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A proof of the mock theta conjectures

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0. Introduction

In his last letter to Hardy ([R1, pp. 354–355] and [R2, pp. 127–131]), Ramanujan gave a list of 17 functions which he called "mock θ-functions". These are functions of a complex variable q, defined by q-series convergent for $|q|<1$. He stated that they have certain asymptotic properties as q approaches a root of unity, similar to the properties of θ-functions, but he conjectured that they are not, in fact, θ-functions. He also stated some identities relating some of the functions to each other.

Ramanujan's list was divided into four groups of functions, which were described as being of orders 3, 5, 5, and 7. In [W1], Watson studied the 3rd order functions, and introduced three new ones. He began by proving some identities which give simpler formulas for the functions. For example, one of Ramanujan's functions is

$$f(q)=\sum_{n\geqq 0}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\dots(1+q^n)^2}. \tag{0.0}$$

Watson gave the following identity for $f(q)$:

$$f(q)\prod_{n\geqq 1}(1-q^n)=1+4\sum_{n\geqq 1}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}. \tag{0.1}$$

Using such identities, he proved not only that the 3rd order functions have the asymptotic properties asserted by Ramanujan, but also that they are not θ-functions.

In [W2], Watson proved the asymptotic formulas for the 5th order functions. In [S], Selberg did the same for the 7th order functions. Neither author found formulas similar to (0.1), nor did they prove that the functions are not expressible as θ-functions.

In 1976, Andrews discovered Ramanujan's "Lost" Notebook (see [A2] and [R2]). It contained many identities involving q-series, including one for each of the ten 5th order functions. In [A-G], Andrews and Garvan discuss these identities, showing that they lead to formulas analogous to (0.1). Presumably these can be used to prove that the 5th order functions are not θ-functions, though this has not yet been done. Andrews and Garvan show that the identities for the functions in Ramanujan's first group of 5th order functions are equivalent to each other; they call these the "First Mock Theta Conjecture". Similarly, the identities for the second group are equivalent; they call these the "Second Mock Theta Conjecture". They also present combinatorial interpretations of the conjectures, in terms of the ranks of partitions.

To state the mock theta conjectures, we need some notation: If q and x are complex numbers with $|q|<1$ and n is an integer, let

$$(x)_\infty=(x;q)_\infty=\prod_{i\geqq 0}(1-q^i x) \tag{0.2}$$

and

$$(x)_n=(x;q)_n=\frac{(x)_\infty}{(q^n x)_\infty}. \tag{0.3}$$

In particular, for $n\geqq 0$,

$$(x)_n=(1-x)(1-qx)\dots(1-q^{n-1}x). \tag{0.4}$$

Two of the 5th order functions are

$$f_0(q)=\sum_{n\geqq 0}\frac{q^{n^2}}{(-q)_n} \tag{0.5}$$

and

$$f_1(q)=\sum_{n\geqq 0}\frac{q^{n^2+n}}{(-q)_n}. \tag{0.6}$$

We also define

$$\Phi(q)=-1+\sum_{n\geqq 0}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \tag{0.7}$$

and

$$\Psi(q)=-1+\sum_{n\geqq 0}\frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}. \tag{0.8}$$

Then the mock theta conjectures state that

$$f_0(q)=\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}-2\Phi(q^2) \tag{0.9}$$

and

$$f_1(q)=\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}-\frac{2}{q}\Psi(q^2). \tag{0.10}$$

In this paper, we prove these conjectures.

The proof relies on a pair of Hecke type identities discovered by Andrews (Eqs. (3.2) and (3.3) below); these express $(q)_\infty f_0(q)$ and $(q)_\infty f_1(q)$ as double sums of powers of q in which the exponents are given by indefinite quadratic forms. In [A5] (and see also [A3]), Andrews showed how such identities could be used to express the 5th order functions as constant terms of θ-functions. In this paper, we use a modification of this technique.

Section 1 presents some preliminary results concerning θ-functions and functional equations. In Sect. 2 we define a function $g(x,q)$ which generalizes $\Phi(q)$ and $\Psi(q)$, express it as the constant term of a θ-function $A(z,x,q)$, and obtain an identity relating $A(z,x,q)$, $g(x,q)$, and two generalized Lambert series. By similar methods, Sect. 3 derives an identity relating a θ-function $B(z,q)$, the functions $f_0(q)$ and $f_1(q)$, and generalized Lambert series. In Sect. 4, we state a θ-function identity which decomposes $B(z,q)$ into a sum of nine other θ-functions and use it, along with the results of Sects. 2 and 3, to prove the mock theta conjectures. Section 5 proves the decomposition of $B(z,q)$.

In a subsequent paper we will prove analogous formulas for the 7th order functions.

I wish to thank Dr. Andrews for several helpful suggestions.

1. Preliminaries

We will use the following notations for θ-functions:

Definition 1.0. If $|q|<1$ and $x \neq 0$, then

$$j(x,q)=(x)_\infty (q/x)_\infty (q)_\infty. \tag{1.0}$$

If m is a positive integer and a is an integer, then

$$J_{a,m}=j(q^a,q^m), \tag{1.1}$$

$$\bar{J}_{a,m}=j(-q^a,q^m), \tag{1.2}$$

and

$$J_m=j(q^m,q^{3m})=(q^m;q^m)_\infty. \quad \square \tag{1.3}$$

By Jacobi's triple product identity [H-W, p. 282], we have

$$j(x,q)=\sum_n (-1)^n q^{n(n-1)/2} x^n. \tag{1.4}$$

(Here and throughout the paper, summation indices are to run through all integers, or through all integers satisfying the conditions listed under the summation sign.)

Definition 1.1. For $r \geqq 0$, a *θ-product* of the variables $q, x_1, \ldots, x_r$ is an expression of the form

$$C q^e x_1^{f_1} \ldots x_r^{f_r} L_1^{g_1} \ldots L_s^{g_s}, \tag{1.5}$$

where C is a complex number, $s \geqq 0$, e, f_i, and g_i are integers, and each L_i has the form

$$j(D\,q^h x_1^{k_1} \dots x_r^{k_r}, \pm q^m) \tag{1.6}$$

for some complex number D and integers h, k_i, and $m \geqq 1$. (In this paper, D will always be ± 1.) A *θ-function* is a sum of finitely many θ-products. □

The representation of a θ-product in the form (1.5) is not unique. For example, the following identities follow easily from the definitions:

$$j(q/x, q) = j(x, q), \tag{1.7}$$

$$j(q^n x, q) = (-1)^n q^{-n(n-1)/2} x^{-n} j(x, q) \quad \text{if } n \text{ is an integer}, \tag{1.8}$$

$$j(-x, q) = \frac{J_{1,2}\, j(x^2, q^2)}{j(x, q)} \quad \text{if } x \text{ is not an integral power of } q, \tag{1.9}$$

$$j(-1, q) = \frac{2J_2^2}{J_1}, \tag{1.10}$$

$$j(x, -q) = \frac{j(x, q^2)\, j(-q\,x, q^2)}{J_{1,4}}, \tag{1.11}$$

$$j(x, q) = \frac{J_1}{J_n^n}\, j(x, q^n)\, j(q\,x, q^n) \dots j(q^{n-1} x, q^n) \quad \text{if } n \geqq 1. \tag{1.12}$$

Many other θ-product identities can be derived from these and the definitions. We will often use such identities without proof; they can be verified by the following method: First apply (1.9), (1.10), and (1.11) to eliminate any minus signs from factors of the form (1.6). Then use (1.8) to ensure that $0 \leqq h < m$ in each such factor. Rewrite each factor using Definition 1.0. What results is an identity involving factors of the form $(X; q^m)_\infty$ for various values of m. Let M be the least common multiple of the m's and rewrite each factor using

$$(X; q^m)_\infty = \prod_{k=0}^{M/m-1} (q^{mk} X; q^M)_\infty; \tag{1.13}$$

the resulting identity will be obvious. (Usually this process can be shortened by judicious use of (1.7) and (1.12).)

