RAMANUJAN'S PARTIAL THETA SERIES AND PARITY IN PARTITIONS

AE JA YEE

Dedicated to George Andrews for his 70th birthday

Abstract. A partial theta series identity from Ramanujan's lost notebook has a connection with some parity problems in partitions studied by Andrews in [3], where 15 open problems are listed. In this paper, the partial theta series identity of Ramanujan is revisited and answers to Questions 9 and 10 of Andrews are provided.

1. Introduction

In his recent paper [3], George Andrews investigated a variety of parity questions in partition identities. At the end of the paper, he then listed 15 open problems, two of which have a connection with the following partial theta series from Ramanujan's lost notebook.

Theorem 1.1. [8, p. 28], [4, Entry 1.6.2] *For any complex number a,*

$$1+\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}a^nq^{n(n+1)/2}}{(-aq^2;q^2)_n}=\sum_{n=0}^{\infty}a^nq^{n^2}. \tag{1.1}$$

As customary, here and in the sequel, we employ the standard notation

$$(a;q)_0=1, \qquad (a;q)_n:=(1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n\geq 1,$$

and

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

For a partition λ, we define $I_{UO}(\lambda)$ (resp. $I_{UE}(\lambda)$) by the maximum length of weakly decreasing subsequences of the parts whose terms alternate in parity starting with an odd (resp. even) part.

Example 1. $\lambda=(7,7,5,4,4,3,2,1,1)$. *Then*

$$I_{UO}(\lambda)=5, \quad I_{UE}(\lambda)=4.$$

Let $\delta_o(N,r,m,n)$ (resp. $\delta_e(N,r,m,n)$) denote the number of partitions of n into m distinct parts $\leq N$ with upper odd (resp. even) parity index equal to r. We define

$$D_o(N,y,x;q):=D_o(N)=\sum_{r,m,n\geq 0}\delta_o(N,r,m,n)y^rx^mq^n,$$
$$D_e(N,y,x;q):=D_e(N)=\sum_{r,m,n\geq 0}\delta_e(N,r,m,n)y^rx^mq^n.$$

Then we have the following theorem.

---

[1]Partially supported by National Science Foundation Grant DMS-0801184.

[2]Keywords: Integer partitions, Parity indices in partitions, partial theta series, Ramanujan's lost notebook

[3]2000 AMS Classification Numbers: Primary, 05A17; Secondary, 11P81

Theorem 1.2 (Andrews [3])**.**

$$D_o(\infty)=\sum_{i,j\geq 0}\frac{x^iy^{2j}q^{(i-j)^2+j^2+i+j}}{(-q;q)_i(q)_{2j}(q)_{i-2j}}+\sum_{i,j\geq 0}\frac{x^iy^{2j-1}q^{(i-j)^2+j^2+i-j}}{(-q;q)_i(q)_{2j-1}(q)_{i-2j+1}}, \tag{1.2}$$

$$D_e(\infty)=\sum_{i,j\geq 0}\frac{x^iy^{2j}q^{(i-j)^2+j^2+j}}{(-q;q)_i(q)_{2j}(q)_{i-2j}}+\sum_{i,j\geq 0}\frac{x^iy^{2j+1}q^{(i-j)^2+j^2+3j+1}}{(-q;q)_i(q)_{2j+1}(q)_{i-2j-1}}. \tag{1.3}$$

The ninth and tenth questions of Andrews are as follows.

Question 9. Prove that if $x=-1$ and $y=1$, then the second sum of (1.2) is

$$\sum_{n=1}^{\infty}(-1)^nq^{n^2}.$$

Question 10. Prove that if $x=-1$ and $y=1$, then the first sum of (1.3) is

$$\sum_{n=0}^{\infty}(-1)^nq^{n^2}.$$

The primary purpose of this paper is to provide answers to these two questions of Andrews, which involve Ramanujan's partial theta series (1.1).

In section 2, we give answers to Andrews' questions. Then, a combinatorial proof of Theorem 1.1 will be given in Section 3. Theorem 1.2 will be proved combinatorially in Section 4. In the last section, some remarks are made, in particular, a sketch of a solution to Question 5 of Andrews [3] is given.

2. Open questions 9 and 10 of Andrews

In this section, we will provide answers to his two open questions. We first need the following lemma.

