\frac{1}{120}}\phi_0(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\,q_1^{-\frac{1}{30}}\psi_0\big(q_1^2\big) + \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\,q_1^{\frac{11}{30}}\psi_1\big(q_1^2\big) + \sqrt{\frac{15\alpha}{2\pi}}\,J\left(\frac{1}{5}, 5\alpha\right),$$

$$q^{-\frac{49}{120}}\phi_1(-q) = -\sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\,q_1^{-\frac{1}{30}}\psi_0\big(q_1^2\big) + \sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\,q_1^{\frac{11}{30}}\psi_1\big(q_1^2\big) - \sqrt{\frac{15\alpha}{2\pi}}\,J\left(\frac{2}{5}, 5\alpha\right),$$

$$q^{-\frac{1}{60}}\psi_0(q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{60}}\phi_0\big(-q_1^2\big) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{49}{60}}\phi_1\big(-q_1^2\big) - \sqrt{\frac{15\alpha}{\pi}}\,J\left(\frac{1}{5}, 10\alpha\right),$$

$$q^{\frac{11}{60}}\psi_1(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{60}}\phi_0\big(-q_1^2\big) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{49}{60}}\phi_1\big(-q_1^2\big) - \sqrt{\frac{15\alpha}{\pi}}\,J\left(\frac{2}{5}, 10\alpha\right),$$

$$q^{-\frac{1}{60}}\psi_0(-q) = -\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{60}}\psi_0(-q_1) + \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{\frac{11}{60}}\psi_1(-q_1) - \sqrt{\frac{15\alpha}{\pi}}\,J\left(\frac{1}{5}, 10\alpha\right),$$

$$q^{\frac{11}{60}}\psi_1(-q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{60}}\psi_0(-q_1) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{\frac{11}{60}}\psi_1(-q_1) + \sqrt{\frac{15\alpha}{\pi}}\,J\left(\frac{2}{5}, 10\alpha\right),$$

$$q^{-\frac{1}{120}}(\chi_0(q)-2) = -\sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\,q_1^{-\frac{1}{30}}\big(\chi_0\big(q_1^4\big)-2\big) - \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\,q_1^{\frac{71}{30}}\chi_1\big(q_1^4\big) - \sqrt{\frac{135\alpha}{2\pi}}\,J\left(\frac{1}{5}, 5\alpha\right),$$

$$q^{\frac{71}{120}}\chi_1(q) = -\sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\,q_1^{-\frac{1}{30}}\big(\chi_0\big(q_1^4\big)-2\big) + \sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\,q_1^{\frac{71}{30}}\chi_1\big(q_1^4\big) - \sqrt{\frac{135\alpha}{2\pi}}\,J\left(\frac{2}{5}, 5\alpha\right),$$

$$q^{-\frac{1}{120}}(\chi_0(-q)-2) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}(\chi_0(-q_1)-2) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{\frac{71}{120}}\chi_1(-q_1) + \sqrt{\frac{135\alpha}{2\pi}}\,J_2\left(\frac{4}{5}, 5\alpha\right),$$

$$q^{\frac{71}{120}}\chi_1(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}(\chi_0(-q_1)-2) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{\frac{71}{120}}\chi_1(-q_1) + \sqrt{\frac{135\alpha}{2\pi}}\,J_2\left(\frac{2}{5}, 5\alpha\right).$$

We will prove the first transformation formula. The other 19 are similar.

Using the mock theta conjectures to eliminate $f_0(q)$, $F_0(q_1^2)$, $F_1(q_1^2)$ in the first transformation formula we obtain

$$\begin{aligned}
&-2q^{\frac{119}{60}}M\left(\frac{1}{5},q^{10}\right)+q^{-\frac{1}{60}}\theta_4(0,q^5)G(q)\\
&\quad=\sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}}\left[q_1^{\frac{119}{60}}M\left(\frac{1}{5},q_1^{10}\right)-q_1^{\frac{119}{60}}\psi\left(q_1^{10}\right)H\left(q_1^4\right)\right]\\
&\qquad+\sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}}\left[q_1^{\frac{191}{60}}M\left(\frac{2}{5},q_1^{10}\right)+q_1^{\frac{71}{60}}\psi\left(q_1^{10}\right)G\left(q_1^4\right)\right]+\sqrt{\frac{60\alpha}{\pi}}J\left(\frac{1}{5},10\alpha\right). \qquad (3.3)
\end{aligned}$$

Equation (2.1) with $r = 1/5$ and $q \to q^{10}$ becomes

$$q^{\frac{119}{60}}M\left(\frac{1}{5},q^{10}\right)=\sqrt{\frac{\pi(5+\sqrt{5})}{50\alpha}}q_1^{-\frac{1}{60}}N\left(\frac{1}{5},q_1^{\frac{2}{5}}\right)-\sqrt{\frac{15\alpha}{\pi}}J\left(\frac{1}{5},10\alpha\right).$$

Substituting this into (3.3) yields

$$\begin{aligned}
&-\sqrt{\frac{2\pi(5+\sqrt{5})}{25\alpha}}q_1^{-\frac{1}{60}}N\left(\frac{1}{5},q_1^{\frac{2}{5}}\right)+q^{-\frac{1}{60}}\theta_4(0,q^5)G(q)\\
&\quad=\sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}}\left[q_1^{\frac{119}{60}}M\left(\frac{1}{5},q_1^{10}\right)-q_1^{\frac{119}{60}}\psi\left(q_1^{10}\right)H\left(q_1^4\right)\right]\\
&\qquad+\sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}}\left[q_1^{\frac{191}{60}}M\left(\frac{2}{5},q_1^{10}\right)+q_1^{\frac{71}{60}}\psi\left(q_1^{10}\right)G\left(q_1^4\right)\right]. \qquad (3.4)
\end{aligned}$$