For example, in Theorem 1.0 below we need the identity

$$\frac{j(-x, q)\, j(q\,x^2, q^2)}{J_2} = \frac{J_1\, j(x^2, q)}{j(x, q)}. \tag{1.14}$$

By (1.9), this is equivalent to

$$J_{1,2}\, j(x^2, q^2)\, j(q\,x^2, q^2) = J_1 J_2\, j(x^2, q).$$

By (1.12) with $n=2$ and x replaced by x^2, this becomes

$$J_2 J_{1,2}=J_1^2;$$

that is,

$$(q^2;q^2)_\infty \cdot (q;q^2)_\infty^2 (q^2;q^2)_\infty=(q;q)_\infty^2,$$

which follows from (1.13) with $m=1$, $M=2$, and $X=q$.

In addition to such rearrangements of θ-products, we will also need three identities which allow us to replace certain sums of two θ-products by single θ-products.

Theorem 1.0. *If* $0<|q|<1$ *and x is neither* 0 *nor an integral power of* q, *then*

$$j(qx^3,q^3)+xj(q^2x^3,q^3)=\frac{J_1 j(x^2,q)}{j(x,q)}. \tag{1.15}$$

Proof. The quintuple product identity [A1, Thm. 3.9] states that, for $|q|<1$ and $x\neq 0$,

$$\sum_n (-1)^n q^{n(3n-1)/2} x^{3n}(1+q^n x)$$
$$=(-x)_\infty(-q/x)_\infty(q)_\infty(qx^2;q^2)_\infty(q/x^2;q^2)_\infty. \tag{1.16}$$

But the left-hand side of this equals

$$\sum_n (-1)^n q^{n(3n-1)/2} x^{3n}+x\sum_n (-1)^n q^{n(3n+1)/2} x^{3n}$$
$$=j(qx^3,q^3)+xj(q^2x^3,q^3),$$

while the right-hand side equals

$$j(-x,q)\frac{j(qx^2,q^2)}{J_2}=\frac{J_1 j(x^2,q)}{j(x,q)},$$

by (1.14). □

Theorem 1.1. *For* $0<|q|<1$, $x\neq 0$, *and* $y\neq 0$,

$$j(x,q)j(y,q)=j(-xy,q^2)j(-qx^{-1}y,q^2)-xj(-qxy,q^2)j(-x^{-1}y,q^2). \tag{1.17}$$

Proof. By (1.4), the left-hand side equals

$$\sum_m (-1)^m q^{m(m-1)/2} x^m \sum_n (-1)^n q^{n(n-1)/2} y^n$$
$$=\sum_{m,n} (-1)^{m+n} q^{(m^2-m+n^2-n)/2} x^m y^n. \tag{1.18}$$

Break this double sum into two parts, depending on whether $m+n$ is even or odd. In the 'even' part, write $m=r-s$ and $n=r+s$. In the 'odd' part, write $m=r-s+1$ and $n=r+s$. Then (1.18) equals

$$\sum_{r,s} q^{r^2+s^2-r} x^{r-s} y^{r+s} - \sum_{r,s} q^{r^2+s^2-s} x^{r-s+1} y^{r+s}$$
$$=\sum_r q^{r^2-r} x^r y^r \sum_s q^{s^2} x^{-s} y^s - x \sum_r q^{r^2} x^r y^r \sum_s q^{s^2-s} x^{-s} y^s$$
$$=j(-xy, q^2)\, j(-qx^{-1}y, q^2) - x\, j(-qxy, q^2)\, j(-x^{-1}y, q^2),$$

as required. □

Theorem 1.2. *For* $0<|q|<1$, $x \neq 0$, *and* $y \neq 0$,

$$j(-x, q)\, j(y, q) - j(x, q)\, j(-y, q) = 2x\, j(x^{-1}y, q^2)\, j(qxy, q^2). \tag{1.19}$$

Proof. Apply Theorem 1.1 twice, once with x replaced by $-x$ and once with y replaced by $-y$, and subtract. □

We will need to know the residues of various θ-functions at their poles. The following result enables us to compute these.

Theorem 1.3. *Let q be fixed, $0<|q|<1$. Let a, b, and m be fixed integers with $b \neq 0$ and $m \geqq 1$. Define*

$$F(z) = \frac{1}{j(q^a z^b, q^m)}. \tag{1.20}$$

Then F is meromorphic for $z \neq 0$, with simple poles at all points z_0 such that $z_0^b = q^{km-a}$ for some integer k. The residue of $F(z)$ at such a point z_0 is

$$\frac{(-1)^{k+1} q^{mk(k-1)/2} z_0}{b J_m^3}. \tag{1.21}$$

Proof. That F has only simple poles is clear. To compute the residue, write $z = z_0 x$ and let $x \to 1$. Then

$$F(z) = \frac{1}{j(q^{km} x^b, q^m)}$$
$$= \frac{1}{(-1)^k q^{-mk(k-1)/2} x^{-bk} j(x^b, q^m)},$$

by (1.8). But as $x \to 1$,

$$j(x^b, q^m) = (1-x^b)(q^m x^b; q^m)_\infty (q^m x^{-b}; q^m)_\infty (q^m; q^m)_\infty$$
$$\sim (1-x^b)(q^m; q^m)_\infty^3 = J_m^3 (1-x^b).$$

Hence the residue is

$$\lim_{z \to z_0} (z-z_0) F(z) = \lim_{x \to 1} \frac{(-1)^k q^{mk(k-1)/2} z_0 (x-1)}{J_m^3 (1-x^b)}$$
$$= \frac{(-1)^{k+1} q^{mk(k-1)/2} z_0}{b J_m^3},$$

as claimed. □

The next few results show that certain θ-products can be written in terms of generalized Lambert series and some related double sums. We begin with Ramanujan's ${}_1\Psi_1$ summation [A-A]:

$$\sum_r \frac{(a)_r x^r}{(b)_r} = \frac{(b/a)_\infty (q)_\infty (q/a x)_\infty (a x)_\infty}{(b)_\infty (b/a x)_\infty (q/a)_\infty (x)_\infty}, \tag{1.22}$$

provided that $0<|q|<1$, $a \neq 0$, $|b/a|<|x|<1$, and neither b nor q/a has the form q^{-k} where k is a nonnegative integer. Setting $a=y$ and $b=qy$ and dividing by $1-y$ gives

$$\sum_r \frac{x^r}{1-q^r y} = \frac{(q)_\infty^2 (x y)_\infty (q/x y)_\infty}{(x)_\infty (q/x)_\infty (y)_\infty (q/y)_\infty}$$

for $0<|q|<|x|<1$ and y neither 0 nor an integral power of q. Rewriting this in "j" notation, we have:

Theorem 1.4. *For $0<|q|<|x|<1$ and y neither 0 nor an integral power of q,*

$$\sum_r \frac{x^r}{1-q^r y} = \frac{J_1^3 j(x y, q)}{j(x, q) j(y, q)}. \quad \square \tag{1.23}$$

If, in addition, $|q|<|y|<1$, we can rewrite the left-hand side in a more symmetric form, by expanding it in powers of y. We need some notation.