Lemma 2.1. *For any positive integer n,*

$$\sum_{k=0}^{n}q^{(n-k)^2+k^2+n-k}\begin{bmatrix}n\\2k-1\end{bmatrix}_q=\sum_{k=0}^{n}q^{(n-k)^2+k^2+k}\begin{bmatrix}n\\2k\end{bmatrix}_q=(-q;q)_{n-1}q^{n(n+1)/2}, \tag{2.1}$$

where

$$\begin{bmatrix}a\\b\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_a}{(q;q)_b(q;q)_{a-b}}, & \text{if } 0\leq b\leq a,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. We will only show that

$$\sum_{k=0}^{n}q^{(n-k)^2+k^2+k}\begin{bmatrix}n\\2k\end{bmatrix}_q=(-q;q)_{n-1}q^{n(n+1)/2},$$

which is equivalent to

$$\sum_{k=0}^{n}q^{(n-2k)(n-2k-1)/2}\begin{bmatrix}n\\2k\end{bmatrix}_q=(-q;q)_{n-1}.$$

This follows from the recurrences of q-binomial coefficient, namely

$$\begin{aligned}\sum_{k=0}^{n} q^{(n-2k)(n-2k-1)/2} \begin{bmatrix} n \\ 2k \end{bmatrix}_q &= \sum_{k=0}^{n} q^{(n-2k)(n-2k-1)/2} \left(\begin{bmatrix} n-1 \\ n-2k-1 \end{bmatrix}_q + q^{n-2k} \begin{bmatrix} n-1 \\ n-2k \end{bmatrix}_q \right) \\ &= \sum_{k=0}^{n-1} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \\ &= (-q;q)_{n-1}.\end{aligned}$$

□

We now give answers to Questions 9 and 10 in the following theorem.

Theorem 2.2. *We have*

$$\sum_{n,k\geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+n-k}}{(-q;q)_n (q)_{2k-1} (q)_{n-2k+1}} = \sum_{n=1}^{\infty} (-1)^n q^{n^2},$$

$$\sum_{n,k\geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+k}}{(-q;q)_n (q)_{2n} (q)_{n-2k}} = \sum_{n=0}^{\infty} (-1)^n q^{n^2}.$$

Proof. Rewrite the left-hand side of the first identity as

$$\begin{aligned}\sum_{n,k\geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+n-k}}{(-q;q)_n (q)_{2k-1} (q)_{n-2k+1}} &= \sum_{n,k\geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+n-k}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k-1 \end{bmatrix}_q \\ &= \sum_{n\geq 1} \frac{(-1)^n (-q;q)_{n-1} q^{n(n+1)/2}}{(q^2;q^2)_n},\end{aligned}$$

where the second equality follows from Lemma 2.1. Setting $a=-1$ in (1.1), we arrive at

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n (-q;q)_{n-1} q^{n(n+1)/2}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} (-1)^n q^{n^2}.$$

which completes the proof.

Since the proof of the second identity is similar, we omit it. □

3. Combinatorics of Ramanaujan's partial theta series

In this section, we provide a combinatorial proof of the partial theta series identity (1.1) from Ramanujan's lost notebook.

We introduce some terminology needed in the rest of this paper. For a partition λ, $\ell(\lambda)$ denotes the number of positive parts, and we define $\lambda_i = 0$ for $i > \ell(\lambda)$. We also denote by $\emptyset$ the empty partition of 0. For partitions λ and μ, we define the sum $\lambda + \mu$ of λ and μ to be the partition whose ith part is $\lambda_i + \mu_i$. We denote the conjugate of a partition λ by λ'.

Theorem 3.1. *For any positive integer n, the generating function of partitions λ into n distinct parts with $\lambda_i - \lambda_{i+1} \leq 2$ and even upper even parity index is*

$$(-q;q)_{n-1} q^{n(n+1)/2}.$$

Note that the least part of the partitions λ stated in Theorem 3.1 must be 1 since $\lambda_{\ell(\lambda)} \le 2$ and their even upper parity index is even.

Proof. Let $\tau = (n, n-1, \dots, 1)$ and μ be a partition generated by $(-q;q)_{n-1}$. We add each part of μ to τ vertically from the largest part and denote the resulting partition λ. That is,

$$\lambda = \tau + \mu'.$$

Since the distinct parts of μ are added, the adjacent parts of λ differ by at most 2. This process is reversible. Let i be the smallest integer such that $\lambda_i - \lambda_{i+1} = 2$. We then subtract 1 from each of the largest i parts of λ. Repeating this until there is no such i, i.e., the resulting partition is τ.

