is the quotient of two polynomials in e^{2iz}. The elementary theory of partial fractions then shows that, when $3N+r+1 \geqslant 0$, we have

$$\frac{e^{(2r-1)iz}}{\vartheta_{2;N}(z-\alpha)\,\vartheta_{2;N}(z-\beta)\,\vartheta_{2;N}(z-\gamma)}$$
$$=\sum_{m=-N}^{N}\frac{A_{m;N}}{e^{2iz}+q^{2m}e^{2i\alpha}}+\sum_{m=-N}^{N}\frac{B_{m;N}}{e^{2iz}+q^{2m}e^{2i\beta}}+\sum_{m=-N}^{N}\frac{C_{m;N}}{e^{2iz}+q^{2m}e^{2i\gamma}},$$

where

$$A_{m;N}=A_m\prod_{n=N-m+1}^{\infty}[(1-q^{2n})(1-q^{2n}e^{-2i(\alpha-\beta)})(1-q^{2n}e^{-2i(\alpha-\gamma)})]$$
$$\times\prod_{n=N+m+1}^{\infty}[(1-q^{2n})(1-q^{2n}e^{2i(\alpha-\beta)})(1-q^{2n}e^{2i(\alpha-\gamma)})]$$

with corresponding values for $B_{m;N}$ and $C_{m;N}$.

We now make $N\to\infty$. The observation that

$$A_{m;N}/A_m$$

is a bounded function of m and N which, for any fixed m, tends to unity as $N\to\infty$, combined with the obvious remark that the series

$$\sum_{m=-\infty}^{\infty}\frac{A_m}{e^{2iz}+q^{2m}e^{2i\alpha}}$$

is absolutely convergent, justifies an appeal to Tannery's theorem; we thus have

$$\lim_{N\to\infty}\sum_{m=-N}^{N}\frac{A_{m;N}}{e^{2iz}+q^{2m}e^{2i\alpha}}=\sum_{m=-\infty}^{\infty}\frac{A_m}{e^{2iz}+q^{2m}e^{2i\alpha}},$$

and the other details of the passage to the limit present no difficulties. We have therefore established the expansion

$$\frac{e^{(2r-1)iz}}{\vartheta_2(z-\alpha)\,\vartheta_2(z-\beta)\,\vartheta_2(z-\gamma)}$$
$$=\sum_{\alpha,\beta,\gamma}\sum_{m=-\infty}^{\infty}\frac{2(-)^{m+r}q^{m(3m+1)+2mr}e^{2mi(2\alpha-\beta-\gamma)+(2r+1)i\alpha}}{\vartheta_1'(0)\,\vartheta_1(\alpha-\beta)\,\vartheta_1(\alpha-\gamma)(e^{2iz}+q^{2m}e^{2i\alpha})}.$$

In this expansion take

$$r=0,\quad z=0,\quad \alpha=\tfrac{1}{4}\pi,\quad \beta=-\tfrac{1}{4}\pi,\quad \gamma=0;$$

we get

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)}=\frac{2\vartheta_2(\frac{1}{4}\pi,q)}{\vartheta_2(0,q)}\sum_{m=-\infty}^{\infty}(-)^m q^{m(3m+1)}\left\{\frac{e^{(\frac{3}{2}m+\frac{1}{4})\pi i}}{1+iq^{2m}}+\frac{e^{-(\frac{3}{2}m+\frac{1}{4})\pi i}}{1-iq^{2m}}\right\}$$
$$-2\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{m(3m+1)}}{1+q^{2m}}.$$

Now it is easy to verify that

$$e^{(\frac{3}{2}m+\frac{1}{4})\pi i}(1-iq^{2m})+e^{-(\frac{3}{2}m+\frac{1}{4})\pi i}(1+iq^{2m})$$
$$=2\cos(\tfrac{1}{2}m-\tfrac{1}{4})\pi-2q^{2m}\sin(\tfrac{1}{2}m-\tfrac{1}{4})\pi$$
$$=(-)^{\frac{1}{2}m(3m+1)}\{1+(-q^2)^m\}\sqrt{2},$$

and hence we have

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)}=\frac{2\vartheta_2(\frac{1}{4}\pi,q)\sqrt{2}}{\vartheta_2(0,q)}\phi(-q^2)\prod_{r=1}^{\infty}\{1-(-q^2)^r\}-f(q^2)\prod_{r=1}^{\infty}(1-q^{2r}).$$

Further,

$$\frac{\vartheta_2(\frac{1}{4}\pi,q)\sqrt{2}}{\vartheta_2(0,q)}\prod_{r=1}^{\infty}\{1-(-q^2)^r\}=\prod_{n=1}^{\infty}\left[\frac{1+q^{4n}}{(1+q^{2n})^2}\right]\prod_{n=1}^{\infty}[(1+q^{4n-2})(1-q^{4n})]$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1+q^{2n})(1-q^{4n})}{(1+q^{2n})^2}\right]$$
$$=\prod_{n=1}^{\infty}(1-q^{2n}),$$

so that we have

$$2\phi(-q^2)-f(q^2)=\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)}\prod_{n=1}^{\infty}(1-q^{2n})^{-1}$$
$$=\prod_{n=1}^{\infty}\left[\frac{1-q^{2n}}{(1+q^{2n})^2}\right]=\vartheta_4(0,q^2)\prod_{n=1}^{\infty}(1+q^{2n})^{-1},$$

whence Ramanujan's relation connecting $\phi(-q)$ with $f(q)$ follows immediately.

