Partition Identities for Ramanujan's Third Order Mock Theta Functions

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Abstract. We find two involutions on partitions that lead to partition identities for Ramanujan's third order mock theta functions $\phi(-q)$ and $\psi(-q)$. We also give an involution for Fine's partition identity on the mock theta function $f(q)$. The two classical identities of Ramanujan on third order mock theta functions are consequences of these partition identities. Our combinatorial constructions also apply to Andrews' generalizations of Ramanujan's identities.

Keywords: mock theta function, Ramanujan's identities, partition identity, Fine's theorem, involution.

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1   Introduction

This paper is concerned with the following three mock theta functions of order 3 defined by Ramanujan,

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1.1}$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \tag{1.2}$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}. \tag{1.3}$$

Mock theta functions have been extensively studied, see, for example, Andrews [8], Fine [17, Chapters 2-3], Gordon and McIntosh [18], and Ono [19]. These functions not only have remarkable analytic properties, but also are closely connected to the theory of partitions, see, for example, Agarwal [1], Andrews [4, 9], Andrews and Garvan [7], Andrews, Eriksson, Petrov, and Romik [10], and Choi and Kim [14].

In this paper, we find two involutions on partitions that imply two partition identities for Ramanujan's third order mock theta functions $\phi(-q)$ and $\psi(-q)$. We also give an involution for

Fine's partition theorem on the mock theta function $f(q)$. These three partition identities lead to the following two identities (1.4) and (1.5) of Ramanujan

$$\phi(-q) - 2\psi(-q) = f(q), \tag{1.4}$$

$$\phi(-q) + 2\psi(-q) = \frac{(q;q)_\infty}{(-q;q)_\infty^2}, \tag{1.5}$$

where we have adopted the standard notation

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.6}$$

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1. \tag{1.7}$$

The first proofs of (1.4) and (1.5) were given by Watson [20]. Fine [17, p. 60] found another proof by using transformation formulas.

Andrews [3] defined the following functions as generalizations of Ramanujan's mock theta functions and he later found that these generalizations were already in Ramanujan's "lost" notebook [6],

$$f(\alpha;q) = \sum_{n=0}^{\infty} \frac{q^{n^2-n}\alpha^n}{(-q;q)_n(-\alpha;q)_n}, \tag{1.8}$$

$$\phi(\alpha;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\alpha q;q^2)_n}, \tag{1.9}$$

$$\psi(\alpha;q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\alpha;q^2)_n}. \tag{1.10}$$

When $\alpha = q$, the above functions reduce to Ramanujan's mock theta functions. Furthermore, Andrews showed these three functions turn out to be mock theta functions for $\alpha = q^r$, where r is any positive integer. More importantly, Ramanujan's identities (1.4) and (1.5) can be extended to the functions $f(\alpha;q)$, $\phi(\alpha;q)$ and $\psi(\alpha;q)$,

$$\phi(-\alpha;-q) - (1+\alpha q^{-1})\psi(-\alpha;-q) = f(\alpha;q), \tag{1.11}$$

$$\phi(-\alpha;-q) + (1+\alpha q^{-1})\psi(-\alpha;-q) = \frac{(q;q)_\infty}{(-q;q)_\infty(-\alpha;q)_\infty}, \tag{1.12}$$

see Andrews [3, p. 78, Eqs. (3a)–(3b)]. Clearly, the above identities (1.11) and (1.12) specialize to (1.4) and (1.5) by setting $\alpha = q$.

The connection between Ramanujan's third order mock theta function and the theory of partitions was first explored by Fine. In [17, p. 55, Chapter 3], he derived the following identity from his transformation formula:

$$f(q) = 1 + \sum_{k\geq 1} \frac{(-1)^{k-1}q^k}{(-q;q)_k} = 1 + \frac{1}{(-q;q)_\infty}\sum_{k\geq 1}(-1)^{k-1}q^k(-q^{k+1};q)_\infty. \tag{1.13}$$

In fact, (1.13) can be easily established from the combinatorial definition (2.21) of $f(q)$. The following partition identity for $f(q)$ can be deduced from (1.13).