By the formula for the sum of a geometric series, we have

$$\frac{1}{1-z} = \sum_{s \geqq 0} z^s \tag{1.24}$$

for $|z|<1$, while for $|z|>1$ we have

$$\frac{1}{1-z} = -\sum_{s<0} z^s. \tag{1.25}$$

It will be convenient to have a single formula which works for both cases.

Definition 1.2. For $|x| \neq 1$, let

$$\varepsilon(x) = \begin{cases} 1 & \text{if } |x|<1; \\ -1 & \text{if } |x|>1. \end{cases} \quad \square \tag{1.26}$$

Definition 1.3. If s is an integer, let

$$\operatorname{sg}(s)=\begin{cases} 1 & \text{if } s\geqq 0;\\ -1 & \text{if } s<0. \quad \square\end{cases} \tag{1.27}$$

Using these definitions, Eqs. (1.24) and (1.25) can be combined to give

$$\frac{1}{1-z}=\varepsilon(z)\sum_{\substack{s\\ \operatorname{sg}(s)=\varepsilon(z)}} z^s. \tag{1.28}$$

Now suppose that $|q|<|y|<1$ in Theorem 1.4. Then $|q^r y|<1$ if and only if $r\geqq 0$, so $\varepsilon(q^r y)=\operatorname{sg}(r)$. Hence the left-hand side of (1.23) equals

$$\begin{aligned}\sum_r \frac{x^r}{1-q^r y}&=\sum_r x^r \operatorname{sg}(r)\sum_{\substack{s\\ \operatorname{sg}(s)=\operatorname{sg}(r)}} (q^r y)^s\\ &=\sum_{\substack{r,s\\ \operatorname{sg}(r)=\operatorname{sg}(s)}} \operatorname{sg}(r)\, q^{rs} x^r y^s.\end{aligned}$$

Therefore:

Theorem 1.5. *For* $|q|<|x|<1$ *and* $|q|<|y|<1$,

$$\sum_{\operatorname{sg}(r)=\operatorname{sg}(s)} \operatorname{sg}(r)\, q^{rs} x^r y^s=\frac{J_1^3\, j(xy,q)}{j(x,q)\, j(y,q)}. \quad \square \tag{1.29}$$

We will also need the corresponding result in which the summation indices r and s are required to have the same parity.

Theorem 1.6. *For* $|q|<|x|<1$ *and* $|q|<|y|<1$,

$$\sum_{\substack{\operatorname{sg}(r)=\operatorname{sg}(s)\\ r\equiv s \pmod 2}} \operatorname{sg}(r)\, q^{rs} x^r y^s=\frac{J_{2,4}\, j(qxy,q^2)\, j(-qxy^{-1},q^2)\, j(x^2y^2,q^4)}{j(x^2,q^2)\, j(y^2,q^2)}. \tag{1.30}$$

Proof. Break up the sum on the left into two parts, depending on whether r and s are even or odd. In the 'even' part replace r and s by $2r$ and $2s$; in the 'odd' part replace them by $2r+1$ and $2s+1$:

$$\begin{aligned}&\sum_{\substack{\operatorname{sg}(r)=\operatorname{sg}(s)\\ r\equiv s \pmod 2}} \operatorname{sg}(r)\, q^{rs} x^r y^s\\ &=\sum_{\operatorname{sg}(r)=\operatorname{sg}(s)} \operatorname{sg}(r)\, q^{4rs} x^{2r} y^{2s}+\sum_{\operatorname{sg}(r)=\operatorname{sg}(s)} \operatorname{sg}(r)\, q^{4rs+2r+2s+1} x^{2r+1} y^{2s+1}\\ &=\sum_{\operatorname{sg}(r)=\operatorname{sg}(s)} \operatorname{sg}(r)\,(q^4)^{rs}(x^2)^r(y^2)^s+qxy\sum_{\operatorname{sg}(r)=\operatorname{sg}(s)} \operatorname{sg}(r)\,(q^4)^{rs}(q^2x^2)^r(q^2y^2)^s.\end{aligned}$$

By Theorem 1.5, this equals

$$\frac{J_4^3 j(x^2 y^2, q^4)}{j(x^2, q^4) j(y^2, q^4)} + q x y \frac{J_4^3 j(q^4 x^2 y^2, q^4)}{j(q^2 x^2, q^4) j(q^2 y^2, q^4)}$$
$$= \frac{J_{2,4} j(x^2 y^2, q^4)}{j(x^2, q^2) j(y^2, q^2)} [j(q^2 x^2, q^4) j(q^2 y^2, q^4) - q x^{-1} y^{-1} j(x^2, q^4) j(y^2, q^4)], \quad (1.31)$$

by product rearrangements. The quantity in brackets can be simplified by using Theorem 1.1 with q, x, and y replaced by q^2, $q x^{-1} y^{-1}$, and $-q x y^{-1}$; (1.31) becomes

$$\frac{J_{2,4} j(x^2 y^2, q^4)}{j(x^2, q^2) j(y^2, q^2)} j(q x y, q^2) j(-q x y^{-1}, q^2),$$

which equals the right-hand side of (1.30). □

Although we will not need it, it is interesting to note that there is a similar result for sums in which r and s are required to have opposite parity:

$$\sum_{\substack{\operatorname{sg}(r) = \operatorname{sg}(s) \\ r \not\equiv s \pmod 2}} \operatorname{sg}(r) q^{rs} x^r y^s = \frac{y J_{2,4} j(x y, q^2) j(-x y^{-1}, q^2) j(q^2 x^2 y^2, q^4)}{j(x^2, q^2) j(y^2, q^2)}. \quad (1.32)$$

Finally, we consider the functional equation

$$F(q z) = C z^{-n} F(z). \quad (1.33)$$

Our first result follows immediately from Lemma 2 of [A-S].

Theorem 1.7. *Let q and C be complex numbers with $0 < |q| < 1$ and $C \neq 0$, and let n be a nonnegative integer. Suppose that $F(z)$ is analytic for $z \neq 0$ and satisfies (1.33). Then either $F(z)$ has exactly n zeros in the annulus $|q| < |z| \leqq 1$ or $F(z) = 0$ for all z.* □

Such a function $F(z)$ must have a Laurent expansion valid for all $z \neq 0$. The next result expresses $F(z)$ in terms of a finite number of its coefficients.