Again, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=0,\quad \beta=\tfrac{1}{2}\pi\tau,\quad \gamma=\tfrac{1}{4}\pi\tau;$$

we get

$$\frac{\vartheta_1'(0,q)\,\vartheta_1^2(\tfrac14\pi\tau,q)}{\vartheta_2(0,q)\,\vartheta_2(\tfrac14\pi\tau,q)\,\vartheta_2(\tfrac12\pi\tau,q)}$$

$$=-\frac{2\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\left[\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+\frac12)}}{1+q^{2m}}+\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3(m+\frac12)(m+1)}}{1+q^{2m+1}}\right]$$
$$+2\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+1)+\frac32}}{1+q^{2m+\frac12}}$$

$$=-\frac{2\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\sum_{n=-\infty}^{\infty}\frac{(-)^{\frac12 n(3n+1)}q^{\frac34 n(n+1)}}{1+q^n}+2q^{\frac34}\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+1)+\frac12}}{1+q^{2m+\frac12}}$$
$$(n=2m \text{ or } 2m+1)$$

$$=-\frac{\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\,\phi(-\sqrt{q})\prod_{r=1}^{\infty}\{1-(-\sqrt{q})^r\}-2q^{\frac34}\psi(-\sqrt{q})\prod_{r=1}^{\infty}(1-q^{2r}).$$

Now evidently

$$\frac{q^{-\frac14}\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\prod_{r=1}^{\infty}\{1-(-\sqrt{q})^r\}=\prod_{n=1}^{\infty}\left[\frac{1-q^{n-\frac12}}{(1-q^{2n-1})^2}\right]\prod_{n=1}^{\infty}[(1+q^{n-\frac12})(1-q^n)]$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})(1-q^n)}{(1-q^{2n-1})^2}\right]$$
$$=\prod_{n=1}^{\infty}(1-q^{2n}).$$

Hence we have

$$\phi(-\sqrt{q})+2\psi(-\sqrt{q})=-\frac{q^{-\frac14}\vartheta_1'(0,q)\,\vartheta_1^2(\tfrac14\pi\tau,q)}{\vartheta_2(0,q)\,\vartheta_2(\tfrac14\pi\tau,q)\,\vartheta_2(\tfrac12\pi\tau,q)}\prod_{n=1}^{\infty}(1-q^{2n})^{-1}$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})(1-q^{n-\frac12})^2}{(1+q^{2n})^2(1+q^{n-\frac12})(1+q^{2n-1})^2}\right]$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})(1-q^{n-\frac12})^2}{(1+q^n)^2(1+q^{n-\frac12})}\right]$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^n)(1-q^{n-\frac12})^2}{(1+q^n)(1+q^{n-\frac12})}\right]$$
$$=\prod_{n=1}^{\infty}\left(\frac{1-q^{\frac12 n}}{1+q^{\frac12 n}}\right)\prod_{n=1}^{\infty}(1-q^{n-\frac12})$$
$$=\vartheta_4(0,\sqrt{q})\prod_{n=1}^{\infty}(1+q^{\frac12 n})^{-1}.$$

We have therefore obtained the two results

$$\vartheta_4(0,q)\prod_{n=1}^{\infty}(1+q^n)^{-1}=2\phi(-q)-f(q)=\phi(-q)+2\psi(-q),$$

and from them the formula

$$f(q)+4\psi(-q)=\vartheta_4(0,q)\prod_{n=1}^{\infty}(1+q^n)^{-1}$$

follows at once.

Next, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=\tfrac12\pi+\tfrac12\pi\tau,\quad \beta=\tfrac14\pi\tau,\quad \gamma=\pi+\tfrac34\pi\tau;$$

we get

$$\frac{\vartheta_1'(0,q)}{2q^{\frac34}\vartheta_2(\tfrac34\pi\tau,q)}=-\frac{iq^{\frac34}\vartheta_1(\tfrac12\pi\tau,q)}{\vartheta_2(\tfrac14\pi\tau,q)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}$$
$$+\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+\frac12)}}{1+q^{2m+\frac12}}-\sum_{m=-\infty}^{\infty}\frac{q^{3(m+\frac12)(m+1)}}{1+q^{2m+\frac32}}$$
$$=\frac{q^{\frac12}\vartheta_4(0,q)}{\vartheta_2(\tfrac14\pi\tau,q)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{\frac34 n(n+1)}}{1+q^{n+\frac12}}$$
$$(n=2m \text{ or } 2m+1)$$
$$=\frac{q^{\frac12}\vartheta_4(0,q)}{\vartheta_2(\tfrac14\pi\tau,q)}\,\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})+\upsilon(\sqrt{q})\prod_{r=1}^{\infty}(1-q^{\frac12 r}).$$