Equation (3.1) with $q \to q^5$ (and hence $\alpha \to 5\alpha$ and $q_1 \to q_1^{\frac{1}{5}}$) becomes

$$\theta_4(0,q^5)=\sqrt{\frac{4\pi}{5\alpha}}q_1^{\frac{1}{20}}\psi\left(q_1^{\frac{2}{5}}\right).$$

Using this equation and (3.2) to eliminate the variable q in (3.4) we get

$$\begin{aligned}
&-\sqrt{\frac{2\pi(5+\sqrt{5})}{25\alpha}}q_1^{-\frac{1}{60}}N\left(\frac{1}{5},q_1^{\frac{2}{5}}\right)+\sqrt{\frac{2\pi(5+\sqrt{5})}{25\alpha}}q_1^{-\frac{1}{60}}\psi\left(q_1^{\frac{2}{5}}\right)G\left(q_1^4\right)\\
&\quad+\sqrt{\frac{2\pi(5-\sqrt{5})}{25\alpha}}q_1^{\frac{47}{60}}\psi\left(q_1^{\frac{2}{5}}\right)H\left(q_1^4\right)
\end{aligned}$$

$$= \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}}\left[q_1^{\frac{119}{60}} M\left(\frac{1}{5}, q_1^{10}\right) - q_1^{\frac{119}{60}} \psi\left(q_1^{10}\right) H\left(q_1^4\right)\right]$$
$$+ \sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}}\left[q_1^{\frac{191}{60}} M\left(\frac{2}{5}, q_1^{10}\right) + q_1^{\frac{71}{60}} \psi\left(q_1^{10}\right) G\left(q_1^4\right)\right]$$

and dividing by $-q_1^{-\frac{1}{60}}\sqrt{2\pi(5+\sqrt{5})/25\alpha}$ we obtain

$$N\left(\frac{1}{5}, q_1^{\frac{2}{5}}\right) = \psi\left(q_1^{\frac{2}{5}}\right) G\left(q_1^4\right) + \frac{\sqrt{5}-1}{2} q_1^{\frac{4}{5}} \psi\left(q_1^{\frac{2}{5}}\right) H\left(q_1^4\right)$$
$$- \frac{5-\sqrt{5}}{2}\left[q_1^2\left(\frac{1}{5}, q_1^{10}\right) - q_1^2 \psi\left(q_1^{10}\right) H\left(q_1^4\right)\right]$$
$$- \sqrt{5}\left[q_1^{\frac{16}{5}} M\left(\frac{2}{5}, q_1^{10}\right) + q_1^{\frac{6}{5}} \psi\left(q_1^{10}\right) G\left(q_1^4\right)\right].$$

For brevity we will drop the subscript form q_1 and then replace q by $q^{\frac{5}{2}}$. This gives

$$N\left(\frac{1}{5}, q\right) = \psi(q) G(q^{10}) + \frac{\sqrt{5}-1}{2} q^2 \psi(q) H(q^{10})$$
$$- \frac{5-\sqrt{5}}{2}\left[q^5 M\left(\frac{1}{5}, q^{25}\right) - q^5 \psi(q^{25}) H(q^{10})\right]$$
$$- \sqrt{5}\left[q^8 M\left(\frac{2}{5}, q^{25}\right) + q^3 \psi(q^{25}) G(q^{10})\right]. \tag{3.5}$$

The coefficients in the taylor series of $N(\frac{1}{5}, q)$ lie in the field $\mathbf{Q}(\sqrt{5})$. The idea is to show that (3.5) is equivalent to a pair of q-series identities with integer coefficients. One of these will involve $M(\frac{1}{5}, q)$ and the other will involve $M(\frac{2}{5}, q)$.

By the generalized Lambert series (1.6) we have

$$N\left(\frac{1}{5}, q\right) = \frac{1}{(q)_\infty}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)\left(2 - 2\cos\frac{2\pi}{5}\right)}{1 - 2q^n\cos\frac{2\pi}{5} + q^{2n}} q^{\frac{1}{2}n(3n+1)}\right]$$
$$= \frac{1}{(q)_\infty}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)\left(2 - 2\cos\frac{2\pi}{5}\right)}{\left(1 - e^{\frac{2\pi i}{5}} q^n\right)\left(1 - e^{-\frac{2\pi i}{5}} q^n\right)} q^{\frac{1}{2}n(3n+1)}\right]. \tag{3.6}$$

Since

$$\frac{1}{\left(1 - e^{\frac{2\pi i}{5}} q^n\right)\left(1 - e^{-\frac{2\pi i}{5}} q^n\right)}$$
$$= \left(1 + e^{\frac{2\pi i}{5}} q^n + e^{\frac{4\pi i}{5}} q^{2n} + \cdots\right)\left(1 + e^{-\frac{2\pi i}{5}} q^n + e^{-\frac{4\pi i}{5}} q^{2n} + \cdots\right)$$
$$= 1 + \left(2\cos\frac{2\pi}{5}\right) q^n + \left(1 + 2\cos\frac{4\pi}{5}\right) q^{2n} + \left(2\cos\frac{2\pi}{5} + 2\cos\frac{6\pi}{5}\right) q^{3n}$$

$$
\begin{aligned}
&+\left(1+2\cos\frac{4\pi}{5}+2\cos\frac{8\pi}{5}\right)q^{4n}+\cdots\\
&= 1+\tau q^n-\tau q^{2n}-q^{3n}+q^{5n}+\tau q^{6n}-\tau q^{7n}-q^{8n}+q^{10n}+\cdots\\
&= (1+\tau q^n-\tau q^{2n}-q^{3n})(1+q^{5n}+q^{10n}+q^{15n}+\cdots)\\
&= \frac{1+\tau q^n-\tau q^{2n}-q^{3n}}{1-q^{5n}},
\end{aligned}
$$