Theorem 1.8. *Suppose that*

$$F(z) = \sum_r F_r z^r \quad (1.34)$$

for all $z \neq 0$ and that $F(z)$ satisfies (1.33) *where $0 < |q| < 1$ and $C \neq 0$.*

(a) *Then*

$$F(z) = \sum_{r=0}^{n-1} F_r z^r j(-C^{-1} q^r z^n, q^n). \quad (1.35)$$

(b) *If, in addition, n is odd,* $C = \pm 1$, *and* $F(z)$ *satisfies*

$$F(z^{-1}) = -C z^{-n} F(z), \tag{1.36}$$

then

$$F(z) = \sum_{r=1}^{\frac{n-1}{2}} F_r [z^r j(-C q^r z^n, q^n) - C z^{n-r} j(-C q^{n-r} z^n, q^n)]. \tag{1.37}$$

Proof. (a) Substituting (1.34) in (1.33) and equating coefficients of z^r gives

$$F_{r+n} = C^{-1} q^r F_r.$$

An inductive argument then yields

$$F_{r+kn} = C^{-k} q^{rk+nk(k-1)/2} F_r$$

for all integers r and k. Consequently,

$$\begin{aligned} F(z) &= \sum_{r=0}^{n-1} \sum_k F_{r+kn} z^{r+kn} \\ &= \sum_{r=0}^{n-1} \sum_k C^{-k} q^{rk+nk(k-1)/2} F_r z^{r+kn} \\ &= \sum_{r=0}^{n-1} F_r z^r \sum_k q^{nk(k-1)/2} (C^{-1} q^r z^n)^k \\ &= \sum_{r=0}^{n-1} F_r z^r j(-C^{-1} q^r z^n, q^n), \end{aligned}$$

proving (1.35).

(b) Similarly, substituting (1.34) in (1.36) and equating coefficients of z^{-r} gives

$$F_r = -C F_{n-r}. \tag{1.38}$$

Hence $F_0 = -C F_n = -C C^{-1} q^0 F_0 = -F_0$; that is, $F_0 = 0$. Substituting this in (1.35) and using (1.38) for $r > n/2$ implies (1.37). □

2. Generalization of Φ and Ψ

In this section we will define a function $g(x, q)$ which generalizes $\Phi(q)$ and $\Psi(q)$, and prove an identity relating $g(x, q)$, a θ-function, and two generalized Lambert series.

Definition 2.0. If $|q| < 1$ and x is neither 0 nor an integral power of q, let

$$g(x, q) = x^{-1}\left(-1 + \sum_{n \geq 0} \frac{q^{n^2}}{(x)_{n+1} (q/x)_n}\right). \quad \square \tag{2.0}$$

Obviously

$$\Phi(q) = q\, g(q, q^5) \tag{2.1}$$

and

$$\Psi(q) = q^2\, g(q^2, q^5). \tag{2.2}$$

Note. $g(x, q)$ can be defined more simply: It is not hard to show that

$$g(x, q) = \sum_{n \geqq 1} \frac{q^{n(n-1)}}{(x)_n (q/x)_n}. \tag{2.3}$$

However, (2.0) is more closely related to the mock theta conjectures, so we will not use (2.3).

Lemma (7.9) of [G] implies that, for $|q| < |x| < |q|^{-1}$ and $x \neq 1$,

$$-1 + \frac{1}{1-x} \sum_{n \geqq 0} \frac{q^{n^2}}{(xq)_n (x^{-1}q)_n} = \frac{x}{(q)_\infty} \sum_n (-1)^n \frac{q^{3n(n+1)/2}}{1 - x q^n}. \tag{2.4}$$

Analytic continuation then shows that this is true whenever $0 < |q| < 1$ and x is neither 0 nor an integral power of q. The left-hand side is $x\, g(x, q)$, so we obtain:

Theorem 2.0. *For $0 < |q| < 1$ and x not 0 or an integral power of q,*

$$J_1\, g(x, q) = \sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1 - q^n x}. \quad \square \tag{2.5}$$

We will use this result to express $g(x, q)$ as the constant term of a θ-function.

Definition 2.1. If $|q| < 1$ and neither x nor z is 0 or an integral power of q, let

$$A(z) = A(z, x, q) = \frac{J_1^2\, j(xz, q)\, j(z, q^3)}{j(x, q)\, j(z, q)}. \quad \square \tag{2.6}$$

Theorem 2.1. *Let q and x be fixed with $0 < |q| < 1$ and x neither 0 nor an integral power of q. Then $g(x, q)$ is the coefficient of z^0 in the Laurent series expansion of $A(z)$ in the annulus $|q| < |z| < 1$.*

Proof. By Theorems 2.0 and 1.4,

$$\begin{aligned} J_1\, g(x, q) &= \sum_n \frac{1}{1 - q^n x} (-1)^n q^{3n(n+1)/2} \\ &= \text{coefficient of } z^0 \text{ in} \sum_n \frac{z^n}{1 - q^n x} \sum_s (-1)^s q^{3s(s+1)/2} z^{-s} \\ &= \text{coefficient of } z^0 \text{ in} \frac{J_1^3\, j(xz, q)}{j(x, q)\, j(z, q)} j(z, q^3). \end{aligned}$$

Dividing by J_1 gives the theorem. $\square$

For fixed q and x, $A(z)$ is meromorphic for $z \neq 0$ with simple poles at $z=q^{3k\pm 1}$ for integers k. (Because of the factor $j(z, q^3)$ in (2.6), the singularities at $z=q^{3k}$ are removable.) Using (1.8), it is easy to show that $A(z)$ satisfies the functional equation

$$A(q^3 z) = -x^{-3} z^{-1} A(z). \tag{2.7}$$

Since $A(z)$ has singularities, we cannot apply the results of Sect. 1 directly. However, by adding two generalized Lambert series to $A(z)$, we will obtain a function to which Theorem 1.8 applies.

Theorem 2.2. *If* $0<|q|<1$ *and neither* x *nor* z *is* 0 *or an integral power of* q, *then*

$$\begin{aligned} A(z, x, q) = {} & j(x^3 z, q^3)\, g(x, q) \\ & -\sum_r \frac{(-1)^r q^{3r(r+1)/2} x^{3r+1} z^{r+1}}{1-q^{3r+1} z} \\ & -\sum_r \frac{(-1)^r q^{3r(r+3)/2+1} x^{-3r-1} z^{-r-1}}{1-q^{3r+1} z^{-1}}. \end{aligned} \tag{2.8}$$

Proof. Let q and x be fixed. Define

$$L(z) = \sum_r \frac{(-1)^r q^{3r(r+1)/2} x^{3r+1} z^{r+1}}{1-q^{3r+1} z}, \tag{2.9}$$

$$M(z) = \sum_r \frac{(-1)^r q^{3r(r+3)/2+1} x^{-3r-1} z^{-r-1}}{1-q^{3r+1} z^{-1}}, \tag{2.10}$$

and

$$F(z) = A(z) + L(z) + M(z). \tag{2.11}$$

It is easy to verify that

$$F(q^3 z) = -x^{-3} z^{-1} F(z), \tag{2.12}$$

since each of A, L, and M satisfies this functional equation.

We next show that $F(z)$ is analytic for all $z \neq 0$. Clearly L and M are meromorphic for $z \neq 0$, L has simple poles at $z=q^{3k-1}$ and M has simple poles at $z=q^{3k+1}$. Hence $F(z)$ is meromorphic for $z \neq 0$ with, at most, simple poles at $z=q^{3k\pm 1}$.

By Theorem 1.3 with $a=0$, $b=m=k=1$, and $z_0=q$, the residue of $A(z)$ at $z=q$ is

$$\frac{J_1^2\, j(xq, q)\, j(q, q^3)}{j(x, q)} \frac{q}{J_1^3} = \frac{q\, j(xq, q)}{j(x, q)} = -q x^{-1}.$$

The residue of $M(z)$ at $z=q$ is found by considering the $r=0$ term in (2.10). As $z \to q$,

$$M(z) = \frac{q x^{-1} z^{-1}}{1-q z^{-1}} + O(1) = \frac{q x^{-1}}{z-q} + O(1),$$

so the residue is qx^{-1}. By (2.11), the residue of $F(z)$ at $z=q$ is $(-qx^{-1})+0+qx^{-1}=0$; i.e. $F(z)$ is analytic at $z=q$.