Now evidently

$$\frac{\vartheta_4(0,q)}{\vartheta_2(\tfrac14\pi\tau,q)}\prod_{r=1}^{\infty}\left(\frac{1-q^{2r}}{1-q^{\frac12 r}}\right)=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})^2}{1+q^{n-\frac12}}\right]\prod_{n=1}^{\infty}\left[\frac{1-q^{2n}}{(1-q^{n-\frac12})(1-q^n)}\right]$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})(1-q^{2n})}{1-q^n}\right]=1.$$

Hence we have

$$\upsilon(\sqrt{q})+q^{\frac12}\omega(q)=\frac{\vartheta_1'(0,q)}{2q^{\frac34}\vartheta_2(\tfrac34\pi\tau,q)}\prod_{r=1}^{\infty}(1-q^{\frac12 r})^{-1}$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})^2}{(1+q^{n-\frac12})(1-q^{n-\frac12})(1-q^n)}\right]$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})^2}{(1-q^{2n-1})(1-q^n)}\right]$$
$$=\prod_{n=1}^{\infty}[(1+q^n)^3(1-q^n)]$$
$$=\tfrac12 q^{-\frac18}\vartheta_2(0,\sqrt{q})\prod_{n=1}^{\infty}(1+q^n).$$

Since the expression on the right is a one-valued function of q, we immediately deduce the pair of formulae

$$v(q)+q\omega(q^2)=v(-q)-q\omega(q)^2=\tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,\,q)\prod_{n=1}^{\infty}(1+q^{2n}).$$

Lastly, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\tau,\quad \beta=\tfrac{1}{4}\pi,\quad \gamma=\tfrac{3}{4}\pi+\pi\tau;$$

we get

$$\frac{e^{-\frac{1}{4}\pi i}\,\vartheta_1'(0)\,\vartheta_1(\frac{1}{4}\pi+\frac{1}{2}\pi\tau)\,\vartheta_2(\pi\tau)}{2\vartheta_1(\frac{1}{2}\pi\tau)\,\vartheta_2(\frac{1}{4}\pi)\,\vartheta_2(\frac{3}{4}\pi+\pi\tau)}=\frac{e^{\frac{1}{4}\pi i}\,q^{\frac{3}{4}}\,\vartheta_1(\frac{1}{2}\pi+\pi\tau)}{\vartheta_1(\frac{1}{4}\pi+\frac{1}{2}\pi\tau)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}$$
$$+i\sum_{m=-\infty}^{\infty}\frac{q^{3m^2}e^{-\frac{1}{2}m\pi i}}{1+iq^{2m}}+\sum_{m=-\infty}^{\infty}\frac{q^{3(m+1)^2}e^{\frac{1}{2}m\pi i}}{1-iq^{2m+2}},$$

so that, replacing m by n or $n-1$, we have

$$\frac{\vartheta_1'(0)\,\vartheta_4(\frac{1}{4}\pi)\,\vartheta_2(0)}{2\vartheta_4(0)\,\vartheta_2^2(\frac{1}{4}\pi)}$$
$$=\frac{q^{\frac{1}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+i\sum_{n=-\infty}^{\infty}q^{3n^2}\left\{\frac{e^{-\frac{1}{2}n\pi i}}{1+iq^{2n}}-\frac{e^{\frac{1}{2}n\pi i}}{1-iq^{2n}}\right\}$$
$$=\frac{q^{\frac{1}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+2\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{4m(3m+1)}}{1+q^{8m}}$$
$$+2q^3\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{12m(m+1)}}{1+q^{8m+4}}$$
$$(n=2m\ \text{ or }\ 2m+1)$$
$$=\frac{q^{\frac{1}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\,\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})+[f(q^8)+2q^3\omega(-q^4)]\prod_{r=1}^{\infty}(1-q^{8r}).$$

When we reduce this in the usual manner, we find that

$$f(q^8)+2q\,\omega(q)+2q^3\omega(-q^4)=\vartheta_3(0,\,q)\,\vartheta_3^2(0,\,q^2)\prod_{n=1}^{\infty}(1-q^{4n})^{-2},$$

and hence, by changing the sign of q,

$$f(q^8)-2q\,\omega(-q)-2q^3\omega(-q^4)=\vartheta_4(0,\,q)\,\vartheta_3^2(0,\,q^2)\prod_{n=1}^{\infty}(1-q^{4n})^{-2}.$$

Any other relations of this kind connecting mock ϑ-functions of order 3 would appear to be derivable from the relations now obtained.