We now show that the upper even parity index of the λ is $2\lfloor (n-\ell(\mu))/2 \rfloor$. We use induction on the number of parts of μ. If $\mu = \emptyset$, then $\lambda = \tau$, the upper even parity index of which is $2\lfloor n/2 \rfloor$. Suppose that μ has k parts, and let λ be the partition resulting from insertion of all parts but the smallest one of μ. Since the insertion process is performed vertically, we see that for $i < \mu_{k-1}$,

$$\lambda_i - \lambda_{i+1} = 1.$$

We also see that vertical insertion of μ_k changes the parity of only the μ_k largest parts of λ. Thus, the μ_kth part and the (μ_k+1)th part have the same parity, from which we see that the upper even parity index reduces by 1. Furthermore, if $n+k$ is even, the parity index increases by 1; while if $n+k$ is odd, the parity index decreases by 1. Hence, we see that the parity index is

$$\begin{cases} 2\lfloor (n-k+1)/2 \rfloor, & \text{if } n+k = \text{even}, \\ 2\lfloor (n-k+1)/2 \rfloor - 2, & \text{if } n+k = \text{odd}, \end{cases}$$

which is equivalent to

$$2\lfloor (n-k)/2 \rfloor.$$

Therefore, the generating function is

$$(-q;q)_{n-1} q^{n(n+1)/2}.$$

□

Note that a partition with even upper even parity index has odd upper odd parity index, and vise versa. Thus, a similar result on the generating function of partitions into distinct parts and odd upper odd parity index follows from Theorem 3.1. We state this in the following corollary.

Corollary 3.2. *For any positive integer n, the generating function of partitions λ into n distinct parts with $\lambda_i - \lambda_{i+1} \le 2$, $\lambda_n = 1$, and odd upper odd parity index is*

$$(-q;q)_{n-1} q^{n(n+1)/2}.$$

It follows from Theorem 3.1 and Corollary 3.2 that the generating function of partitions into n distinct parts with even (resp. odd) upper even (resp. odd) parity index is

$$\frac{(-q;q)_{n-1} q^{n(n+1)/2}}{(q^2;q^2)_n}.$$

We now recall Theorem 1.1 with a replaced by $-a$.

Theorem 3.3. [8, p. 28], [4, Entry 1.6.2]

$$1+\sum_{n=1}^{\infty}\frac{(-a)^n(-q;q)_{n-1}q^{n(n+1)/2}}{(aq^2;q^2)_n}=\sum_{n=0}^{\infty}(-a)^nq^{n^2}. \tag{3.1}$$

Proof. Let $D_e(E)$ be the set of partitions into distinct parts with even upper even parity index. As noted after Corollary 3.2, the left-hand side of (3.1) generates partitions π in $D_e(E)$. We will prove the theorem by setting up a sign reversing involution.

Let λ be a partition into distinct parts with $\lambda_i - \lambda_{i+1} \le 2$ and even upper even parity index, and let σ be a partition into even parts $\le 2\ell(\lambda)$. Then it follows from the remark made after Corollary 3.2 that

$$\lambda + \sigma' \in D_e(E).$$

For a partition $\pi \in D_e(E)$, by taking out

$$2i\lfloor(\pi_i - \pi_{i+1})/2\rfloor$$

boxes in columns from right to left in the Ferrers graph of π, we can also decompose it into λ and σ, where λ is a partition into distinct parts with $\lambda_i - \lambda_{i+1} \le 2$ and even upper even parity index, and σ is a partition into even parts $\le 2\ell(\lambda)$. Note that it follows from the decomposition of π into λ and σ that π is counted with $(-1)^{\ell(\pi)}a^{w(\pi)}$, where

$$w(\pi) = \sum_{i=1}^{\ell(\pi)} \lceil(\pi_i - \pi_{i+1})/2\rceil = \ell(\lambda) + \ell(\sigma).$$

