

BIJECTIVE PROOFS OF A THEOREM OF FINE AND RELATED PARTITION IDENTITIES

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Abstract. In this paper, we prove a theorem of Fine bijectively. Stacks with summits and gradual stacks with summits are also discussed.

1. INTRODUCTION

The two main combinatorial methods we can employ in the partition theory are involutions and bijections, which are two sides of a coin in many cases. Existence of an involutive proof connotes existence of a bijective proof and vice versa. In 2003, C. Bessenrodt and I. Pak wrote a very beautiful paper [3], where they discussed the nature of involutive proofs and proved interesting identities. In this paper, we focus on bijective proofs of partition identities.

As a corollary of their main theorem [3], Bessenrodt and Pak obtained one of the partition theorems proposed by Fine in his short note [4]. In Section 2, we will present a bijective proof of Fine's theorem and related identities. G. E. Andrews defined stacks with summits and gradual stacks with summits in [2] studying some identities from Ramanujan's Lost Notebook. The generating function for stacks with summits is related to Fine's theorem, so we will discuss stacks with summits in Section 2.

The idea of the proofs of the Fine theorem and related identities can be naturally generalized. In Section 3, we will discuss a generalization, which is a bijective proof of the main theorem of Bessenrodt and Pak.

Although the identity for the generating function of gradual stacks [2] is not derived from the generalization in Section 3, it has a nice bijective proof. We will establish the proof in Section 4. We can generalize the concept of gradual stacks with summits as much as we do with stacks with summits. We will prove the generating function for generalized gradual stacks with summits in the last section.

In the sequel, we assume that $|q| < 1$ and use the customary notation for q -series

$$\begin{aligned}(a; q)_0 &:= 1, \\ (a; q)_\infty &:= \prod_{k=0}^{\infty} (1 - aq^k), \\ (a; q)_n &:= \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad \text{for any } n.\end{aligned}$$

For a given partition π of n , we denote the sum of all parts of π by $|\pi|$, the largest part by $m(\pi)$, the smallest part by $s(\pi)$, and the number of parts of π by $\ell(\pi)$.

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For every pair of partitions, we define their sum and union. The sum $\lambda + \mu$ is the partition whose parts are $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$ and $\lambda \cup \mu$ is the partition with parts $\lambda_1, \lambda_2, \dots, \mu_1, \mu_2, \dots$ arranged in weakly decreasing order.

Throughout this paper, we denote the partition with no parts by ϵ and let

$$\begin{aligned} \mathcal{P} &= \{ \text{all partitions } \pi \}, & \mathcal{D} &= \{ \text{all partitions } \pi \text{ into distinct parts} \}, \\ \mathcal{P}_n &= \{ \text{partitions } \pi \mid m(\pi) = n \}, & \mathcal{D}_n &= \{ \text{partitions } \pi \text{ into distinct parts} \mid m(\pi) = n \}, \\ \mathcal{P}_{\leq n} &= \{ \text{partitions } \pi \mid m(\pi) \leq n \}, & \mathcal{D}_{\leq n} &= \{ \text{partitions } \pi \text{ into distinct parts} \mid m(\pi) \leq n \}, \\ \mathcal{P}_{\geq n} &= \{ \text{partitions } \pi \mid s(\pi) \geq n \}, & \mathcal{D}_{\geq n} &= \{ \text{partitions } \pi \text{ into distinct parts} \mid s(\pi) \geq n \}, \end{aligned}$$

for any $n \geq 0$. Define $\tau_0 = \epsilon$ and $\tau_n = (n, n-1, \dots, 1)$, a partition of $n(n+1)/2$. Let

$$\Delta = \{ \tau_n \mid n = 0, 1, 2, \dots \}.$$

2. FINE'S THEOREM, STACKS WITH SUMMITS, AND RELATED IDENTITIES

In [5], Fine derived the following identity

$$\sum_{n=0}^{\infty} \frac{q^n}{(bq; q)_n (q; q)_n} = \frac{1}{(bq; q)_{\infty} (q; q)_{\infty}} \sum_{n=0}^{\infty} (-b)^n q^{(n^2+n)/2}, \quad (2.1)$$

which can be used to derive Gauss's well-known identity

$$\frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \sum_{n=0}^{\infty} q^{(n^2+n)/2}$$

and Jacobi's triple product identity

$$(zq; q)_{\infty} (z^{-1}; q)_{\infty} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-z)^n q^{(n^2+n)/2}$$

for $z \neq 0$. E. M. Wright [8] gave the first combinatorial proof of Jacobi's triple product identity. In this section, we will present a combinatorial proof of (2.1), which is very similar to Wright's proof.