It is now feasible to construct the linear transformations of the mock ϑ-functions. Since any substitution of the modular group can be resolved into a number of substitutions of the forms

$$\tau'=\tau+1,\quad \tau'=-1/\tau,$$

it is sufficient to construct the transformations which express the fourteen functions* $f(\pm q)$, ... in terms of similar functions of q_1 (or powers of q_1), where q and q_1 are connected by the relations†

$$q=e^{-\alpha},\quad \alpha\beta=\pi^2,\quad q_1=e^{-\beta}.$$

The general similarity between the series involved in the new definitions of the mock ϑ-functions and the series which are generating functions of class-numbers of binary quadratic forms suggests that it may be possible to construct the required transformations by means of functional equations such as have been used by Mordell‡ in connection with class-numbers. Since, however, I lacked the ingenuity necessary for the construction of the functional equations (if indeed they exist), I decided to use the more prosaic methods of contour integration by which a writer subsequent to Mordell has treated the generating functions of class-numbers§.

It is unnecessary to work out all the fourteen transformation formulae by contour integration; when the transformation formulae for $f(q)$ and $\phi(q)$ have been constructed, the remainder can be deduced immediately from the relations connecting the various mock ϑ-functions.

First consider $f(q)$. We have, by Cauchy's theorem,

$$f(q)\prod_{r=1}^{\infty}(1-q^r)=\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic}+\int_{\infty+ic}^{-\infty+ic}\right\}\frac{\pi}{\sin\pi z}\,\frac{\exp(-\frac{3}{2}\alpha z^2)}{\cosh\frac{1}{2}\alpha z}\,dz,$$

where c is a positive number so small that the zeros of $\sin\pi z$ are the only poles of the integrand between the lines forming the contour. On the higher of these two lines we write

$$\frac{1}{\sin\pi z}=-2i\sum_{n=0}^{\infty}e^{(2n+1)\pi iz},$$

* Actually I do not trouble to deal with the functions $\chi(\pm q)$ and $\rho(\pm q)$ which are less interesting than the rest.

† The numbers α and β, which are positive when q is positive, are slightly easier to work with than the complex τ.

‡ L. J. Mordell, *Quart. J. of Math.*, 48 (1920), 329–342.

§ G. N. Watson, *Compositio Math.*, 1 (1934), 39–68.

so that

$$\frac{1}{2\pi i}\int_{\infty+ic}^{-\infty+ic} \frac{\pi}{\sin \pi z}\, \frac{\exp(-\frac{3}{2}az^2)}{\cosh \frac{1}{2}az}\, dz$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{-\infty+ic}^{\infty+ic} 4\pi i \exp\{(2n+1)\pi i z - \tfrac{3}{2}az^2\} \frac{e^{az}+e^{-az}-1}{e^{\frac{3}{2}az}+e^{-\frac{3}{2}az}}\, dz$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{-\infty+ic}^{\infty+ic} F_n(z)\, dz,$$

say. We calculate these integrals in the following manner. The poles of $F_n(z)$ are (at most) simple poles at the points

$$z_m = \frac{(2m+1)\pi i}{3a} \quad (m = -\infty, \ldots, -1, 0, 1, \ldots, +\infty),$$

and the residue at z_m is

$$\frac{4\pi}{3a} (-)^m \exp\{(2n+1)\pi i z_m - \tfrac{3}{2}a z_m{}^2\} . (2\cosh a z_m - 1) = \lambda_{n,m},$$

say. Now, by Cauchy's theorem,

$$\frac{1}{2\pi i} \left\{ \int_{-\infty+ic}^{\infty+ic} - P \int_{-\infty+z_n}^{\infty+z_n} \right\} F_n(z)\, dz = \lambda_{n,0} + \lambda_{n,1} + \ldots + \lambda_{n,n-1} + \tfrac{1}{2}\lambda_{n,n},$$

where P denotes the "principal value" of the integral. Next, by rearrangement of repeated series,

$$\tfrac{1}{2}\lambda_{0,0} + \sum_{n=1}^{\infty} (\lambda_{n,0} + \lambda_{n,1} + \ldots + \lambda_{n,n-1} + \tfrac{1}{2}\lambda_{n,n})$$

$$= \sum_{m=0}^{\infty} (\tfrac{1}{2}\lambda_{m,m} + \lambda_{m+1,m} + \lambda_{m+2,m} + \ldots)$$

$$= \tfrac{1}{2} \sum_{m=0}^{\infty} \lambda_{m,m} \frac{1+\exp 2\pi i z_m}{1-\exp 2\pi i z_m}$$

$$= \frac{2\pi}{3a} \sum_{m=0}^{\infty} (-)^m \{2\cos \tfrac{1}{3}(2m+1)\pi - 1\}\, q_1^{\frac{1}{6}(2m+1)^2} \frac{1+q_1^{\frac{2}{3}(2m+1)}}{1-q_1^{\frac{2}{3}(2m+1)}}$$

$$= \frac{2\pi}{a} \sum_{p=0}^{\infty} (-)^p\, q_1^{\frac{3}{2}(2p+1)^2} \frac{1+q_1^{4p+2}}{1-q_1^{4p+2}},$$

where $m = 3p+1$, the terms for which $m \neq 3p+1$ vanishing.