where $\tau = 2\cos(2\pi/5) = (\sqrt{5}-1)/2$, (3.6) simplifies to

$$
\begin{aligned}
N\left(\frac{1}{5},q\right) &= \frac{1}{(q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{5}\tau(1+q^n)(1+\tau q^n-\tau q^{2n}-q^{3n})}{1-q^{5n}}q^{\frac{1}{2}n(3n+1)}\right]\\
&= \frac{1}{(q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{5}(\tau+q^n-q^{3n}-\tau q^{4n})}{1-q^{5n}}q^{\frac{1}{2}n(3n+1)}\right]\\
&= \frac{\sqrt{5}\tau}{(q)_\infty}\left[\frac{2}{5}+\sum_{n=1}^{\infty}\frac{(-1)^n(1-q^{4n})}{1-q^{5n}}q^{\frac{1}{2}n(3n+1)}\right]\\
&\quad+\frac{\sqrt{5}}{(q)_\infty}\left[\frac{1}{5}+\sum_{n=1}^{\infty}\frac{(-1)^n(1-q^{2n})}{1-q^{5n}}q^{\frac{3}{2}n(n+1)}\right]\\
&= \frac{\sqrt{5}\tau A(q)}{5}+\frac{\sqrt{5}B(q)}{5},
\end{aligned}
\tag{3.7}
$$

where

$$
A(q) = \frac{1}{(q)_\infty}\left[2+5\sum_{n=1}^{\infty}\frac{(-1)^n(1-q^{4n})}{1-q^{5n}}q^{\frac{1}{2}n(3n+1)}\right]
$$

and

$$
B(q) = \frac{1}{(q)_\infty}\left[1+5\sum_{n=1}^{\infty}\frac{(-1)^n(1-q^{2n})}{1-q^{5n}}q^{\frac{3}{2}n(n+1)}\right].
$$

Returning to Eq. (3.5), we view $\mathbf{Q}(\sqrt{5})$ as a vector space over $\mathbf{Q}$ with basis $\{\sqrt{5}\tau, \sqrt{5}\}$. Hence (3.5) is equivalent to the following pair of equations:

$$
A(q)+5q^5M\left(\frac{1}{5},q^{25}\right) = 2\psi(q)G(q^{10})-q^2\psi(q)H(q^{10})+5q^5\psi(q^{25})H(q^{10}) \tag{3.8}
$$

and

$$
B(q)+5q^8M\left(\frac{2}{5},q^{25}\right) = \psi(q)G(q^{10})+2q^2\psi(q)H(q^{10})-5q^3\psi(q^{25})G(q^{10}). \tag{3.9}
$$

We now prove (3.8). The proof of (3.9) is similar. We will need the function $j(x, q)$ defined by

$$j(x, q) = (x)_\infty (q/x)_\infty (q)_\infty$$

and equal to

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} x^n$$

by Jacobi's well-known triple product identity (see, for example [8, p. 12]). Equation (2.13) in [6] with $m = 0$, $q = 5$ and $x = q$ becomes

$$\sum_{n=1}^{\infty} N(0, 5, n) q^n = \frac{1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n (1 + q^{5n})}{1 - q^{5n}} q^{\frac{1}{2}n(3n+1)},$$

where $N(0, 5, n)$ (not to be confused with our $N(r, q)$) denotes the number of partitions of n with rank congruent to 0 modulo 5. The rank of a partition is defined as the largest part minus the number of parts. Atkin and Swinnerton-Dyer did not define the rank of the empty partition. We prefer to define it to have rank 0. Since

$$(q)_\infty = j(q, q^3) = 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{\frac{1}{2}n(3n+1)},$$

we get

$$\begin{aligned}
\sum_{n=0}^{\infty} N(0, 5, n) q^n &= \frac{1}{(q)_\infty} \left[1 + 2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{1 - q^{5n}} q^{\frac{1}{2}n(3n+1)} \right] \\
&= \frac{1}{(q)_\infty} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n (1 - q^{4n})}{1 - q^{5n}} q^{\frac{1}{2}n(3n+1)} \right],
\end{aligned}$$

from which it follows that

$$2A(q) = -\frac{1}{(q)_\infty} + 5 \sum_{n=0}^{\infty} N(0, 5, n) q^n = \sum_{n=0}^{\infty} [5N(0, 5, n) - p(n)] q^n,$$

where $p(n)$ denotes the number of unrestricted partitions of n. As usual we take $p(0) = 1$. Since $p(n)$ is the sum of $N(i, 5, n)$ for i from 0 to 4, and since $N(i, 5, n) = N(5 - i, 5, n)$ for $i = 1, 2$ [6, p. 85], we can rewrite this as

$$A(q) = \sum_{n=0}^{\infty} \{[N(0, 5, n) - N(1, 5, n)] + [N(0, 5, n) - N(2, 5, n)]\} q^n.$$

We break this sum into 5 parts depending on n modulo 5, and use the results of Atkin and Swinnerton-Dyer's Theorem 4. Since we define the rank of the empty partition to be 0, the "-1" term is dropped from the rightside of their Eq. (6.12). Therefore

$$\begin{aligned}
A(q) &= \sum_{d=0}^{4} q^d \left[\sum_{n=0}^{\infty} [n(0,5,5n+d) - N(1,5,5n+d)] q^{5n} \right. \\
&\quad \left. + \sum_{n=0}^{\infty} [N(0,5,5n+d) - N(2,5,5n+d)] q^{5n} \right] \\
&= \sum_{d=0}^{4} q^d [r_{01}(d,q^5) + r_{02}(d,q^5)] \\
&= \frac{2P(0)P(2)}{P^2(1)} - \frac{5q^5\Sigma(1,0)}{P(0)} + \frac{2qP(0)}{P(1)} - \frac{q^2P(0)}{P(2)} + \frac{q^3P(0)P(1)}{P^2(2)} \\
&= \frac{2(q^{25};q^{25})_\infty^2 j(q^{10},q^{25})}{j(q^5,q^{25})^2} - 5q^5 M\left(\frac{1}{5},q^{25}\right) + \frac{2q(q^{25};q^{25})_\infty^2}{j(q^5,q^{25})} - \frac{q^2(q^{25};q^{25})_\infty^2}{j(q^{10},q^{25})} \\
&\quad + \frac{q^3(q^{25};q^{25})_\infty^2 j(q^5,q^{25})}{j(q^{10},q^{25})^2},
\end{aligned}$$