To prove (2.1), we need to put it in another form. By multiplying both sides of (2.1) by $(bq; q)_{\infty}$ and replacing b by $-b$, we obtain

$$\sum_{n=0}^{\infty} \frac{q^n (-bq^{n+1}; q)_{\infty}}{(q; q)_n} = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} b^n q^{(n^2+n)/2}, \quad (2.2)$$

which gives the following theorem.

Theorem 2.1. *We have*

$$\bigcup_{n=0}^{\infty} \mathcal{D}_{\geq n+1} \times \mathcal{P}_n = \Delta \times \mathcal{P}.$$

Proof. Let (λ, μ) be a pair of partitions in $\mathcal{D}_{\geq n+1} \times \mathcal{P}_n$ for some n . Since

$$\lambda_1 > \lambda_2 > \dots > \lambda_{\ell(\lambda)} > n,$$

we can subtract $n - i + 1$ from each λ_i to obtain a partition λ^* in $\mathcal{P}_{\geq n}$. Thus we can write λ as $\tau_n + \lambda^*$, where $s(\lambda^*) \geq n$. We now take $\lambda^* \cup \mu$, which is an ordinary partition. Thus there is one to one correspondence between (λ, μ) and $(\tau_n, \lambda^* \cup \mu)$. \square

We denote by φ the map defined in the proof. Figure 1 illustrates φ with an example.

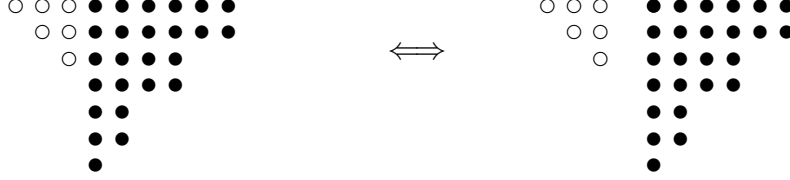


FIGURE 1. $n = 4$, $((9, 8, 5), (4, 2, 2, 1)) \leftrightarrow ((3, 2, 1), (6, 6, 4, 4, 2, 2, 1))$

In [2], Andrews defined stacks with summits and discussed the generating function. His definition of a stack with summit of size k can be associated with a pair of partitions $(\mu, \nu) \in \mathcal{S}$ such that $|\mu| + |\nu| = k$, where

$$\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{P}_n \times \mathcal{P}_{\leq n}.$$

In this paper, for convenience, we regard stacks with summits as pairs of partitions $(\mu, \nu) \in \mathcal{S}$. Let $\sigma\sigma(k)$ be the number of stacks with summits of size k . Then

$$\sum_{k=0}^{\infty} \sigma\sigma(k) q^k = \sum_{j=0}^{\infty} \frac{q^j}{(q; q)_j^2}. \quad (2.3)$$

In [2], Andrews gave the formula

$$\sum_{k=0}^{\infty} \sigma\sigma(k) q^k = \sum_{j=0}^{\infty} \frac{q^j}{(q; q)_j^2} = \frac{1}{(q; q)_{\infty}^2} \sum_{k=0}^{\infty} (-1)^k q^{(k^2+k)/2}, \quad (2.4)$$

which is a special case when $b = 1$ in (2.1). Thus the left side of (2.1) generates stacks with summits, where the power of b equals $\ell(\nu)$. Meanwhile, we see from the map φ that on the right side of (2.1),

$$\frac{1}{(q; q)_{\infty}} \sum_{k=0}^{\infty} (-b)^k q^{(k^2+k)/2}$$

generates $\mathcal{D}_{\geq n+1} \times \mathcal{P}_n$ for any $n \geq 0$. Thus the right side of (2.1) generates triples of partitions $(\lambda, \mu, \nu) \in \mathcal{D}_{\geq n+1} \times \mathcal{P}_n \times \mathcal{P}$ for any n , where the power of b equals $\ell(\lambda) + \ell(\nu)$.