Further, we have

$$P \int_{-\infty+z_n}^{\infty+z_n} F_n(z)\, dz$$

$$= P \int_{-\infty}^{\infty} F_n(z_n + x)\, dx$$

$$= P \int_{-\infty}^{\infty} 4\pi i \exp\left\{ -\frac{(2n+1)^2\pi^2}{6a} - \frac{3ax^2}{2} \right\} \frac{\cosh\{ax + \frac{1}{3}(2n+1)\pi i\} - \frac{1}{2}}{(-)^n i \sinh \frac{3}{2}ax}\, dx$$

$$= 4\pi i (-)^n \sin \tfrac{1}{3}(2n+1)\pi . q_1^{\frac{1}{6}(2n+1)^2} \int_{-\infty}^{\infty} e^{-\frac{3}{2}ax^2} \frac{\sinh ax}{\sinh \frac{3}{2}ax}\, dx.$$

This simplification in the integral under consideration is due to the modified contour having been chosen to pass through the stationary point of the function

$$\exp\{(2n+1)\pi i z - \tfrac{3}{2}az^2\},$$

which occurs in the integrand, in the manner of the "method of steepest descents".

The integral along the lower line can be evaluated at once by changing the sign of i throughout the previous work. On combining the results we get

$$f(q) \prod_{r=1}^{\infty} (1-q^r) = \frac{4\pi}{a} q_1^{\frac{3}{2}} \omega(q_1{}^2) \prod_{r=1}^{\infty} (1-q_1^{4r}) + 4\vartheta_1(\tfrac{1}{3}\pi, q_1^{\frac{2}{3}}) \int_0^{\infty} e^{-\frac{3}{2}ax^2} \frac{\sinh ax}{\sinh \frac{3}{2}ax}\, dx.$$

By Jacobi's imaginary transformation this reduces to

$$q^{-\frac{1}{24}} f(q) = 2\sqrt{\left(\frac{2\pi}{a}\right)}\, q_1^{\frac{4}{3}} \omega(q_1{}^2) + 4\sqrt{\left(\frac{3a}{2\pi}\right)} \int_0^{\infty} e^{-\frac{3}{2}ax^2} \frac{\sinh ax}{\sinh \frac{3}{2}ax}\, dx,$$

which is the transformation for $f(q)$. We shall consider the integral on the right presently.

We now turn to $\phi(q)$. We have, by Cauchy's theorem,

$$\phi(q) \prod_{r=1}^{\infty} (1-q^r) = \frac{1}{2\pi i} \left\{ \int_{-\infty-ic}^{\infty-ic} + \int_{\infty+ic}^{-\infty+ic} \right\} \frac{\pi}{\sin \pi z}\, \frac{\cosh \frac{1}{2}az}{\cosh az} \exp(-\tfrac{3}{2}az^2)\, dz,$$

and, as before, we get

$$\frac{1}{2\pi i} \int_{\infty+ic}^{-\infty+ic} \frac{\pi}{\sin \pi z}\, \frac{\cosh \frac{1}{2}az}{\cosh az} \exp(-\tfrac{3}{2}az^2)\, dz = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{-\infty+ic}^{\infty+ic} \Phi_n(z)\, dz,$$

where

$$\Phi_n(z) = 2\pi i \exp\{(2n+1)\pi i z - \tfrac{3}{2}az^2\} \frac{\cosh \frac{1}{2}az \{2\cosh 2az - 1\}}{\cosh 3az}.$$

The poles of $\Phi_n(z)$ are (at most) simple poles at the points

$$\zeta_m = \frac{(4m+1)\pi i}{6\alpha}, \quad \eta_m = \frac{(4m-1)\pi i}{6\alpha} \quad (m = -\infty, \ldots, -1, 0, 1, \ldots, \infty),$$

and the residues at ζ_m and η_m are

$$\frac{2\pi}{3\alpha} \exp\{(2n+1)\pi i \zeta_m - \tfrac{3}{2}\alpha\zeta_m^2\} . \cosh \tfrac{1}{2}\alpha\zeta_m \{2 \cosh 2\alpha\zeta_m - 1\} = \mu_{n,m},$$

$$-\frac{2\pi}{3\alpha} \exp\{(2n+1)\pi i \eta_m - \tfrac{3}{2}\alpha\eta_m^2\} . \cosh \tfrac{1}{2}\alpha\eta_m \{2 \cosh 2\alpha\eta_m - 1\} = \nu_{n,m},$$

say. Now, by Cauchy's theorem,

$$\frac{1}{2\pi i}\left\{\int_{-\infty+ic}^{\infty+ic} - \int_{-\infty+z_n}^{\infty+z_n}\right\} \Phi_n(z)\,dz$$

$$= \mu_{n,0}+\mu_{n,1}+\mu_{n,2}+\ldots+\mu_{n,n}+\nu_{n,1}+\nu_{n,2}+\ldots+\nu_{n,n},$$