Let $\ell(\lambda) = n$. By Theorem 3.1, λ can be decomposed uniquely into $\tau = (n, n-1, \dots, 1)$ and a partition μ generated by $(-q;q)_{n-1}$, namely

$$\lambda = \tau + \mu'.$$

If $\mu \ne \emptyset$, then we now consider the sequence s_i,

$$s_i = n + i - 1 + \mu_i,$$

which is weakly decreasing since $\mu_j > \mu_{j+1}$. Also,

$$s_{\ell(\mu)} = n + \ell(\mu) - 1 + \mu_{\ell(\mu)} \ge n + \ell(\mu) = \lambda_1. \tag{3.2}$$

Let e be the least i such that s_e is even. For convenience, we define $s_e = 0$ if there is no such even s_e or $\mu = \emptyset$. Let λ_E be the largest even part of λ. We also define $\lambda_E = 0$ if there is no even part in π. We now compare $m = \max(s_e, \lambda_E)$ and the largest part of σ, namely σ_1.

Case 1: If $\sigma \ne \emptyset$ and $\sigma_1 > m$, then we remove σ_1 from σ and add σ_1 boxes to λ as follows. If $\sigma_1 \le \lambda_1$, then we just add σ_1 to λ as a part. Since $\lambda_E \le m < \sigma_1 \le \lambda_1$, in this case λ_1 is odd. Clearly, the resulting partition λ^* satisfies the part difference condition $\lambda^*_i - \lambda^*_{i+1} \le 2$. If $\sigma_1 > \lambda_1$, then since $\sigma_1 \le 2n$, we see that

$$\ell(\mu) - i = \lambda_1 - n - i < \sigma_1 - n - i \le n - i,$$

from which it follows that

$$\sigma_1 - n - \ell(\mu) > 0 = \mu_{\ell(\mu)+1}.$$

Since $\sigma_1 - n - i$ is a strictly decreasing sequence between 1 and $n-1$, there exists a unique c such that

$$\mu_c < \sigma_1 - n - c \le \mu_{c-1}.$$

We now define

$$\begin{aligned} \mu^* &= (\mu_1+1, \mu_2+1, \ldots, \mu_{c-1}+1, \sigma_1-n-c, \mu_c, \mu_{c+1}, \ldots), \\ \tau^* &= (n+1, n, \ldots, 1), \\ \lambda^* &= \tau^* + (\mu^*)'. \end{aligned} \tag{3.3}$$

Clearly, the parts of μ^* are distinct and less than $n+1$. Thus the adjacent parts of λ^* differ by at most 2. The largest part of the resulting partition σ^* is less than or equal to $2n$.

Case 2: $m \neq 0$ and $\sigma_1 \le m$. In this case, if $m = \lambda_E \neq s_e$, then we remove the part λ_E from λ and denote the resulting partition by λ^*. If $\lambda_1 = \lambda_E$, then any adjacent parts of λ^* still differ by 2. If λ_1 is odd, then by (3.2)

$$\lambda_E > s_e \ge \lambda_1,$$

which is a contradiction. So, $s_e = 0$. That is, $\mu = \emptyset$ or every s_i is odd. If $\mu = \emptyset$, then $\lambda = (n, n-1, \ldots, 1)$, so clearly λ^* satisfies the part difference condition. If every s_i is odd, then

$$\lambda_i - \lambda_{i+1} = 2 \text{ iff } \lambda_i \text{ is odd.}$$

Since $s_1 = n + \mu_1$ is odd, the least two adjacent parts differing by 2 are both odd. Also, since s_2 is odd, the next least two adjacent parts differing by 2 are both odd, and so on. Thus, the part after λ_E has to be odd. Thus any two adjacent parts of λ^* still differ by at most 2. If $m = s_e$, then we subtract s_e boxes from λ as follows. Let

$$\begin{aligned} \mu^* &= (\mu_1-1, \mu_2-1, \ldots, \mu_{e-1}-1, \mu_{e+1}, \mu_{e+2}, \ldots), \\ \tau^* &= (n-1, n-2, \ldots, 1), \\ \lambda^* &= \tau^* + (\mu^*)'. \end{aligned} \tag{3.4}$$

Since the parts of μ are distinct and less than n, the parts of μ^* are distinct and less than $n-1$. Thus the adjacent parts of λ^* differ by at most 2. In either case, we then add m to σ as a part and denote the resulting partition by σ^*. From the definition of m, we see that

$$m = \max(\lambda_E, s_e) \le \max(\lambda_1, s_1) \le 2n-1,$$

so $m \le 2n-2$ since m is even. Thus the largest part of the resulting partition σ^* is less than or equal to $2n-2$.