Corollary 2.2. *We have*

$$\sum_{k=0}^{\infty} \sigma\sigma(k) q^k = \sum_{n=0}^{\infty} \sum_{(\lambda, \mu, \nu)} (-1)^{\ell(\lambda)} q^{|\lambda| + |\mu| + |\nu|},$$

where $(\lambda, \mu, \nu) \in \mathcal{D}_{\geq n+1} \times \mathcal{P}_n \times \mathcal{P}$.

Proof. Let

$$T = \bigcup_{n=0}^{\infty} \mathcal{D}_{\geq n+1} \times \mathcal{P}_n \times \mathcal{P}.$$

We define a sign reversing involution ϕ on T under which a subset equinumerous with \mathcal{S} is invariant.

For a given $(\lambda, \mu, \nu) \in T$, let r be the largest number such that $\nu_r > m(\mu)$ if any; otherwise, let $r = 0$ and define $\nu_0 = \infty$. If $\lambda \neq \epsilon$ and $s(\lambda) \leq \nu_r$, then we move $s(\lambda)$ to ν ; if $\lambda = \epsilon$ and $r > 0$, or if $s(\lambda) > \nu_r$, we move ν_r to λ . This process is clearly invertible and changes the number of parts of λ , which changes the sign in the generating function for T . Thus there is cancellation in T so that there remains (ϵ, μ, ν) , where $m(\mu) \geq m(\nu)$. In other words, after cancellation, there remain only pairs of partitions that give stacks with summits. \square

Figure 2 illustrates the process in the proof of Corollary 2.2 with an example.

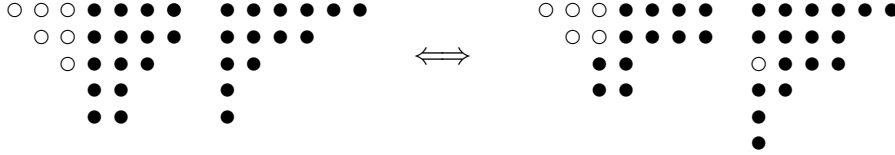


FIGURE 2. $n = 3, ((7, 6, 4), (2, 2), (6, 4, 2, 1, 1)) \leftrightarrow n = 2, ((7, 6), (2, 2), (6, 4, 4, 2, 1, 1))$

By multiplying both sides of (2.2) by $b(q; q)_{\infty}$, we obtain

$$\sum_{n=0}^{\infty} bq^n(q^{n+1}; q)_{\infty}(-bq^{n+1}; q)_{\infty} = \sum_{n=0}^{\infty} b^{n+1}q^{(n^2+n)/2}, \quad (2.5)$$

from which we can derive

$$\sum_{n=1}^{\infty} -q^{2n-1}(q^{2n}; q)_{\infty} = \sum_{n=1}^{\infty} (-1)^n q^{n^2} \quad (2.6)$$

by replacing q , b , and n by q^2 , $-q^{-1}$, and $n - 1$, respectively. Identity (2.6) has a very nice combinatorial interpretation, which was first mentioned by N. J. Fine in his short note [4].

Theorem 2.3 (Fine [4]). *Let $L(n)$ be the number of partitions of n into distinct parts with the minimum part being odd. For $n \geq 1$, $L(n)$ is odd if and only if n is a square.*

Theorem 2.3 can be analytically proved from theorems in the book [5], as indicated in [4]. However, since equation (2.2) has a nice combinatorial proof, we can devise a combinatorial proof of (2.5), which leads to a combinatorial proof of (2.6). In [3], Bessenrodt and Pak combinatorially proved a more general case of (2.6) and obtained (2.6) as a corollary.