In the same way we find that the residues of $A(z)$ and $L(z)$ at $z=q^{-1}$ are $q^{-2}x$ and $-q^{-2}x$, so $F(z)$ is analytic at $z=q^{-1}$.

Since F satisfies (2.12), it follows that $F(z)$ is analytic at all points of the form $z=q^{3k\pm 1}$ and hence for all $z\neq 0$. Now apply Theorem 1.8(a) with $n=1$, $C=-x^{-3}$, and q replaced by q^3:

$$F(z)=F_0\, j(x^3 z, q^3), \tag{2.13}$$

where F_0 is the coefficient of z^0 in the Laurent expansion of $F(z)$ for $z\neq 0$.

For $|q|<|z|<1$, the coefficient of z^0 in $A(z)$ is $g(x,q)$, by Theorem 2.1. For such z, $|q^{3r+1}z|<1$ if and only if $r\geqq 0$; i.e. $\varepsilon(q^{3r+1}z)=\mathrm{sg}(r)$. By (1.28),

$$\frac{1}{1-q^{3r+1}z}=\mathrm{sg}(r)\sum_{\substack{s\\ \mathrm{sg}(r)=\mathrm{sg}(s)}} q^{(3r+1)s} z^s,$$

so

$$L(z)=\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r)\,(-1)^r q^{3r(r+1)/2+(3r+1)s} x^{3r+1} z^{r+s+1}.$$

But if $\mathrm{sg}(r)=\mathrm{sg}(s)$ then $r+s+1$ is either $\geqq 1$ or $\leqq -1$, so the coefficient of z^0 in $L(z)$ is 0. Similarly, the coefficient of z^0 in $M(z)$ is 0. Hence the coefficient of z^0 in $F(z)$ is $g(x,q)$. By (2.13), we have

$$F(z)=g(x,q)\, j(x^3 z, q^3),$$

which implies the theorem. □

3. A relation between f_0, f_1, and a theta function

We now do for $f_0(q)$ and $f_1(q)$ what we just did for $g(x,q)$. We begin by rewriting two Hecke type identities due to Andrews.

Theorem 3.0.

$$J_1 f_0(q)=\sum_{\substack{\mathrm{sg}(r)=\mathrm{sg}(s)\\ r\equiv s \pmod 2}} \mathrm{sg}(r)\,(-1)^{\frac{r-s}{2}} q^{rs+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)} \tag{3.0}$$

and

$$J_1 f_1(q)=\sum_{\substack{\mathrm{sg}(r)=\mathrm{sg}(s)\\ r\equiv s \pmod 2}} \mathrm{sg}(r)\,(-1)^{\frac{r-s}{2}} q^{rs+\frac{3}{8}(r+s)^2+\frac{3}{4}(r+s)}. \tag{3.1}$$

Proof. By Eq. (6.1) of [A4],

$$\begin{aligned} J_1 f_0(q) &= \sum_{\substack{n\geqq 0\\ |j|\leqq n}} (-1)^j q^{n(5n+1)/2-j^2}(1-q^{4n+2}) \\ &= \sum_{\substack{n\geqq 0\\ |j|\leqq n}} (-1)^j q^{n(5n+1)/2-j^2} - \sum_{\substack{n\geqq 0\\ |j|\leqq n}} (-1)^j q^{(n+1)(5n+4)/2-j^2}. \end{aligned} \tag{3.2}$$

Writing $n=(r+s)/2$ and $j=(r-s)/2$ in the first sum on the right and $n=-(r+s+2)/2$ and $j=(r-s)/2$ in the second, this becomes

$$\sum_{\substack{r\geq 0\\ s\geq 0\\ r\equiv s\,(\mathrm{mod}\,2)}} (-1)^{\frac{r-s}{2}} q^{rs+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)} - \sum_{\substack{r\leq -1\\ s\leq -1\\ r\equiv s\,(\mathrm{mod}\,2)}} (-1)^{\frac{r-s}{2}} q^{rs+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)},$$

which equals the right-hand side of (3.0).

Similarly, Eq. (6.5) of [A4] gives

$$J_1 f_1(q) = \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^j q^{n(5n+3)/2-j^2}(1-q^{2n+1}). \tag{3.3}$$

Rearranging this in the same way yields (3.1). □

Definition 3.0. For $|q|<1$ and z not an integral power of q^2, let

$$B(z)=B(z,q)=\frac{z^2 J_2 j(-z,q) j(z,q^3)}{j(z,q^2)}. \quad \square \tag{3.4}$$

Note that $B(z)$ is meromorphic for $z\neq 0$, with simple poles at $z=q^{6k\pm 2}$. (The singularities at $z=q^{6k}$ are removable.) Further, $B(z)$ satisfies the functional equations

$$B(q^6 z) = -z^{-5} B(z) \quad \text{and} \quad B(z^{-1}) = z^{-5} B(z). \tag{3.5}$$

Theorem 3.1. *Let q be fixed, $0<|q|<1$. Then, in the annulus $|q|^2<|z|<1$, the coefficient of z^1 in the Laurent series expansion of $B(z)$ is $q f_0(q)$ and the coefficient of z^2 is $f_1(q)$.*

Proof. By Theorem 1.6 with $x=-z^{1/2}$ and $y=z^{1/2}$, we have

$$\sum_{\substack{\mathrm{sg}(r)=\mathrm{sg}(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} \mathrm{sg}(r)(-1)^r q^{rs} z^{(r+s)/2} = \frac{J_{2,4}\, j(-qz,q^2) j(q,q^2) j(z^2,q^4)}{j(z,q^2)^2}$$

$$= \frac{J_1 J_2 j(-z,q)}{j(z,q^2)}. \tag{3.6}$$

Hence

$$J_1 B(z) = z^2 \frac{J_1 J_2 j(-z,q)}{j(z,q^2)} j(z,q^3)$$

$$= z^2 \sum_{\substack{\mathrm{sg}(r)=\mathrm{sg}(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} \mathrm{sg}(r)(-1)^r q^{rs} z^{(r+s)/2} \sum_t (-1)^t q^{3t(t-1)/2} z^t. \tag{3.7}$$

The coefficient of z^1 in (3.7) is obtained by setting $t=-(r+s+2)/2$; it equals

$$\sum_{\substack{\mathrm{sg}(r)=\mathrm{sg}(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} \mathrm{sg}(r)(-1)^{\frac{r-s}{2}+1} q^{rs+\frac{3}{8}(r+s)^2+\frac{9}{4}(r+s)+3}.$$

Replacing r and s by $-1-r$ and $-1-s$ and noting that $\operatorname{sg}(-1-r)=-\operatorname{sg}(r)$, this becomes

$$\sum_{\substack{\operatorname{sg}(r)=\operatorname{sg}(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} \operatorname{sg}(r)\,(-1)^{\frac{r-s}{2}}\, q^{rs+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)+1}=q\,J_1\,f_0(q),$$

by (3.0). Dividing by J_1 shows that the coefficient of z^1 in $B(z)$ is $q\,f_0(q)$.