where the functions r, P and Σ are defined in [6]. Hence the leftside of (3.8) is equal to

$$\begin{aligned}
&\frac{2(q^{25};q^{25})_\infty^2 j(q^{10},q^{25})}{j(q^5,q^{25})^2} + \frac{2q(q^{25};q^{25})_\infty^2}{j(q^5,q^{25})} - \frac{q^2(q^{25};q^{25})_\infty^2}{j(q^{10},q^{25})} + \frac{q^3(q^{25};q^{25})_\infty^2 j(q^5,q^{25})}{j(q^{10},q^{25})^2} \\
&\quad = \frac{2(q^{25};q^{25})_\infty G(q^5)^2}{H(q^5)} + 2q(q^{25};q^{25})_\infty G(q^5) - q^2(q^{25};q^{25})_\infty H(q^5) \\
&\qquad + \frac{q^3(q^{25};q^{25})_\infty H(q^5)^2}{G(q^5)}.
\end{aligned}$$

We now show that the right side of (3.8) is also equal to this sum. Observe that

$$\begin{aligned}
\psi(q) &= \sum_{n=0}^{\infty} \left(q^{\frac{5}{2}n(5n+1)} + q^{\frac{1}{2}(5n+4)(5n+5)} \right) + \sum_{n=0}^{\infty} \left(q^{\frac{1}{2}(5n+1)(5n+2)} + q^{\frac{1}{2}(5n+3)(5n+4)} \right) \\
&\quad + \sum_{n=0}^{\infty} q^{\frac{1}{2}(5n+2)(5n+3)} \\
&= \sum_{n=-\infty}^{\infty} q^{\frac{5}{2}n(5n+1)} + \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(5n+1)(5n+2)} + \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(5n+2)(5n+3)} \\
&= j(-q^{10},q^{25}) + qj(-q^5,q^{25}) + \frac{1}{2}q^3 j(-1,q^{25}) \\
&= \frac{(q^{25};q^{25})_\infty H(q^5)}{H(q^{10})} + \frac{q(q^{25};q^{25})_\infty G(q^5)}{G(q^{10})} + q^3\psi(q^{25}),
\end{aligned}$$

where the exponents of q in the first sum are congruent to 0 modulo 5, the exponents of q in the second sum are congruent to 1 modulo 5 and the exponents of q in the third sum are congruent to 3 modulo 5. Hence the right side of (3.8) is equal to

$$\begin{aligned}
&\frac{2(q^{25};q^{25})_\infty H(q^5)G(q^{10})}{H(q^{10})} + 4q^5\psi(q^{25})H(q^{10}) + 2q(q^{25};q^{25})_\infty G(q^5) \\
&\quad - q^2(q^{25};q^{25})_\infty H(q^5) + 2q^3\psi(q^{25})G(q^{10}) - \frac{q^3(q^{25};q^{25})_\infty G(q^5)H(q^{10})}{G(q^{10})} \\
&= \frac{2(q^{25};q^{25})_\infty H(q^5)G(q^{10})}{H(q^{10})} + 4q^5(q^{25};q^{25})_\infty(-q^{25};q^{25})_\infty^2 H(q^{10}) \\
&\quad + 2q(q^{25};q^{25})_\infty G(q^5) - q^2(q^{25};q^{25})_\infty H(q^5) \\
&\quad + 2q^3(q^{25};q^{25})_\infty(-q^{25};q^{25})_\infty^2 G(q^{10}) - \frac{q^3(q^{25};q^{25})_\infty G(q^5)H(q^{10})}{G(q^{10})} \\
&= \frac{2(q^{25};q^{25})_\infty G(q^5)^2}{H(q^5)} + 2q(q^{25};q^{25})_\infty G(q^5) - q^2(q^{25};q^{25})_\infty H(q^5) \\
&\quad + \frac{q^3(q^{25};q^{25})_\infty H(q^5)^2}{G(q^5)},
\end{aligned}$$

where we made use of the identities

$$G(q)^2H(q^2) + H(q)^2G(q^2) = 2(-q^5;q^5)_\infty^2 G(q)G(q^2)^2 \tag{3.10}$$

and

$$G(q)^2H(q^2) - H(q)^2G(q^2) = 2q(-q^5;q^5)_\infty^2 H(q)H(q^2)^2. \tag{3.11}$$

These identities can be easily obtained by Theorem 1.2 of [**10**], which states that

$$j(-x,q)j(y,q) - j(x,q)j(-y,q) = 2xj(y/x,q^2)j(xyq,q^2) \tag{3.12}$$

for $0 < |q| < 1$ and $xy \neq 0$. Replacing q by q^5, x by q, and y by q^2 yields

$$j(-q,q^5)j(q^2,q^5) - j(q,q^5)j(-q^2,q^5) = 2qj(q,q^{10})j(q^2,q^{10}).$$

If we replace q by q^5, x by $1/q$, and y by q^2 in (3.12), and multiply by q we get

$$j(-q,q^5)j(q^2,q^5) + j(q,q^5)j(-q^2,q^5) = 2j(q^3,q^{10})j(q^4,q^{10}).$$

Identities (3.10) and (3.11) are obtained by multiplying these equations by

$$\frac{(q^{10};q^{10})_\infty^2}{j(q,q^5)j(q^2,q^5)j(q^2,q^{10})j(q^4,q^{10})}.$$

The proof of (3.8) and hence the proof of the first transformation formula for the fifth order mock theta functions is now complete.