In (2.1), by replacing q and b by q^2 and q^{-1} , respectively, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q; q)_{2n}} = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{n^2}. \quad (2.7)$$

As Fine indicated in [5], the left side of (2.7) generates partitions into an even number of parts. Thus, we obtain

$$p_E(n) = p(n) - p(n - 1^2) + p(n - 2^2) - p(n - 3^2) + \cdots, \quad (2.8)$$

where $p_E(n)$ and $p(n)$ denote the number of partitions into an even number of parts and the number of partitions, respectively. By observing the proof of (2.1), we can show (2.8) using the principle of inclusion and exclusion.

3. BIJECTIVE ASPECT OF A RESULT OF BESSENRODT AND PAK

Bessenrodt and Pak [3] generalized their method for Theorem 2.3. In this section, we prove their results by generalizing the bijective method we used in the previous section. We follow the notation defined by Bessenrodt and Pak in [3]. Let $A = (a_0, a_1, a_2, \dots)$ be an infinite integer sequence, define $P(A)$ to be the set of partitions satisfying

$$\lambda_i - \lambda_{i+1} \geq a_{\ell-i} \quad (3.1)$$

for all $i = 1, \dots, \ell$, where $\ell = \ell(\lambda)$, and $\lambda_{\ell+1} = 0$, and define

$$h_k(A) = a_{k-1} + 2a_{k-2} + \cdots + ka_0$$

for any $k \geq 1$. For convenience, set $s(\epsilon) = \infty$, where ϵ is the partition with no parts.

Given $A = (a_0, a_1, a_2, \dots)$, define $T(A)$ to be the subset of $P(A) \times \mathcal{P}$ such that $s(\lambda) - m(\mu) \geq a_0$ for $\lambda \in P(A)$ and $\mu \in \mathcal{P}$. Then the generating function for $T(A)$ is

$$\sum_{(\lambda, \mu) \in T(A)} z^{\ell(\lambda)} q^{|\lambda|+|\mu|} = \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k} \sum_{\nu \in P(A)} (zq^k)^{\ell(\nu)} q^{|\nu|}, \quad (3.2)$$

where we obtain the second summation by subtracting k from each part of λ when $m(\mu) = k$.

Theorem 3.1. *Given $A = (a_0, a_1, a_2, \dots)$, we have*

$$\sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k} \sum_{\nu \in P(A)} (zq^k)^{\ell(\nu)} q^{|\nu|} = \frac{1}{(q; q)_{\infty}} \sum_{k=0}^{\infty} z^k q^{h_k(A)}. \quad (3.3)$$

Proof. By (3.2), we need only show that $(\lambda, \mu) \in T(A)$ is generated by the right hand side of (3.3). Let $\alpha_j = a_0 + a_1 + a_2 + \cdots + a_{j-1}$ for any $j \geq 1$. Given $(\lambda, \mu) \in T(A)$ with $k = \ell(\lambda)$, since $\lambda \in P(A)$, by (3.1) we can write λ as $\alpha + \lambda^*$, where $\alpha = (\alpha_k, \alpha_{k-1}, \dots, \alpha_1)$ and λ^* is an ordinary partition. Since $\lambda_{\ell} \geq a_0 + \mu_1$, we see that $\lambda_1^* \geq \mu_1$. We take $\lambda^* \cup \mu$, which is an ordinary partition. Since $h_k(A) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, we see that $(\alpha, \lambda^* \cup \mu)$ is generated by the right hand side of (3.3) as desired. \square

When $A = (1, 1, 1, \dots)$, we obtain (2.2). We multiply both sides of (3.3) by $(q; q)_\infty$ to obtain

$$\sum_{k=0}^{\infty} q^k (q^{k+1}; q)_\infty \sum_{\nu \in P(A)} (zq^k)^{\ell(\nu)} q^{|\nu|} = \sum_{k=0}^{\infty} z^k q^{h_k(A)},$$

which is equivalent to Theorem 1 in [3].