Similarly, the coefficient of z^2 in (3.7) equals

$$\sum_{\substack{\operatorname{sg}(r)=\operatorname{sg}(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} \operatorname{sg}(r)\,(-1)^{\frac{r-s}{2}}\, q^{rs+\frac{3}{8}(r+s)^2+\frac{3}{4}(r+s)}=J_1\,f_1(q),$$

so the coefficient of z^2 in $B(z)$ is $f_1(q)$. □

Theorem 3.2. *If* $0<|q|<1$ *and* z *is neither* 0 *nor an integral power of* q^2, *then*

$$\begin{aligned} B(z)=&\,q\,f_0(q)\,[z\,j(q^6 z^5, q^{30})+z^4\,j(q^{24} z^5, q^{30})]\\ &+f_1(q)\,[z^2\,j(q^{12} z^5, q^{30})+z^3\,j(q^{18} z^5, q^{30})]\\ &+2\sum_r \frac{(-1)^r q^{15r^2+15r+3} z^{5r+5}}{1-q^{6r+2} z}\\ &+2\sum_r \frac{(-1)^r q^{15r^2+15r+3} z^{-5r}}{1-q^{6r+2} z^{-1}}. \end{aligned} \tag{3.8}$$

Proof. Let

$$L(z)=2\sum_r \frac{(-1)^r q^{15r^2+15r+3} z^{5r+5}}{1-q^{6r+2} z}, \tag{3.9}$$

$$M(z)=2\sum_r \frac{(-1)^r q^{15r^2+15r+3} z^{-5r}}{1-q^{6r+2} z^{-1}}, \tag{3.10}$$

and

$$F(z)=B(z)-L(z)-M(z). \tag{3.11}$$

It is easy to verify that

$$F(q^6 z)=-z^{-5}\,F(z), \tag{3.12}$$

since each of the functions B, L, and M satisfies this functional equation. Further, $L(z^{-1})=z^{-5}\,M(z)$ and $M(z^{-1})=z^{-5}\,L(z)$, so F also satisfies

$$F(z^{-1})=z^{-5}\,F(z). \tag{3.13}$$

$F(z)$ is meromorphic for $z\neq 0$ with, at most, simple poles at $z=q^{6k\pm 2}$. The residue of $B(z)$ at $z=q^2$ is

$$q^4\,J_2\,j(-q^2, q)\,j(q^2, q^3)\,\frac{q^2}{J_2^3}=2q^5. \tag{3.14}$$

The residue of $M(z)$ at $z=q^2$ is given by the $r=0$ term in (3.10): As $z \to q^2$,

$$M(z)=\frac{2q^3}{1-q^2 z^{-1}}+O(1)=\frac{2q^3 z}{z-q^2}+O(1)=\frac{2q^5}{z-q^2}+O(1),$$

so the residue is $2q^5$. Hence the residue of $F(z)$ at $z=q^2$ is 0; i.e. F is analytic at $z=q^2$. By (3.12) and (3.13), F is analytic at all points $z=q^{6k\pm 2}$, and hence for all $z \neq 0$. By Theorem 1.8 (b) with q replaced by q^6, $n=5$, and $C=-1$,

$$\begin{aligned} F(z)=&F_1[z\,j(q^6 z^5, q^{30})+z^4 j(q^{24} z^5, q^{30})] \\ &+F_2[z^2 j(q^{12} z^5, q^{30})+z^3 j(q^{18} z^5, q^{30})] \end{aligned} \tag{3.15}$$

where F_1 and F_2 are the coefficients of z^1 and z^2 in the Laurent series of $F(z)$.

Now restrict z to the annulus $|q|^2<|z|<1$. Then $|q^{6r+2} z|<1$ if and only if $r \geqq 0$; i.e. $\varepsilon(q^{6r+2} z)=\operatorname{sg}(r)$. Hence

$$\begin{aligned} L(z)&=2\sum_r (-1)^r q^{15r^2+15r+3} z^{5r+5} \operatorname{sg}(r) \sum_{\substack{s \\ \operatorname{sg}(r)=\operatorname{sg}(s)}} (q^{6r+2} z)^s \\ &=2 \sum_{\operatorname{sg}(r)=\operatorname{sg}(s)} \operatorname{sg}(r)(-1)^r q^{15r^2+15r+3+(6r+2)s} z^{5r+s+5}. \end{aligned}$$

But if $\operatorname{sg}(r)=\operatorname{sg}(s)$ then $5r+s+5$ is either $\geqq 5$ or $\leqq -1$, so the coefficients of z^1 and z^2 in $L(z)$ equal 0. Similarly, the coefficients of z^1 and z^2 in $M(z)$ equal 0. Hence F_1 and F_2 equal the coefficients of z^1 and z^2 in $B(z)$; by Theorem 3.1 these are $q f_0(q)$ and $f_1(q)$, respectively. Substituting these values in (3.15) gives the theorem. □

4. The mock theta conjectures

Theorem 3.2 gives an identity involving both f_0 and f_1. To obtain the mock theta conjectures from it, we will decompose both sides of (3.8) in order to separate f_0 from f_1. We write

$$B(z)=\sum_{i=0}^{4} z^i B_i(z^5), \tag{4.0}$$

where each B_i is a single-valued function. This decomposition is clearly unique. We begin by finding B_1 and B_2 from the right-hand side of (3.8). (The other B_i's will not be needed.)

We have

$$\frac{1}{1-q^{6r+2} z}=\frac{1}{1-q^{30r+10} z^5} \sum_{i=0}^{4} z^i q^{(6r+2)i}$$

and

$$\frac{1}{1-q^{6r+2} z^{-1}}=\frac{z^{-5}}{1-q^{30r+10} z^{-5}} \sum_{i=1}^{5} z^i q^{(6r+2)(5-i)}.$$

By substituting these equations into the right-hand side of (3.8), we can read off the values of B_1 and B_2:

$$B_1(z^5)=q\,f_0(q)\,j(q^6 z^5, q^{30}) +2\sum_r \frac{(-1)^r q^{15r^2+21r+5} z^{5r+5}}{1-q^{30r+10} z^5} +2\sum_r \frac{(-1)^r q^{15r^2+39r+11} z^{-5r-5}}{1-q^{30r+10} z^{-5}} \tag{4.1}$$

and

$$B_2(z^5)=f_1(q)\,j(q^{12} z^5, q^{30}) +2\sum_r \frac{(-1)^r q^{15r^2+27r+7} z^{5r+5}}{1-q^{30r+10} z^5} +2\sum_r \frac{(-1)^r q^{15r^2+33r+9} z^{-5r-5}}{1-q^{30r+10} z^{-5}}. \tag{4.2}$$

To obtain B and B_2 from the left-hand side of (3.8), we use the following θ-function identity, which will be proved in the next section:

Theorem 4.0. *Let $|q|<1$. For $1\leqq r\leqq 4$, let*

$$G_r(z)=\frac{q^{(r-2)^2} z^r J_{5,10} J_{2r,5}\, j(q^{6r} z^5, q^{30})}{J_1} \tag{4.3}$$

and

$$H_r(z)=-2q^3 z^r A(z^5, q^{2r}, q^{10}) = -\frac{2q^3 z^r J_{10}^2\, j(q^{2r} z^5, q^{10})\, j(z^5, q^{30})}{J_{2r,10}\, j(z^5, q^{10})}. \tag{4.4}$$