Let $\pi^* = \lambda^* + {\sigma^*}'$. Indeed,

$$w(\pi) = \ell(\lambda) + \ell(\sigma) = \ell(\lambda^*) + \ell(\sigma^*) = w(\pi^*).$$

It now suffices to show that this is a sign reversing involution under which $\pi = (2n-1, 2n-3, \ldots, 3, 1)$ remains fixed. The generating function of such partitions is the right-hand side of (3.1).

If $\sigma_1 > m$ and $\sigma_1 > \lambda_1$, then $\sigma_1 > s_e$, and by (3.3) we get

$$\begin{aligned} s_i^* &= n+1+i-1+\mu_i^* = n+1+i-1+\mu_i+1 = s_i+2 \quad \text{for } i < c, \\ s_c^* &= n+1+c-1+\mu_c^* = n+1+c-1+\sigma_1-n-c = \sigma_1, \end{aligned}$$

which implies that s_c^* is the largest even number in the sequence since s_i^*'s are weakly decreasing and s_i is odd for $i < e$. So, since $\sigma_1^* \le \sigma_1$, we subtract σ_1 boxes back from λ^* adding to σ^* as defined in (3.4).

If $\sigma \neq \emptyset$ and $m < \sigma_1 \le \lambda_1$, then λ_1 is odd and we added σ_1 to λ as a part. In this case, $s_e = 0$ since $m = \max(s_e, \lambda_E)$ and $s_e \ge \lambda_1$ if $s_e > 0$. Since the adjacent parts of λ differ by at most 2, there exists a unique j such that

$$\lambda_j = \sigma_1 + 1 = \lambda_{j+1} + 2.$$

Then $\lambda^* = (\lambda_1, \ldots, \lambda_j, \sigma_1, \lambda_{j+1}, \ldots, \lambda_n)$. By the definition,

$$(\mu^*)' = (\mu_1' - 1, \ldots, \mu_j' - 1, \mu_j' - 1, \mu_{j+1}', \ldots),$$

which is equivalent to

$$\mu_i^* = \begin{cases} \mu_i + 1, & \text{if } \mu_i > j, \\ \mu_{i+1}, & \text{if } \mu_i < j. \end{cases}$$

We compute

$$\begin{aligned} s_i^* &= n + 1 + i - 1 + \mu_i^* = n + 1 + i - 1 + \mu_i + 1 = s_i + 2 \quad \text{if } \mu_i > j, \\ s_i^* &= n + 1 + i - 1 + \mu_i^* = n + 1 + i - 1 + \mu_{i+1} = s_{i+1} \quad \text{if } \mu_i < j, \end{aligned}$$

from which it follows that s_i^* are all odd. Thus $m = \lambda_E = \sigma_1 \neq s_e^*$. By subtracting σ_1 from λ^* and adding it to σ^*, we recover the original λ and σ.

Therefore, the identity holds true. ☐

4. Upper parity indices in partitions into distinct parts

In this section, we prove Theorem 1.2 combinatorially. We first rewrite the theorem as follows.

$$D_o(\infty) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^n y^{2k} q^{(n-k)^2+k^2+n+k}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^n y^{2k-1} q^{(n-k)^2+k^2+n-k}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k-1 \end{bmatrix}_q, \quad (4.1)$$

$$D_e(\infty) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^n y^{2k} q^{(n-k)^2+k^2+k}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^n y^{2k+1} q^{(n-k)^2+k^2+3k+1}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q. \quad (4.2)$$

In the following theorem, we first show that the inner summation of the first double summations in (4.2) is the generating function of partitions into n distinct parts with even upper even parity index.