4. GRADUAL STACKS WITH SUMMITS

Andrews [2] defined gradual stacks with summits of size k , which can be associated with triples of partitions $(\lambda, \pi, \tau_n) \in G$ such that $|\lambda| + |\pi| + |\tau_n| = k$, where

$$G = \bigcup_{n=0}^{\infty} \mathcal{P}_{\leq n} \times \mathcal{P}_{\leq n} \times \{\tau_n\}.$$

Let $g\sigma(k)$ be the number of gradual stacks with summits of size k . Then

$$\sum_{k=0}^{\infty} g\sigma(k) q^k = \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2}}{(q; q)_j^2}.$$

In this section, we will show combinatorially that

$$\sum_{k=0}^{\infty} g\sigma(k) q^k = \frac{1}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q^2; q^2)_n}, \quad (4.1)$$

which was asserted by Ramanujan [2] and proved by Watson [7].

We denote by \mathcal{DE} the set of partitions σ into an even number of distinct parts such that $\sigma_{2i-1} = \sigma_{2i} + 1$ for $i = 1, \dots, \ell(\sigma)/2$. In the following lemma, we show that the summation on the right hand side of (4.1) generates partitions in \mathcal{DE} .

Lemma 4.1. *We have*

$$\sum_{\sigma \in \mathcal{DE}} q^{|\sigma|} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q^2; q^2)_n}.$$

Proof. For a $\sigma \in \mathcal{DE}$, let $\ell(\sigma) = 2n$. Since the parts of σ are distinct, we can subtract $2n - i + 1$ from each σ_i to obtain a partition σ^* . Thus we can write σ as $\tau_{2n} + \sigma^*$. Since $\sigma_{2i-1} = \sigma_{2i} + 1$, $\sigma_{2i-1}^* = \sigma_{2i}^*$, from which it follows that the generating function of (τ_{2n}, σ^*) is the right hand side. This completes the proof. \square

For a given integer $k \geq 0$, the Durfee rectangle of a partition with width minus height equal to k is defined by the largest rectangle $d \times (k + d)$ that fits inside the Ferrers graph of the partition. We call the Durfee rectangle with width minus height equal to k the k -Durfee rectangle. The 0-Durfee rectangle becomes the Durfee square (see [1]).

We now prove (4.1) by establishing a bijection between G and $\mathcal{P} \times \mathcal{DE}$.

Theorem 4.2. *There is a one-to-one correspondence between gradual stacks with summits of size k and pairs of partitions (μ, σ) , where $|\mu| + |\sigma| = k$, $\mu \in \mathcal{P}$ and $\sigma \in \mathcal{DE}$.*

Proof. We define a bijection from G to \mathcal{DE} . For a given $(\lambda, \pi, \tau_n) \in G$, we take the conjugate π' of π . If $\ell(\pi') < n$, we add 0 to π' as parts so that π' has n nonnegative parts. We rearrange the parts of π' as follows. If an integer k appears an odd number of times in π' , we take a part of size k from π' . We then list the taken parts in decreasing order. Let r be the number of the remaining parts. Note that r is even. The remaining parts in π' are then inserted in weakly increasing order before the aforementioned strictly decreasing parts. We call the resulting sequence π^* . For instance, we get $\pi^* = (1, 1, 3, 3, 6, 6, 6, 4, 2, 1)$ from $\pi' = (6, 6, 6, 4, 3, 3, 2, 1, 1, 1)$. It is clear that the algorithm is invertible. We define $\sigma = (r + \pi_r^*, \dots, 1 + \pi_1^*)$, which is indeed a partition into distinct parts since $\pi_r^* \geq \dots \geq \pi_1^*$. Furthermore, since $\pi_{2i-1}^* = \pi_{2i}^*$, we see that σ is a partition into an even number of parts, where $\sigma_{2i-1} = \sigma_{2i} + 1$ for $j = 1, \dots, r/2$, i.e., $\sigma \in \mathcal{DE}$. We now define $\mu = (r + 1 + \pi_{r+1}^*, \dots, n + \pi_n^*)$. Since the last $n - r$ parts of π^* are distinct and strictly decreasing, we see that μ_i are weakly decreasing. Thus μ is a partition into parts greater than or equal to n . Since $\mu \in \mathcal{P}_{\geq n}$ and $\lambda \in \mathcal{P}_{\leq n}$, the union $\mu \cup \lambda$ is the partition $(\mu_1, \dots, \mu_{n-r}, \lambda_1, \dots, \lambda_{\ell(\lambda)})$ and the r -Durfee rectangle has size $(n - r) \times n$.