Let

$$H_0(z)=\frac{2q^3 J_{10}^2\, j(z^{10}, q^{30})}{j(z^5, q^{10})}. \tag{4.5}$$

Then

$$B(z)=G_1(z)+G_2(z)-G_3(z)-G_4(z)+\sum_{r=0}^{4} H_r(z), \tag{4.6}$$

provided that $z\neq 0$ and z is not of the form ωq^{2k} where $\omega^5=1$ and k is an integer. $\square$

Assuming this identity, we find that

$$B_1(z^5)=z^{-1}[G_1(z)+H_1(z)] =\frac{q J_{5,10} J_{2,5}\, j(q^6 z^5, q^{30})}{J_1}-2q^3 A(z^5, q^2, q^{10}) \tag{4.7}$$

and

$$B_2(z^5)=z^{-2}[G_2(z)+H_2(z)] =\frac{J_{5,10} J_{4,5}\, j(q^{12} z^5, q^{30})}{J_1}-2q^3 A(z^5, q^4, q^{10}). \tag{4.8}$$

By Theorem 2.2,

$$A(z^5,q^2,q^{10})=j(q^6z^5,q^{30})\,g(q^2,q^{10})-\sum_r\frac{(-1)^r q^{15r^2+21r+2}z^{5r+5}}{1-q^{30r+10}z^5}-\sum_r\frac{(-1)^r q^{15r^2+39r+8}z^{-5r-5}}{1-q^{30r+10}z^{-5}} \tag{4.9}$$

and

$$A(z^5,q^4,q^{10})=j(q^{12}z^5,q^{30})\,g(q^4,q^{10})-\sum_r\frac{(-1)^r q^{15r^2+27r+4}z^{5r+5}}{1-q^{30r+10}z^5}-\sum_r\frac{(-1)^r q^{15r^2+33r+6}z^{-5r-5}}{1-q^{30r+10}z^{-5}}. \tag{4.10}$$

Combining (4.1), (4.7), and (4.9) yields

$$f_0(q)=\frac{J_{5,10}J_{2,5}}{J_1}-2q^2\,g(q^2,q^{10}). \tag{4.11}$$

By (2.1), this is equivalent to the first mock theta conjecture (0.9). Similarly, combining (4.2), (4.8), and (4.10) gives

$$f_1(q)=\frac{J_{5,10}J_{4,5}}{J_1}-2q^3\,g(q^4,q^{10}), \tag{4.12}$$

which is equivalent to the second mock theta conjecture (0.10).

5. Proof of Theorem 4.0

Let

$$V(z)=B(z)-G_1(z)-G_2(z)+G_3(z)+G_4(z)-\sum_{r=0}^{4}H_r(z). \tag{5.0}$$

We wish to prove that $V(z)$ is identically 0. Each of the functions B, G_r, and H_r satisfies the functional equation

$$f(q^6z)=-z^{-5}f(z).$$

In addition, B and H_0 satisfy

$$f(z^{-1})=z^{-5}f(z),$$

and, for $1\leqq r\leqq 4$, we have $G_r(z^{-1})=-z^{-5}G_{5-r}(z)$ and $H_r(z^{-1})=z^{-5}H_{5-r}(z)$. Therefore,

$$V(q^6z)=-z^{-5}V(z)\quad\text{and}\quad V(z^{-1})=z^{-5}V(z). \tag{5.1}$$

$V(z)$ is meromorphic for $z \neq 0$ with, at most, simple poles at the points $z = \omega q^{6k \pm 2}$, where $\omega^5 = 1$. We will show that $V(z)$ is, in fact, analytic for $z \neq 0$, by finding its residue at the points $z = \omega q^2$.

By (3.14), the residue of $B(z)$ at $z = q^2$ is $2q^5$; $B(z)$ is analytic at $z = \omega q^2$ for $\omega \neq 1$. $G_r(z)$ is analytic for $z \neq 0$. For $1 \leqq r \leqq 4$, the residue of $H_r(z)$ at $z = \omega q^2$ is

$$-\frac{2q^3 (\omega q^2)^r J_{10}^2 j(q^{2r+10}, q^{10}) j(q^{10}, q^{30})}{J_{2r,10}} \frac{\omega q^2}{5 J_{10}^3} = \frac{2\omega^{r+1}}{5} q^5.$$

This formula also applies when $r = 0$, but the derivation is different: The residue of $H_0(z)$ is

$$2q^3 J_{10}^2 j(q^{20}, q^{30}) \frac{\omega q^2}{5 J_{10}^3} = \frac{2\omega}{5} q^5.$$

Hence the residue of $V(z)$ at $z = q^2$ is

$$2q^5 - \sum_{r=0}^{4} \frac{2}{5} q^5 = 0;$$

the residue at $z = \omega q^2$ for $\omega \neq 1$ is

$$0 - \sum_{r=0}^{4} \frac{2\omega^{r+1}}{5} q^5 = 0.$$

So $V(z)$ is analytic at $z = \omega q^2$, $\omega^5 = 1$. By (5.1), $V(z)$ is analytic at all points $z = \omega q^{6k \pm 2}$ and hence for all $z \neq 0$.

According to Theorem 1.7 with $C = -1$, $n = 5$, and q replaced by q^6, if $V(z)$ is not identically 0, then it has exactly 5 zeros in $|q|^6 < |z| \leqq 1$. But we will show that $V(z) = 0$ at the 6 points $z = -q$, $-q^2$, $\pm q^3$, $-q^4$, and $-q^5$, which will imply the theorem.

By (5.1), $V(q^6 z^{-1}) = -z^5 V(z^{-1}) = -V(z)$. In particular, $V(\pm q^3) = 0$, $V(-q^4) = -V(-q^2)$, and $V(-q^5) = -V(-q)$. Hence it suffices to show that $V(-q) = V(-q^2) = 0$.

By (4.4), we have, for $1 \leqq r \leqq 4$,

$$\frac{H_{5-r}(z)}{H_r(z)} = \frac{z^{5-2r} j(q^{10-2r} z^5, q^{10})}{j(q^{2r} z^5, q^{10})} = \frac{z^{5-2r} j(q^{2r} z^{-5}, q^{10})}{j(q^{2r} z^5, q^{10})}.$$

If $z = -q^k$ where k is an integer, then (1.8) implies

$$\begin{aligned} j(q^{2r} z^5, q^{10}) &= j(q^{10k} \cdot q^{2r} z^{-5}, q^{10}) \\ &= (-1)^k q^{-5k(k-1)} q^{-2rk} z^{5k} j(q^{2r} z^{-5}, q^{10}) \\ &= q^{(5-2r)k} j(q^{2r} z^{-5}, q^{10}) \\ &= -z^{5-2r} j(q^{2r} z^{-5}, q^{10}); \end{aligned}$$

hence $H_{5-r}(-q^k)=-H_r(-q^k)$ and

$$\sum_{r=1}^{4} H_r(-q^k)=0.$$

Also, it is clear from (3.4) that $B(-q^k)=0$; therefore

$$\begin{aligned}
V(-q^k) &= -G_1(-q^k)-G_2(-q^k)+G_3(-q^k)+G_4(-q^k)-H_0(-q^k) \\
&= \frac{J_{5,10}}{J_1}(q^{k+1}J_{2,5}\bar{J}_{5k+6,30}-q^{2k}J_{4,5}\bar{J}_{5k+12,30} \\
&\quad -q^{3k+1}J_{6,5}\bar{J}_{5k+18,30}+q^{4k+4}J_{8,5}\bar{J}_{5k+24,30}) \\
&\quad -\frac{2q^3J_{10}^2J_{10k,30}}{\bar{J}_{5k,10}} \\
&= \frac{J_{5,10}}{J_1}[q^{k+1}J_{2,5}(\bar{J}_{5k+6,30}-q^{3k}\bar{J}_{5k+24,30}) \\
&\quad -q^{2k}J_{1,5}(\bar{J}_{5k+12,30}-q^k\bar{J}_{5k+18,30})] \\
&\quad -\frac{2q^3J_{10}^2J_{10k,30}}{\bar{J}_{5k,10}}.
\end{aligned}$$