and, as before, by rearrangement of repeated series,

$$\sum_{n=0}^{\infty} (\mu_{n,0}+\mu_{n,1}+\ldots+\mu_{n,n}) + \sum_{n=1}^{\infty} (\nu_{n,1}+\nu_{n,2}+\ldots+\nu_{n,n})$$

$$= \sum_{m=0}^{\infty} (\mu_{m,m}+\mu_{m+1,m}+\mu_{m+2,m}+\ldots) + \sum_{m=1}^{\infty} (\nu_{m,m}+\nu_{m+1,m}+\nu_{m+2,m}+\ldots)$$

$$= \sum_{m=0}^{\infty} \frac{\mu_{m,m}}{1-\exp 2\pi i\zeta_m} + \sum_{m=1}^{\infty} \frac{\nu_{m,m}}{1-\exp 2\pi i\eta_m}$$

$$= \frac{2\pi}{3\alpha} \sum_{m=0}^{\infty} \frac{q_1^{(4m+1)(4m+3)/24}}{1+q_1^{(4m+1)/3}} \cos\frac{(4m+1)\pi}{12} \left\{2\cos\frac{(4m+1)\pi}{3} - 1\right\}$$

$$- \frac{2\pi}{3\alpha} \sum_{m=1}^{\infty} \frac{q_1^{(4m-1)(4m+5)/24}}{1-q_1^{(4m-1)/3}} \cos\frac{(4m-1)\pi}{12} \left\{2\cos\frac{4m-1)\pi}{3} - 1\right\}$$

$$= \frac{\pi\sqrt{2}}{\alpha} q_1^{\frac{2}{3}} \left[\sum_{n=0}^{\infty} \frac{(-)^n q_1^{2n(3n+5)+1}}{1-q_1^{4n+3}} + \sum_{p=0}^{\infty} \frac{(-)^p q_1^{6p(p+1)}}{1-q_1^{4p+1}}\right]$$

$$(m = 3n+2;\ m = 3p+1).$$

Further, we have

$$\int_{-\infty+z_n}^{\infty+z_n} \Phi_n(z)\,dz = \int_{-\infty}^{\infty} \Phi_n(z_n+x)\,dx$$

$$= 2\pi i \int_{-\infty}^{\infty} \exp\left\{-\frac{(2n+1)^2\pi^2}{6\alpha} - \frac{3\alpha x^2}{2}\right\}$$

$$\times \frac{\cosh\frac{5}{2}\alpha z + \cosh\frac{3}{2}\alpha z - \cosh\frac{1}{2}\alpha z}{-\cosh 3\alpha x}\,dx$$

$$= 2\pi i q_1^{(2n+1)^2/6} \cos\frac{(2n+1)\pi}{6} \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\,dx.$$

The integral along the lower line can be evaluated by changing the sign of i throughout the previous work. On combining the results we get

$$\phi(q) \prod_{r=1}^{\infty} (1-q^r)$$

$$= \frac{2\pi\sqrt{2}}{\alpha} q_1^{\frac{1}{3}} \psi(q_1) \prod_{r=1}^{\infty} (1-q_1^{4r}) + \vartheta_2(\tfrac{1}{6}\pi, q_1^{\frac{2}{3}}) \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\,dx.$$

This reduces to

$$q^{-\frac{1}{24}}\phi(q) = 2\sqrt{\left(\frac{\pi}{\alpha}\right)}\, q_1^{-\frac{1}{24}} \psi(q_1) + \sqrt{\left(\frac{6\alpha}{\pi}\right)} \int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\,dx,$$

which is the transformation for $\phi(q)$.

We next consider the integrals on the right of the transformation formulae. Let

$$\int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\,dx = J(\alpha);$$

then it is easy to see that

$$J(\alpha) = \sqrt{\left(\frac{6\beta}{\pi}\right)} \int_0^{\infty}\int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \cos 3\pi xy\, \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\,dy\,dx$$

$$= \frac{\pi}{3\alpha}\sqrt{\left(\frac{6\beta}{\pi}\right)} \int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \left\{\frac{\cosh\frac{1}{2}\beta y \cos\frac{5}{12}\pi}{\cosh\beta y + \cos\frac{5}{6}\pi} + \frac{\cosh\frac{1}{2}\beta y \cos\frac{1}{12}\pi}{\cosh\beta y + \cos\frac{1}{6}\pi}\right\} dy$$

$$= \sqrt{\left(\frac{\beta\pi}{\alpha^2}\right)} \int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \frac{2\cosh\frac{3}{2}\beta y \cosh\beta y}{\cosh 3\beta y}\,dy,$$

so that

$$J(\alpha) = \sqrt{\left(\frac{\pi^3}{\alpha^3}\right)}\, J(\beta).$$

Next let

$$\int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\sinh\alpha x}{\sinh\frac{3}{2}\alpha x}\,dx = J_1(\alpha),$$

and it is found in a similar manner that

$$J_1(\alpha) = \sqrt{\left(\frac{2\pi^3}{\beta^3}\right)}\, J_2(\beta),$$

where

$$J_2(\beta) = \int_0^{\infty} e^{-\frac{3}{2}\beta x^2} \frac{\cosh\beta x}{\cosh 3\beta x}\,dx.$$