Remark. It can be shown that

$$A(q) = N\left(\frac{1}{5},q\right) + N\left(\frac{2}{5},q\right)$$

and

$$B(q) = \frac{1+\sqrt{5}}{2} N\left(\frac{1}{5}, q\right) + \frac{1-\sqrt{5}}{2} N\left(\frac{2}{5}, q\right).$$

In his "lost" notebook [14, p. 9] Ramanujan gave four more fifth order (sometimes called tenth order) mock theta functions:

$$\phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(q;q^2)_{n+1}}, \quad \psi_R(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n+1)(n+2)}}{(q;q^2)_{n+1}},$$
$$X_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q)_{2n}}, \quad \chi_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q)_{2n+1}}.$$

Numerical calculations suggest that they satisfy the following transformation formulas:

$$q^{\frac{1}{5}}\phi_R(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{20}} X_R(q_1^2) - \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{-\frac{9}{20}} \chi_R(q_1^2) - \sqrt{\frac{20\alpha}{\pi}} K_5(\alpha),$$

$$q^{-\frac{1}{5}}\psi_R(q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{20}} X_R(q_1^2) + \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{-\frac{9}{20}} \chi_R(q_1^2) - \sqrt{\frac{20\alpha}{\pi}} K_6(\alpha),$$

$$q^{\frac{1}{5}}\phi_R(-q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{\frac{1}{5}} \phi_R(-q_1) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{5}} \psi_R(-q_1) + \sqrt{\frac{20\alpha}{\pi}} K_1(\alpha),$$

$$q^{-\frac{1}{5}}\psi_R(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{\frac{1}{5}} \phi_R(-q_1) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{5}} \psi_R(-q_1) - \sqrt{\frac{20\alpha}{\pi}} K_3(\alpha),$$

$$q^{-\frac{1}{40}} X_R(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}} q_1^{\frac{2}{5}} \phi_R(q_1^2) + \sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}} q_1^{-\frac{2}{5}} \psi_R(q_1^2) + \sqrt{\frac{10\alpha}{\pi}} K_4\left(\frac{\alpha}{2}\right),$$

$$q^{-\frac{9}{40}} \chi_R(q) = -\sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}} q_1^{\frac{2}{5}} \phi_R(q_1^2) + \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}} q_1^{-\frac{2}{5}} \psi_R(q_1^2) + \sqrt{\frac{10\alpha}{\pi}} K_2\left(\frac{\alpha}{2}\right),$$

$$q^{-\frac{1}{40}} X_R(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{40}} X_R(-q_1) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{-\frac{9}{40}} \chi_R(-q_1) + \sqrt{\frac{40\alpha}{\pi}} K_7(\alpha),$$

$$q^{-\frac{9}{40}} \chi_R(-q) = -\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{40}} X_R(-q_1) - \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{-\frac{9}{40}} \chi_R(-q_1) + \sqrt{\frac{40\alpha}{\pi}} K_8(\alpha),$$

where $q = e^{-\alpha}$ and $q_1 = e^{-\beta}$ with $\alpha\beta = \pi^2$. The Mordell integrals $K_1, K_2, K_3, \ldots, K_8$

are defined by

$$K_1(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh \alpha x}{\sinh 5\alpha x}\, dx, \quad K_2(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh 2\alpha x}{\sinh 5\alpha x}\, dx,$$
$$K_3(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh 3\alpha x}{\sinh 5\alpha x}\, dx, \quad K_4(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh 4\alpha x}{\sinh 5\alpha x}\, dx,$$
$$K_5(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\cosh \alpha x}{\cosh 5\alpha x}\, dx, \quad K_6(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\cosh 3\alpha x}{\cosh 5\alpha x}\, dx,$$
$$K_7(\alpha) = \int_0^\infty e^{-10\alpha x^2} \frac{\cosh 9\alpha x - \cosh \alpha x}{\cosh 10\alpha x}\, dx,$$
$$K_8(\alpha) = \int_0^\infty e^{-10\alpha x^2} \frac{\cosh 7\alpha x + \cosh 3\alpha x}{\cosh 10\alpha x}\, dx.$$

Without generalized Lambert series we do not see an easy way to prove these formulas. Since these Mordell integrals are not related to the Mordell integrals involved in the transformation formulas of the other fifth order mock theta functions, it is unlikely that these functions are related to the other fifth order mock theta functions or the functions $M(\frac{1}{5}, q)$ and $M(\frac{2}{5}, q)$.

Ramanujan's seventh order mock theta functions are [13, p. 355] (also see [1, pp. 132–133] and [2, p. 286]):

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1})_n}, \quad \mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1})_{n+1}}, \quad \mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1})_{n+1}}.$$

Analogous to the mock theta conjectures are the following identities proved by Dean Hickerson [11]:

$$\mathcal{F}_0(q) - 2 = 2qM\left(\frac{1}{7}, q^7\right) - \frac{j(q^3, q^7)^2}{(q)_\infty},$$
$$\mathcal{F}_1(q) = 2q^2 M\left(\frac{2}{7}, q^7\right) + \frac{qj(q, q^7)^2}{(q)_\infty},$$
$$\mathcal{F}_2(q) = 2q^2 M\left(\frac{3}{7}, q^7\right) + \frac{j(q^2, q^7)^2}{(q)_\infty}.$$