Theorem 1.1 (Fine) *Let $p_{do}(n)$ denote the number of partitions of n into distinct parts with the smallest part being odd. Then*

$$(-q;q)_\infty f(q) = 1 + 2\sum_{n\geq 1} p_{do}(n)q^n. \qquad (1.14)$$

We obtain the following two partition identities from our involutions.

Theorem 1.2 *We have*

$$(-q;q)\phi(-q) = 1 + \sum_{n=1}^\infty p_{do}(n)q^n + \sum_{k=1}^\infty (-1)^k q^{k^2}. \qquad (1.15)$$

Theorem 1.3 *We have*

$$2(-q;q)\psi(-q) = -\sum_{n=1}^\infty p_{do}(n)q^n + \sum_{k=1}^\infty (-1)^k q^{k^2}. \qquad (1.16)$$

Since the generating function for $p_{do}(n)$ is easy to compute, it would be interesting to establish the above relations as q-series identities without resort to partitions.

It can be seen that the above partition identities lead to Ramanujan's identities (1.4) and (1.5). It follows from (1.15) and (1.16) that

$$(-q;q)_\infty \phi(-q) - 2(-q;q)_\infty \psi(-q)$$

$$= 1 + \sum_{n=1}^\infty p_{do}(n)q^n + \sum_{k=1}^\infty (-1)^k q^{k^2}$$

$$+ \sum_{n=1}^\infty p_{do}(n)q^n - \sum_{k=1}^\infty (-1)^k q^{k^2}$$

$$= 1 + 2\sum_{n\geq 1} p_{do}(n)q^n$$

$$= (-q;q)_\infty f(q),$$

where the last equality is a consequence of (1.14). So we obtain the identity (1.4) by dividing both sides by $(-q;q)_\infty$.

In view of (1.15) and (1.16), we find

$$(-q;q)_\infty \phi(-q) + 2(-q;q)_\infty \psi(-q)$$

$$= 1 + \sum_{n=1}^\infty p_{do}(n)q^n + \sum_{k=1}^\infty (-1)^k q^{k^2}$$

$$- \sum_{n=1}^\infty p_{do}(n)q^n + \sum_{k=1}^\infty (-1)^k q^{k^2}$$

$$= 1 + 2\sum_{k\geq 1} (-1)^k q^{k^2}$$

$$= \frac{(q;q)_\infty}{(-q;q)_\infty},$$

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where the last equality follows from Gauss' identity

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}.$$ (1.17)

This yields Ramanujan's identity (1.5) after dividing both sides by $(-q;q)_\infty$.

In fact, we can deduce two partition identities for $\phi(-q)$ and $\psi(-q)$ analogous to Fine's identity for $f(q)$ by employing the following partition theorem of Bessenrodt and Pak [12] which extends a theorem of Fine in [16, Theorem 5]. It is worth mentioning that there are other involutions which also imply this partition theorem, see, Berndt, Kim and Yee [11], Chen and Liu [13], and Yee [21, 22].

Theorem 1.4 (Bessenrodt and Pak) *Let $p_{do}^e(n)$ $(p_{do}^o(n))$ denote the number of partitions of n into even (odd) distinct parts with the smallest part being odd. Then*

$$\sum_{n=1}^{\infty}[p_{do}^e(n) - p_{do}^o(n)]q^n = \sum_{k=1}^{\infty}(-1)^k q^{k^2}.$$ (1.18)

In light of Theorem 1.4, we can restate Theorems 1.2 and 1.3 as follows.

Theorem 1.5 *We have*

$$(-q;q)_\infty \phi(-q) = 1 + 2\sum_{n\geq 1} p_{do}^e(n)q^n.$$ (1.19)

Theorem 1.6 *We have*

$$(-q;q)_\infty \psi(-q) = -\sum_{n\geq 1} p_{do}^o(n)q^n.$$ (1.20)

This paper is organized as follows. In Section 2, we provide an involution for Fine's theorem 1.1. In Sections 3 and 4, we give two involutions that lead to partition identities (1.15) and (1.16) for $\phi(-q)$ and $\psi(-q)$. Section 5 is devoted to partition identities for $f(\alpha q; q), \phi(-\alpha q; -q)$, and $\psi(-\alpha q; -q)$ based on our involutions, which imply Andrews' identities (1.11) and (1.12).