It is easy to obtain asymptotic expansions for $J(\alpha)$, $J_1(\alpha)$, and $J_2(\alpha)$ proceeding in ascending powers of α and valid when $|\alpha|$ is small and the real part of α is positive; the first few terms of the expansions are

$$J(\alpha)=\sqrt{\left(\frac{2\pi}{3\alpha}\right)}\,[1-\tfrac{23}{24}\alpha+\tfrac{3985}{1152}\alpha^2-\ldots],$$

$$J_1(\alpha)=\sqrt{\left(\frac{2\pi}{27\alpha}\right)}\,[1-\tfrac{5}{72}\alpha+\tfrac{17}{1152}\alpha^2-\ldots],$$

$$J_2(\alpha)=\sqrt{\left(\frac{\pi}{6\alpha}\right)}\,[1-\tfrac{4}{3}\alpha+\tfrac{44}{9}\alpha^2-\ldots].$$

It can be proved that these expansions possess the property that (for α complex) the error due to stopping at any term never exceeds in absolute value the first term neglected; in addition, for α positive, the error is of the same sign as that term*.

I now revert to the construction of the set of transformation formulae; there is no difficulty in verifying that

$$q^{-\frac{1}{24}}f(q)-2\sqrt{\left(\frac{2\pi}{\alpha}\right)}\,q_1^{\frac{4}{3}}\omega(q_1^2)=2\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J_1(\alpha)=\frac{4\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{-\frac{1}{24}}f(-q)+\sqrt{\left(\frac{\pi}{\alpha}\right)}\,q_1^{-\frac{1}{24}}f(-q_1)=2\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J(\alpha)=\frac{2\beta\sqrt{6}}{\pi}\,J(\beta).$$

$$q^{-\frac{1}{24}}\phi(q)-2\sqrt{\left(\frac{\pi}{\alpha}\right)}\,q_1^{-\frac{1}{24}}\psi(q_1)=\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J(\alpha)=\frac{\beta\sqrt{6}}{\pi}\,J(\beta),$$

$$q^{-\frac{1}{24}}\phi(-q)-\sqrt{\left(\frac{2\pi}{\alpha}\right)}\,q_1^{\frac{1}{3}}v(-q_1)=\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J_1(\alpha)=\frac{2\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{-\frac{1}{24}}\psi(-q)-\sqrt{\left(\frac{\pi}{2\alpha}\right)}\,q_1^{\frac{1}{3}}v(q_1)=-\sqrt{\left(\frac{3\alpha}{2\pi}\right)}\,J_1(\alpha)=-\frac{\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{\frac{2}{3}}\omega(q)-\sqrt{\left(\frac{\pi}{4\alpha}\right)}\,q_1^{-\frac{1}{12}}f(q_1^2)=-\sqrt{\left(\frac{3\alpha}{\pi}\right)}\,J_2(\tfrac{1}{2}\alpha)=-\frac{2\beta\sqrt{3}}{\pi}\,J_1(2\beta).$$

The transformation formula for $\omega(-q)$ is a little more troublesome; we need the two relations

$$f(q^8)+2q\,\omega(q)+2q^3\omega(-q^4)=\vartheta_3(0,q)\,\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n-2})^{-2},$$

$$f(q^8)+q\,\omega(q)-q\omega(-q)=\vartheta_3(0,q^4)\,\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n-2})^{-2}.$$

* This property is established by the method given by G. N. Watson, *Compositio Math.*, 1 (1934), 39–68 (64–65). It is the fact that these expansions are asymptotic (and not terminating series) which shows that mock ϑ-functions are of a more complex character than ordinary ϑ-functions.

From these relations we have

$$q_1^{\frac{8}{3}}\omega(-q_1^4)=\tfrac{1}{2}q_1^{-\frac{1}{3}}\vartheta_3(0,q_1)\,\vartheta_3^2(0,q_1^2)\prod_{n=1}^{\infty}(1-q_1^{4n})^{-2}-\tfrac{1}{2}q_1^{-\frac{1}{3}}f(q_1^8)-q_1^{\frac{2}{3}}\omega(q_1)$$

$$=\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{-\frac{1}{12}}\vartheta_3(0,q)\,\vartheta_3^2(0,q^{\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)^{-2}$$

$$-\left[\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{\frac{1}{6}}\omega(q^{\frac{1}{4}})+4\sqrt{\left(\frac{3\beta}{\pi}\right)}\,J_1(8\beta)\right]$$

$$-\left[\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{-\frac{1}{12}}f(q^2)-\sqrt{\left(\frac{3\beta}{\pi}\right)}\,J_2(\tfrac{1}{2}\beta)\right]$$

$$=-\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{\frac{1}{6}}\omega(-q^{\frac{1}{4}})+\sqrt{\left(\frac{3\beta}{\pi}\right)}\,[J_2(\tfrac{1}{2}\beta)-4J_1(8\beta)].$$