To prove the transformation formulas for the seventh order mock theta functions one needs the following transformations of $j(q, q^7)$, $j(q^2, q^7)$, $j(q^3, q^7)$:

$$q^{\frac{25}{56}} j(q, q^7) = \sqrt{\frac{8\pi}{7\alpha}} \sin\left(\frac{2\pi}{7}\right) q_1^{\frac{25}{14}} j\left(q_1^4, q_1^{28}\right) - \sqrt{\frac{8\pi}{7\alpha}} \sin\left(\frac{3\pi}{7}\right) q_1^{\frac{9}{14}} j\left(q_1^8, q_1^{28}\right)$$
$$+ \sqrt{\frac{8\pi}{7\alpha}} \sin\left(\frac{\pi}{7}\right) q_1^{\frac{1}{14}} j\left(q_1^{12}, q_1^{28}\right),$$
$$q^{\frac{9}{56}} j(q^2, q^7) = -\sqrt{\frac{8\pi}{7\alpha}} \sin\left(\frac{3\pi}{7}\right) q_1^{\frac{25}{14}} j\left(q_1^4, q_1^{28}\right) - \sqrt{\frac{8\pi}{7\alpha}} \sin\left(\frac{\pi}{7}\right) q_1^{\frac{9}{14}} j\left(q_1^8, q_1^{28}\right)$$

$$+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{\frac{1}{14}}j\left(q_1^{12},q_1^{28}\right),$$

$$q^{\frac{1}{56}}j(q^3,q^7)=\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{\frac{25}{14}}j\left(q_1^{4},q_1^{28}\right)+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{\frac{9}{14}}j\left(q_1^{8},q_1^{28}\right)$$
$$+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{\frac{1}{14}}j\left(q_1^{12},q_1^{28}\right),$$

where $q=e^{-\alpha}$ and $q_1=e^{-\beta}$ with $\alpha\beta=\pi^2$. These formulas are special instances of the more general transformation formula

$$q^{\frac{B^2}{4A}}\sum_{n=-\infty}^{\infty}(-1)^n q^{An^2+Bn}=\sqrt{\frac{4\pi}{A\alpha}}\sum_{n=1}^{\infty}q_1^{\frac{(2n-1)^2}{4A}}\cos\frac{(2n-1)B\pi}{2A},$$

which follows from the transformation formula for $\theta_3(z,q)$ (see for example [7, p. 4]).

The transformation formulas for Ramanujan's seventh order mock theta functions are:

$$q^{-\frac{1}{168}}(\mathcal{F}_0(q)-2)=\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{1}{42}}\mathcal{F}_0\left(q_1^4\right)+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{25}{42}}\mathcal{F}_1\left(q_1^4\right)$$
$$+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{\frac{47}{42}}\mathcal{F}_2\left(q_1^4\right)-\sqrt{\frac{42\alpha}{\pi}}J\left(\frac{1}{7},7\alpha\right),$$

$$q^{-\frac{25}{168}}\mathcal{F}_1(q)=\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{1}{42}}\mathcal{F}_0\left(q_1^4\right)-\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{25}{42}}\mathcal{F}_1\left(q_1^4\right)$$
$$+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{\frac{47}{42}}\mathcal{F}_2\left(q_1^4\right)-\sqrt{\frac{42\alpha}{\pi}}J\left(\frac{2}{7},7\alpha\right),$$

$$q^{\frac{47}{168}}\mathcal{F}_2(q)=\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{1}{42}}\mathcal{F}_0\left(q_1^4\right)+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{25}{42}}\mathcal{F}_1\left(q_1^4\right)$$
$$-\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{\frac{47}{42}}\mathcal{F}_2\left(q_1^4\right)-\sqrt{\frac{42\alpha}{\pi}}J\left(\frac{3}{7},7\alpha\right),$$

$$q^{-\frac{1}{168}}(\mathcal{F}_0(-q)-2)=-\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{1}{168}}\mathcal{F}_0(-q_1)+\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{25}{168}}\mathcal{F}_1(-q_1)$$
$$+\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{\frac{47}{168}}\mathcal{F}_2(-q_1)+\sqrt{\frac{42\alpha}{\pi}}J_2\left(\frac{6}{7},7\alpha\right),$$

$$q^{-\frac{25}{168}}\mathcal{F}_1(-q)=\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{1}{168}}\mathcal{F}_0(-q_1)-\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{25}{168}}\mathcal{F}_1(-q_1)$$
$$+\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{\frac{47}{168}}\mathcal{F}_2(-q_1)-\sqrt{\frac{42\alpha}{\pi}}J_2\left(\frac{2}{7},7\alpha\right),$$

$$q^{\frac{47}{168}}\mathcal{F}_2(-q) = \sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{1}{168}}\mathcal{F}_0(-q_1) + \sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{25}{168}}\mathcal{F}_1(-q_1)$$
$$+\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{\frac{47}{168}}\mathcal{F}_2(-q_1) - \sqrt{\frac{42\alpha}{\pi}}J_2\left(\frac{4}{7}, 7\alpha\right).$$

The proof of the above formulas is similar to the proof of the fifth order formulas. We will outline the proof of the first seventh order transformation formula.

Proceeding along the same lines used in the proof of the first fifth order transformation formula we obtain the following analogue of Eq. (3.5):