We now show that the mapping from G to $\mathcal{P} \times \mathcal{DE}$ is invertible. Given (ν, σ) with $\nu \in \mathcal{P}$ and $\sigma \in \mathcal{DE}$, let r be the number of parts of σ . We take the r -Durfee rectangle of ν , and we let λ be the partition consisting of the parts below the Durfee rectangle and μ be the partition consisting of the remaining parts. Then we see that $\lambda \in \mathcal{P}_{\leq n}$ and $\mu \in \mathcal{P}_{\geq n}$, where n is the width of the r -Durfee rectangle. Since each partition has a unique r -Durfee rectangle, λ and μ are uniquely determined. Given (μ, σ) with $\mu \in \mathcal{P}_{\geq n}$ and $\sigma \in \mathcal{DE}$, we can return to π^* and τ_n by subtracting $r + i$ from each part μ_i and $r - i + 1$ from σ . Thus the whole process is invertible, which proves (4.1). \square

We present the process for gradual stacks graphically as follows. With the parts of τ_n , we place j dots in row j from right to left for $j = 1, 2, \dots, n$ by forming a triangle. Then, we place λ_i dots in a row below the triangle from the left to right and place π_i dots in a column to the right of the triangle. For instance, the first picture in Figure 3 is the graphical representation of the gradual stack $((4, 3, 3, 1), (5, 5, 4, 4, 4, 1), (5, 4, 3, 2, 1))$. Take the conjugate of $(5, 5, 4, 4, 4, 1)$, which is $(6, 5, 5, 5, 2)$. By putting the first two 5's before 6, we obtain $(5, 5, 6, 5, 2)$, which constitutes the dots to the right of the triangle in the second picture in Figure 3, where the taken parts 5 and 5 are placed in the first two rows, because we are going to add them to the smallest two parts of the triangle. Then we separate the first two rows from the other rows, which are in the third picture in Figure 3.

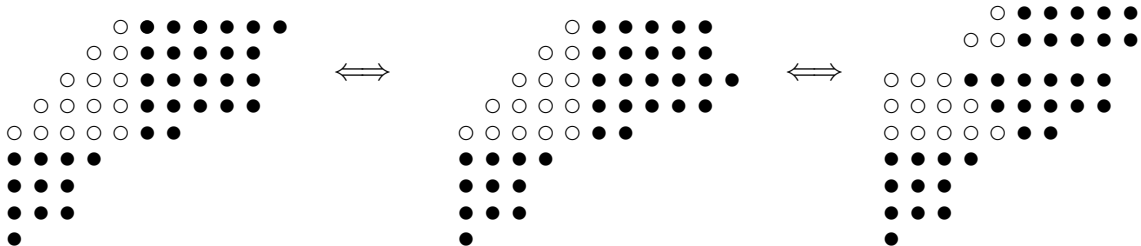


FIGURE 3. $n = 5$, $((4, 3, 3, 1), (5, 5, 4, 4, 4, 1), (5, 4, 3, 2, 1)) \leftrightarrow ((9, 9, 7, 4, 3, 3, 1), (7, 6))$

5. A GENERALIZATION OF GRADUAL STACKS WITH SUMMITS

Let $B = (b_1, b_2, b_3, \dots)$ be an infinite nonnegative integer sequence and define

$$B_n = (b_1 + \dots + b_n, b_1 + \dots + b_{n-1}, \dots, b_1), \quad h_n(B) = b_n + 2b_{n-1} + \dots + nb_1$$

for any $n \geq 1$. We define B -gradual stacks with summits of size k by $(\lambda, \pi, B_n) \in \mathcal{P}_{\leq b_1 + \dots + b_n} \times \mathcal{P}_{\leq n} \times \{B_n\}$, where $|\lambda| + |\pi| + |B_n| = k$. Let

$$G(B) = \bigcup_{n=0}^{\infty} \mathcal{P}_{\leq b_1 + \dots + b_n} \times \mathcal{P}_{\leq n} \times \{B_n\}.$$