Hence, replacing q_1 by $q^{\frac{1}{4}}$, we get, as the last of the required transformations,

$$q^{\frac{2}{3}}\omega(-q)+\sqrt{\left(\frac{\pi}{\alpha}\right)}\,q_1^{\frac{2}{3}}\omega(-q_1)=2\sqrt{\left(\frac{3\alpha}{\pi}\right)}\,J_3(\alpha),$$

where

$$J_3(\alpha)=\tfrac{1}{4}J_2(\tfrac{1}{8}\alpha)-J_1(2\alpha)$$

$$=\tfrac{1}{4}\int_0^\infty e^{-\frac{1}{16}\alpha y^2}\frac{\cosh\frac{1}{8}\alpha y}{\cosh\frac{3}{8}\alpha y}\,dy-\int_0^\infty e^{-3\alpha x^2}\frac{\sinh 2\alpha x}{\sinh 3\alpha x}\,dx$$

$$=\int_0^\infty e^{-3\alpha x^2}\left\{\frac{\cosh\frac{1}{2}\alpha x}{\cosh\frac{3}{2}\alpha x}-\frac{\sinh 2\alpha x}{\sinh 3\alpha x}\right\}dx,$$

i.e.

$$J_3(\alpha)=\int_0^\infty e^{-3\alpha x^2}\frac{\sinh\alpha x}{\sinh 3\alpha x}\,dx.$$

It is easy to prove that

$$J_3(\beta)=\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}J_3(\alpha),$$

and that $J_3(\alpha)$ possesses the asymptotic expansion

$$J_3(\alpha)=\tfrac{1}{6}\sqrt{\left(\frac{\pi}{3\alpha}\right)}\,[1-\tfrac{2}{9}\alpha+\tfrac{17}{108}\alpha^2-\ldots],$$

for small values of α, this expansion having the same general properties as the asymptotic expansions previously obtained.

Now that I have no more to say about the functions of order 3, I conclude with a brief mention of the functions of orders 5 and 7. The basic hypergeometric series which has been used hitherto is of no avail for these functions, and other means must be sought to establish Ramanujan's relations which connect functions of order 5. After spending a fortnight on fruitless attempts, I proceeded to attack the problem by the most elementary methods available, namely applications of Euler's formulae mingled with rearrangements of repeated series; and within the day I had proved not only the five relations set out by Ramanujan but also five other relations whose existence he had merely stated. My proofs of these relations are all so long that I took the trouble to analyse one of the longest in the hope of being able to say that it involved "thirty-nine steps"; it was, however, disappointing to a student of John Buchan to find that a moderately liberal count revealed only twenty-four.

The functions of order 7 seem to possess fewer features of interest, though a study of their behaviour near the unit circle by the process of estimating the sum of those terms of the series by which they are defined which are in the neighbourhood of the greatest terms has raised one question for which it was fascinating to seek the answer.

The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, "on a la chair de poule". I would express my own attitude with more prolixity by saying that such a formula as

$$\int_0^\infty e^{-3\pi x^2} \frac{\sinh \pi x}{\sinh 3\pi x}\, dx = \frac{1}{e^{\frac{2}{3}\pi}\sqrt{3}} \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2 (1+e^{-3\pi})^2 \ldots (1+e^{-(2n+1)\pi})^2}$$

gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing "Day", "Night", "Evening", and "Dawn" which Michelangelo has set over the tombs of Giuliano de' Medici and Lorenzo de' Medici.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where

"Pale, beyond porch and portal,
  Crowned with calm leaves, she stands
Who gathers all things mortal
  With cold immortal hands".

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RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1935—JUNE, 1936.

Thursday, 16 *January*, 1936.

Prof. G. B. JEFFERY, President, in the Chair.

There were present about one hundred and thirty members and visitors.

The minutes of the last meeting were read and confirmed.

Messrs. H. A. Heilbronn and G. J. Whitrow and Dr. O. Taussky were elected members of the Society.

Prof. H. Levy opened a discussion on Probability.

Dr. P. Dienes, Dr. H. Jeffreys, Sir Arthur Eddington, Prof. J. B. S. Haldane, and Dr. L. Isserlis also addressed the meeting.

The following papers were taken as read:—

On the product of two Legendre polynomials with different arguments: W. N. Bailey.
Some uniqueness theorems: M. L. Cartwright.
Eddington's probability problem: H. W. Chapman.
On the zeros of certain Dirichlet series: H. Davenport and H. Heilbronn.
(i) Two canonical forms for a net of quadric surfaces; (ii) The net of quadric surfaces associated with a pair of Möbius tetrads: W. L. Edge.
Class numbers of binary quadratic forms: H. Gupta.
Infinitesimal deformations of an L_m in an L_n: H. A. Hayden.