$$\begin{aligned} N\left(\frac{1}{7}, q\right) &= 4\sin^2\frac{\pi}{7} + 4q^7 M\left(\frac{1}{7}, q^{49}\right)\sin^2\frac{\pi}{7} + 4q^{13}M\left(\frac{2}{7}, q^{49}\right)\sin\frac{\pi}{7}\sin\frac{2\pi}{7} \\ &+ 4q^{16}M\left(\frac{3}{7}, q^{49}\right)\sin\frac{\pi}{7}\sin\frac{3\pi}{7} + \frac{2}{(q^7;q^7)_\infty}\left[-j(q^{21}, q^{49})^2\sin^2\frac{\pi}{7}\right. \\ &\left. + q^6 j(q^7, q^{49})^2\sin\frac{\pi}{7}\sin\frac{2\pi}{7} + q^2 j(q^{14}, q^{49})^2\sin\frac{\pi}{7}\sin\frac{3\pi}{7}\right] \\ &+ \frac{4}{\sqrt{7}(q^7;q^7)_\infty}\left[q^6 j(q^7, q^{49})^2\sin^3\frac{\pi}{7} + q^2 j(q^{14}, q^{49})^2\sin\frac{\pi}{7}\sin^2\frac{2\pi}{7}\right. \\ &+ j(q^{21}, q^{49})^2\sin\frac{\pi}{7}\sin^2\frac{3\pi}{7} + 2q^4 j(q^7, q^{49})j(q^{14}, q^{49})\sin^2\frac{\pi}{7}\sin\frac{2\pi}{7} \\ &+ 2q^3 j(q^7, q^{49})j(q^{21}, q^{49})\sin^2\frac{\pi}{7}\sin\frac{3\pi}{7} \\ &\left. + 2qj(q^{14}, q^{49})j(q^{21}, q^{49})\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right]. \end{aligned} \tag{3.13}$$

The coefficients in the taylor series of $N(\frac{1}{7}, q)$ lie in the field $\mathbf{Q}(\cos(2\pi/7))$. The idea is to show that (3.13) is equivalent to three q-series identities with integer coefficients; the first involving $M(\frac{1}{7}, q)$, the second involving $M(\frac{2}{7}, q)$ and the third involving $M(\frac{3}{7}, q)$. From the generalized Lambert series (1.6) we obtain

$$\begin{aligned} N\left(\frac{1}{7}, q\right) &= \frac{1}{(q)_\infty}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)q^{\frac{1}{2}n(3n+1)}}{1-q^{7n}}(2 - q^n + q^{4n} - 2q^{5n}\right. \\ &+ u(-1 + 2q^n - 2q^{2n} + q^{3n} - 2q^{4n} + q^{5n}) \\ &\left. + v(-q^n + 3q^{2n} - 3q^{3n} + q^{4n}))\right] \\ &= \frac{4}{7}A(q)\sin^2\frac{\pi}{7} + \frac{4}{7}B(q)\sin\frac{\pi}{7}\sin\frac{2\pi}{7} + \frac{4}{7}C(q)\sin\frac{\pi}{7}\sin\frac{3\pi}{7}, \end{aligned}$$

where $u = 2\cos(2\pi/7)$, $v = 1 + 2\cos(4\pi/7)$ and

$$A(q) = \frac{1}{(q)_\infty}\left[3 + 7\sum_{n=1}^{\infty}\frac{(-1)^n(1-q^{6n})}{1-q^{7n}}q^{\frac{1}{2}n(3n+1)}\right],$$

$$B(q) = \frac{1}{(q)_\infty}\left[1 + 7\sum_{n=1}^{\infty}\frac{(-1)^n(1-q^{2n})}{1-q^{7n}}q^{\frac{1}{2}n(3n+1)}\right],$$
$$C(q) = \frac{1}{(q)_\infty}\left[2 + 7\sum_{n=1}^{\infty}\frac{(-1)^n(1-q^{4n})}{1-q^{7n}}q^{\frac{1}{2}n(3n+1)}\right].$$

Considering the field $\mathbf{Q}(\cos(2\pi/7))$ as a vector space over $\mathbf{Q}$ with basis $\{\sin^2(\pi/7), \sin(\pi/7)\sin(2\pi/7), \sin(\pi/7)\sin(3\pi/7)\}$, we can show that (3.13) is equivalent to the following three equations:

$$\begin{aligned}
&\left[A(q) - 7q^7 M\left(\frac{1}{7}, q^{49}\right) - 7\right](q^7;q^7)_\infty \\
&\quad = -2q^6 j(q^7,q^{49})^2 - q^2 j(q^{14},q^{49})^2 - 4j(q^{21},q^{49})^2 + q^4 j(q^7,q^{49})j(q^{14},q^{49}) \\
&\quad\quad + 2q^3 j(q^7,q^{49})j(q^{21},q^{49}) + 3qj(q^{14},q^{49})j(q^{21},q^{49}), \\
&\left[B(q) - 7q^{13} M\left(\frac{2}{7}, q^{49}\right)\right](q^7;q^7)_\infty \\
&\quad = 4q^6 j(q^7,q^{49})^2 + q^2 j(q^{14},q^{49})^2 + j(q^{21},q^{49})^2 - 2q^4 j(q^7,q^{49})j(q^{14},q^{49}) \\
&\quad\quad + 3q^3 j(q^7,q^{49})j(q^{21},q^{49}) + qj(q^{14},q^{49})j(q^{21},q^{49}), \\
&\left[C(q) - 7q^{16} M\left(\frac{3}{7}, q^{49}\right)\right](q^7;q^7)_\infty \\
&\quad = q^6 j(q^7,q^{49})^2 + 4q^2 j(q^{14},q^{49})^2 + 2j(q^{21},q^{49})^2 + 3q^4 j(q^7,q^{49})j(q^{14},q^{49}) \\
&\quad\quad - q^3 j(q^7,q^{49})j(q^{21},q^{49}) + 2qj(q^{14},q^{49})j(q^{21},q^{49}).
\end{aligned}$$

With much more work a proof of these equations should be obtained by the method used to prove (3.8).