In particular,

$$\begin{aligned}
V(-q) &= \frac{q^2J_{5,10}}{J_1}[J_{2,5}(\bar{J}_{11,30}-q^3\bar{J}_{29,30})-J_{1,5}(\bar{J}_{17,30}-q\bar{J}_{23,30})] \\
&\quad -\frac{2q^3J_{10}^3}{\bar{J}_{5,10}}.
\end{aligned} \tag{5.2}$$

By Theorem 1.0 with q and x replaced by q^{10} and $-q^3$,

$$\bar{J}_{11,30}-q^3\bar{J}_{29,30}=\bar{J}_{19,30}-q^3\bar{J}_{29,30}=\frac{J_{10}J_{6,10}}{\bar{J}_{3,10}}.$$

Similarly, replacing q and x by q^{10} and $-q$ gives

$$\bar{J}_{17,30}-q\bar{J}_{23,30}=\bar{J}_{13,30}-q\bar{J}_{23,30}=\frac{J_{10}J_{2,10}}{\bar{J}_{1,10}}.$$

So the bracketed quantity in (5.2) equals

$$\begin{aligned}
\frac{J_{2,5}J_{10}J_{6,10}}{\bar{J}_{3,10}}-\frac{J_{1,5}J_{10}J_{2,10}}{\bar{J}_{1,10}} &= \frac{J_{1,4}}{J_{20}}(\bar{J}_{1,10}J_{3,10}-J_{1,10}\bar{J}_{3,10}) \\
&= \frac{J_{1,4}}{J_{20}}2qJ_{2,20}J_{14,20}=\frac{2qJ_1J_{20}^2}{J_{10}},
\end{aligned}$$

by Theorem 1.2 with q, x, and y replaced by q^{10}, q, and q^3. Hence

$$V(-q)=\frac{2q^3 J_{5,10} J_{20}^2}{J_{10}}-\frac{2q^3 J_{10}^3}{\bar{J}_{5,10}}=0.$$

Similarly, we find

$$V(-q^2)=\frac{q^3 J_{5,10}}{J_1}[J_{2,5}(\bar{J}_{16,30}-q^6 \bar{J}_{34,30})-q J_{1,5}(\bar{J}_{22,30}-q^2 \bar{J}_{28,30})] \\ -\frac{2q^3 J_{10}^3}{\bar{J}_{10,10}}. \tag{5.3}$$

By Theorem 1.0,

$$\bar{J}_{16,30}-q^6 \bar{J}_{34,30}=\bar{J}_{16,30}-q^2 \bar{J}_{26,30}=\frac{J_{10} J_{4,10}}{\bar{J}_{2,10}}$$

and

$$\bar{J}_{22,30}-q^2 \bar{J}_{28,30}=\bar{J}_{22,30}-q^4 \bar{J}_{32,30}=\frac{J_{10} J_{8,10}}{\bar{J}_{4,10}}$$

so the bracketed quantity in (5.3) equals

$$\frac{J_{2,5} J_{10} J_{4,10}}{\bar{J}_{2,10}}-\frac{q J_{1,5} J_{10} J_{8,10}}{\bar{J}_{4,10}} \\ =\frac{J_5 J_{2,4}}{J_{10} J_{10,20}}(\bar{J}_{6,10} J_{3,10}-q J_{1,10} \bar{J}_{8,10}).$$

By Theorem 1.1 with q, x, and y replaced by $-q^5$, q, and $-q^2$, this equals

$$\frac{J_5 J_{2,4}}{J_{10} J_{10,20}} j(q,-q^5) j(-q^2,-q^5) \\ =\frac{J_5 J_{2,4}}{J_{10} J_{10,20}} \frac{J_{1,10} \bar{J}_{6,10}}{J_{5,20}} \frac{\bar{J}_{2,10} J_{7,10}}{J_{5,20}}=\frac{J_1 J_{10,20}^2}{J_{5,10}}.$$

Hence

$$V(-q^2)=q^3 J_{10,20}^2-\frac{2q^3 J_{10}^3}{\bar{J}_{10,10}}=0,$$

and the proof is complete. □

References

[A1] Andrews, G.E.: Applications of basic hypergeometric functions. SIAM Rev. **16**, 441–484 (1974)

[A2] Andrews, G.E.: An introduction to Ramanujan's "Lost" Notebook. Am. Math. Mon. **86**, 89–108 (1979)

[A3] Andrews, G.E.: Hecke modular forms and the Kac-Peterson identities. Trans. Am. Math. Soc. **283**, 451–458 (1984)

[A4] Andrews, G.E.: The fifth and seventh order mock theta functions. Trans. Am. Math. Soc. **293**, 113–134 (1986)

[A5] Andrews, G.E.: Ramanujan's fifth order mock theta functions as constant terms. In: "Ramanujan Revisited: Proc. of the Centenary Conference, Univ. of Illinois at Urbana-Champaign, June 1–5, 1987". San Diego: Acad. Press 1988

[A-A] Andrews, G.E., Askey, R.: A simple proof of Ramanujan's summation of the ${}_1\psi_1$. Aequationes Math. **18**, 333–337 (1978)

[A-G] Andrews, G.E., Garvan, F.G.: Ramanujan's "Lost" Notebook VI: The mock theta conjectures, Dept. of Math. Research Report no. 87007, The Pennsylvania State Univ., University Park, PA 16802; to appear in Adv. Math.

[A-S] Atkin, A.O.L., Swinnerton-Dyer, P.: Some properties of partitions. Proc. London Math. Soc. (3) **4**, 84–106 (1954)

[G] Garvan, F.G.: New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, and 11. Trans. Am. Math. Soc. **305**, 47–77 (1988)

[H-W] Hardy, G.H., Wright, E.M.: An introduction to the theory of numbers, 4th edition. London: Oxford Univ. Press 1968

[R1] Ramanujan, S.: Collected Papers. Hardy, G.H., Seshu Aiyar, P.V., Wilson, B.M. (eds.). London: Cambridge Univ. Press 1927; reprinted New York: Chelsea Publ. Cy., 1962

[R2] Ramanujan, S.: The lost notebook and other unpublished papers. New Delhi: Narosa Publishing House 1988

[S] Selberg, A.: Über die Mock-Thetafunktionen siebenter Ordnung. Arch. Math. Naturvidenskab **41**, 3–15 (1938)

[W1] Watson, G.N.: The final problem: an account of the mock theta functions. J. London Math. Soc. **11**, 55–80 (1936)

[W2] Watson, G.N.: The mock theta functions (2). Proc. London Math. Soc. (2) **42**, 274–304 (1937)

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