Then the generating function for $G(B)$ is

$$\sum_{(\lambda, \pi, B_n) \in G(B)} q^{|\lambda| + |\pi| + |B_n|} = \sum_{j=0}^{\infty} \frac{q^{h_j(B)}}{(q; q)_{b_1 + \dots + b_j} (q; q)_j}.$$

In this section, we show that

$$\sum_{j=0}^{\infty} \frac{q^{h_j(B)}}{(q; q)_{b_1 + \dots + b_j} (q; q)_j} = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{h_{2n}(B)}}{(q^2; q^2)_n}. \quad (5.1)$$

Lemma 5.1. *Let $\mathcal{DE}(B)$ be the set of partitions σ into an even number of parts such that*

$$\sigma_{2i-1} - \sigma_{2i} = b_{\ell(\sigma)-2i+2},$$

$$\sigma_{2i} - \sigma_{2i+1} \geq b_{\ell(\sigma)-2i+1},$$

for $i = 1, 2, \dots, \ell(\sigma)/2$, where $\sigma_{\ell(\sigma)+1} = 0$. Then the generating function is

$$\sum_{\sigma \in \mathcal{DE}(B)} q^{|\sigma|} = \sum_{n=0}^{\infty} \frac{q^{h_{2n}(B)}}{(q^2; q^2)_n}.$$

Proof. For $\sigma \in \mathcal{DE}(B)$, let $\ell(\sigma) = 2n$. Since $\sigma_i - \sigma_{i+1} \geq b_{2n-i+1}$, we see that

$$\sigma_i \geq b_1 + \dots + b_{2n-i+1}.$$

Thus we can write σ as $B_{2n} + \sigma^*$. Since $\sigma_{2i-1} = \sigma_{2i} + b_{2n-2i+2}$, $\sigma_{2i-1}^* = \sigma_{2i}^*$, from which it follows that the generating function of (B_{2n}, σ^*) is the right hand side. This completes the proof. \square

Theorem 5.2. *Given $B = (b_1, b_2, \dots)$, we have*

$$\sum_{(\lambda, \pi, B_n) \in G(B)} q^{|\lambda| + |\pi| + |B_n|} = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{h_{2n}(B)}}{(q^2; q^2)_n}.$$

Proof. By Lemma 5.1, it is sufficient to show that there is one-to-one correspondence between $G(B)$ and $\mathcal{P} \times \mathcal{DE}(B)$. We define a bijection from $G(B)$ to $\mathcal{P} \times \mathcal{DE}(B)$. The bijection will be almost identical with the bijection defined in Section 4 with τ_n replaced by B_n . We omit the details. \square

Theorem 5.2 is identical to (4.1) when $B = (1, 1, 1, \dots)$.

REFERENCES

- [1] G. E. Andrews, *The Theory of Partitions*, Addison–Wesley, Reading, MA, 1976; reissued: Cambridge University Press, Cambridge, 1998.
- [2] G. E. Andrews, *Ramanujan’s Lost Notebook IV. Stacks and Alternating Parity in Partitions*, Adv. Math., **53** (1984), 55–74.
- [3] C. Bessenrodt and I. Pak, *Partition congruences by involutions*, Euro. J. Combin., **25** (2004), 1139–1149.
- [4] N. J. Fine, *Some new results on partitions*, Proc. Nat. Acad. Sci. USA, **34** (1948), 616–618.
- [5] N. J. Fine, *Basic hypergeometric series and applications*, American Mathematical Society, Providence, RI, 1988.
- [6] I. Pak, *On Fine’s partition theorems, Dyson, Andrews, and missed opportunities*, Math. Intelligencer, **25** (2003), 10–16.
- [7] G. N. Watson, *The final problem: An account of the mock-theta functions*, J. London Math. Soc., **11** (1936), 55–80.
- [8] E. M. Wright, *An enumerative proof of an identity of Jacobi*, J. London Math. Soc., **40** (1965), 55–57.

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