4. Concluding remarks

Ramanujan divided his list of mock theta functions into "third order," "fifth order" and "seventh order" functions, but did not say what he meant by this. We will define the "order" of a mock theta function by its behavior under the action of the modular group. Observe that if a is relatively prime to b, then the coefficients in the Taylor series expansion of $N(a/b, q)$ lie in the field $\mathbf{Q}(\cos(2\pi/b))$. If k is the smallest positive integer such that these coefficients lie in the field $\mathbf{Q}(\cos(2\pi/k))$, then we will assign order k to the mock theta function $\mathcal{M}(a/b, q)$. This definition of order can be extended from the functions $\mathcal{M}(a/b, q)$ to other mock theta functions whose Taylor series expansions have integral coefficients. Suppose the map $\tau \to -1/\tau$ (hence $q \to q_1$) transforms $f(q)$ into $\lambda q^\mu g(q_1)$ (plus a Mordell integral), where $\lambda \in \mathbf{C}$, $\mu \in \mathbf{Q}$ and $g(q_1)$ has Taylor coefficients in $\mathbf{Q}(\cos(2\pi/k))$. The smallest such $k > 0$ is defined to be the order of $f(q)$. In Eq. (2.1) we find a factor $\sqrt{\pi/2\alpha}\ \csc(\pi r)$ in front of $N(r, q_1^4)$. Such factors must be taken into account in determining the transform $g(q_1)$ of $f(q)$. For example, first in our list of transformation formulas for the fifth order mock theta functions is

$$
\begin{aligned}
&q^{-\frac{1}{60}} f_0(q) \\
&= \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}} q_1^{-\frac{1}{60}} \left(F_0\left(q_1^2\right) - 1\right) + \sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}} q_1^{\frac{71}{60}} F_1\left(q_1^2\right) + \sqrt{\frac{60\alpha}{\pi}} J\left(\frac{1}{5}, 10\alpha\right), \\
&= \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}} q_1^{-\frac{1}{60}} \left[F_0\left(q_1^2\right) - 1 + \frac{1+\sqrt{5}}{2} q_1^{\frac{6}{5}} F_1\left(q_1^2\right)\right] + \sqrt{\frac{60\alpha}{\pi}} J\left(\frac{1}{5}, 10\alpha\right), \\
&= \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}} q_1^{-\frac{1}{60}} g\left(q_1^{\frac{1}{5}}\right) + \sqrt{\frac{60\alpha}{\pi}} J\left(\frac{1}{5}, 10\alpha\right),
\end{aligned}
$$

where

$$
g(q) = F_0(q^{10}) - 1 + \frac{1+\sqrt{5}}{2} q^6 F_1(q^{10}).
$$

The Taylor coefficients of $g(q)$ lie in the field $\mathbf{Q}(\sqrt{5}) = \mathbf{Q}(\cos(2\pi/5))$. Hence by the above definition $f_0(q)$ is a fifth order mock theta function.

Since $\mathbf{Q}(\cos(2\pi/k)) = \mathbf{Q}$ when $k = 1, 2, 3, 4$ and 6, mock theta fuctions of second, third, fourth and sixth order are actually of first order by this definition. Since $\mathbf{Q}(\cos(\pi/k)) = \mathbf{Q}(\cos(2\pi/k))$ for odd k, mock theta functions of order $2k$ are actually order k when k is odd.

According to this definition the functions $U_0(q)$, $U_1(q)$, $V_0(q)$, $V_1(q)$ in our paper on some eighth order mock theta functions [9] are actually first order mock theta functions, but the functions $S_0(q)$, $S_1(q)$, $T_0(q)$, $T_1(q)$ are eighth order mock theta functions. By Eq. (4.1) in that paper

$$
\begin{aligned}
q^{-\frac{1}{16}} S_0(q) &= \sqrt{\frac{\pi}{4\alpha}} V_0(q_1) + \sqrt{\frac{2\pi}{\alpha}} q_1^{-\frac{1}{4}} V_1(q_1) + \sqrt{\frac{4\alpha}{\pi}} K_0(\alpha) \\
&= \sqrt{\frac{\pi}{4\alpha}} q_1^{-\frac{1}{4}} \left[q_1^{\frac{1}{4}} V_0(q_1) + 2\sqrt{2} V_1(q_1)\right] + \sqrt{\frac{4\alpha}{\pi}} K_0(\alpha) \\
&= \sqrt{\frac{\pi}{4\alpha}} q_1^{\frac{1}{4}} g\left(q_1^{\frac{1}{4}}\right) + \sqrt{\frac{4\alpha}{\pi}} K_0(\alpha),
\end{aligned}
$$

where $g(q) = q V_0(q^4) + 2\sqrt{2} V_1(q^4)$. Since the Taylor coefficients of $g(q)$ lie in the field $\mathbf{Q}(\sqrt{2}) = \mathbf{Q}(\cos(2\pi/8))$, $S_0(q)$ is an eighth order mock theta function.

Equation (1.11) in [9] is incorrect. It should read

$$
V_1(q) - V_1(-q) = 2q(-q^2; q^2)_\infty (-q^4; q^4)_\infty^2 (q^8; q^8)_\infty.
$$

In [9] we did not express $R_0(q)$ and $R_1(q)$ as linear combinations of Euler products. Such combinations can be easily obtained from our equation

$$R_0(q^2) - qR_1(q^2) = \frac{(q;q^2)^3_\infty(q^2;q^2)_\infty}{(-q^4;q^4)_\infty(-q^2;q^4)_\infty} = \frac{(q;q^2)^3_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} = (q;q^2)^3_\infty\,\theta_4(0,q^2). \tag{4.1}$$

Replacing q by $-q$ we get

$$R_0(q^2) + qR_1(q^2) = (-q;q^2)^3_\infty\,\theta_4(0,q^2). \tag{4.2}$$

Adding (4.1) and (4.2) yields

$$R_0(q) = \frac{1}{2}\Big[\big(-q^{\frac{1}{2}}\big)^3_\infty + \big(q^{\frac{1}{2}}\big)^3_\infty\Big]\theta_4(0,q)$$

and subtracting (4.1) from (4.2) gives

$$R_1(q) = \frac{1}{2}q^{-\frac{1}{2}}\Big[\big(-q^{\frac{1}{2}}\big)^3_\infty - \big(q^{\frac{1}{2}}\big)^3_\infty\Big]\theta_4(0,q).$$

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