Theorem 4.1. *The generating function of partitions into n distinct parts with even upper even parity index is*

$$\sum_{k=0}^{n} \frac{q^{(n-k)^2+k^2+k}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q .$$

Proof. By Theorem 3.1, it suffices to show that

$$\sum_{k=0}^{n} q^{(n-k)^2+k^2+k} \begin{bmatrix} n \\ 2k \end{bmatrix}_q = (-q;q)_{n-1} q^{n(n+1)/2}, \quad (4.3)$$

which is equivalent to

$$\sum_{k=0}^{n} q^{(n-2k)(n-2k-1)/2} \begin{bmatrix} n \\ 2k \end{bmatrix}_q = (-q;q)_{n-1}.$$

This follows from Lemma 2.1. Indeed, one can fully show identity (4.3) in terms of partitions by rearranging the parts of partitions generated by one side to obtain the partitions generated by the other side. We omit the details. □

Similarly, we can show that the generating function of partitions into n distinct parts with odd upper odd parity index is

$$\sum_{k=0}^{n} \frac{q^{(n-k)^2+k^2+n-k}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k-1 \end{bmatrix}_q .$$

For even upper odd parity index (resp. odd upper even parity index), we first take a partition λ with even upper even parity index (resp. odd upper odd parity index). Then the smallest part of λ must be odd. We now add 1 to each of the parts of λ. Then the resulting partition has even upper odd parity index (resp. odd upper even parity index). By Theorem 4.1, we can show that the generating function of partitions into n distinct parts with even upper odd parity index is

$$\sum_{k=0}^{n} \frac{q^{(n-k)^2+k^2+n+k}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q .$$

Similarly, we can show that the generating function of partitions into n distinct parts with odd upper even parity index is

$$\sum_{k=0}^{n} \frac{q^{(n-k)^2+k^2+3k+1}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q .$$

5. Remarks

The first three open questions of Andrews are settled by S. Kim and the author in [6]. The proof of Theorem 1.2 given in Section 4 is essentially the same as that of K. Kursungoz [7].

Question 5 of Andrews follows from Franklin's involution for the Euler pentagonal number theorem [2, pp. 10–11]. We state the question and sketch a solution. For a partition λ, we denote by $I_{LO}(\lambda)$ (resp. I_{LE}) the maximum length of weakly increasing subsequences of the parts whose terms alternate in parity starting with an odd (resp. even) part. Let $p_o(r,m,n)$ be the number of partitions of n into m distinct parts with $I_{LO}=r$. We define

$$P_o(y,x;q) = \sum_{r,m,n\geq 0} p_o(r,m,n) y^r x^m q^n.$$

Then we have the following theorem.

Theorem 5.1 (Andrews [3])**.**

$$P_o(y,x;q) = \sum_{n=0}^{\infty} \frac{x^n y^n q^{n(n+1)/2}(-q/y;q)_n}{(q^2;q^2)_n}.$$

Question 5. It follows from an old formula of Rogers that

$$P_o(-1,1;q) = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}).$$

Prove combinatorially.

Note that $\lambda_1 \equiv I_{LO}(\lambda) \pmod 2$ for a partition λ. Thus,

$$P_o(-1,1;q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}),$$

where the second equality follows from Franklin's involution.

Another combinatorial proof of Theorem 1.1 is given by B. C. Berndt, B. Kim, and the author in [5]. K. Alladi [1] has devised a completely different proof of Theorem 1.1 and has also provided a number-theoretic interpretation of Theorem 1.1 as a weighted partition theorem.

The author thanks the anonymous referee for helpful comments. The author also would like to thank William Y. C. Chen for pointing out an error in my original proof of Theorem 3.3.

References

[1] K. Alladi, *A partial theta identity of Ramanujan and its number theoretic interpretation*, to appear in Ramanujan J.

[2] G. E. Andrews, *The Theory of Partitions*, Addison–Wesley, Reading, MA, 1976; reissued: Cambridge University Press, Cambridge, 1998.

[3] G. E. Andrews, *Parity in partition identities*, to appear in Ramanujan J.

[4] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part* II, Springer, New York, 2009.

[5] B. C. Berndt, B. Kim, and A. J. Yee, *Ramanujan's Lost Notebook: Combinatorial proofs of identities associated with Heine's transformation or partial theta functions*, to appear in J. Combin. Thy. Ser. A.

[6] S. Kim and A. J. Yee, *Göllnitz-Gordon identities and parity questions in partitions*, preprint.

[7] K. Kursungoz, *Parity considerations in Andrews-Gordon identities, and the k-marked Durfee symbols*, Ph.D. Thesis, Penn Sate University.

[8] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

Department of Mathematics, Penn State University, University Park, PA 16802, USA

E-mail address: yee@math.psu.edu