2   An involution for Fine's theorem

In this section we give an involution for Fine's partition theorem. Let P denote the set of partitions, and let D denote the set of partitions with distinct parts. The rank of a partition λ, denoted by $r(\lambda)$, is defined as the largest part minus the number of parts, as introduced by Dyson [15]. The empty partition is assumed to have rank zero. A pair of partitions (λ, μ) is called a bipartition of n if $|\lambda| + |\mu| = n$. Fine [17, p.49] found the following combinatorial interpretation for $f(q)$:

$$f(q) = 1 + \sum_{\lambda \in P}(-1)^{r(\lambda)}q^{|\lambda|},$$ (2.21)

from which we can construct an involution to prove Theorem 1.1.

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Proof of Theorem 1.1. For a bipartition $(\lambda, \mu) \in P \times D$, let $s(\mu)$ denote the smallest part of μ, $m(\lambda)$ denote the number of occurrences of the largest part of λ and $l(\lambda)$ denote the number of (positive) parts of λ. Let U be the set of two classes of bipartitions $(\emptyset, \mu)$ and bipartitions (λ, μ), where $\lambda = (1, 1, \ldots, 1)$ and $s(\mu) > l(\lambda)$, and let V be the set of bipartitions $(\lambda, \mu) \in P \times D$ of n except for bipartitions in U. We shall construct an involution Υ on the set V. The following two cases are considered.

(1) If $s(\mu) \leq m(\lambda)$, then delete the smallest part in μ and add 1 to each of the first $s(\mu)$ parts $\lambda_1, \lambda_2, \ldots, \lambda_{s(\mu)}$ of λ.

(2) If $s(\mu) > m(\lambda)$, then subtract 1 from each of the first $m(\lambda)$ parts $\lambda_1, \lambda_2, \ldots, \lambda_{m(\lambda)}$ of λ and add a part of size $m(\lambda)$ to μ.

It is easy to check that the above mapping is an involution. Moreover, Υ changes the parity of the rank of λ in V.

Let (λ, μ) be a bipartition in U. It is easily seen that if $l(\lambda)$ is even, then $r(\lambda)$ is odd. In this case, we obtain a bipartition $(\emptyset, \nu)$ with $s(\nu)$ being even by moving all the parts of λ to μ as a single part which cancels with (λ, μ). In the case that $l(\lambda)$ is odd, we see that $r(\lambda)$ is even. Thus we get bipartition $(\emptyset, \nu)$ with $s(\nu)$ being odd by moving all the parts of λ to μ as a single part. Now, we are left with two types of bipartitions $(\emptyset, \nu)$ such that $s(\nu)$ is odd, which correspond to the right side of (1.14). This completes the proof. ∎

Here is an example. There are eight bipartitions $(\lambda, \mu) \in P \times D$ of 4 with the rank of λ being even,

$$((3), (1)), ((2, 1), (1)), ((1, 1, 1), (1)), ((1), (2, 1)), ((2, 2), \emptyset), ((1), (3)), (\emptyset, (3, 1)), (\emptyset, (4)).$$

Meanwhile, there are six bipartitions $(\lambda, \mu) \in P \times D$ of 4 with the rank of λ being odd,

$$((4), \emptyset), ((3, 1), \emptyset), ((2, 1, 1), \emptyset), ((2), (2)), ((1, 1), (2)), ((1, 1, 1, 1), \emptyset),$$

and there is only one partition of 4 into distinct parts with the smallest part being odd, i.e., $(3, 1)$.

The involution Υ gives the following correspondence:

$$((3), (1)) \leftrightarrow ((4), \emptyset), \quad ((2, 1), (1)) \leftrightarrow ((3, 1), \emptyset), \quad ((1, 1, 1), (1)) \leftrightarrow ((2, 1, 1), \emptyset),$$

$$((1), (2, 1)) \leftrightarrow ((2), (2)), \quad ((2, 2), \emptyset) \leftrightarrow ((1, 1), (2)).$$

For the remaining four bipartitions $((1), (3))$, $(\emptyset, (3, 1))$, $(\emptyset, (4))$, and $((1, 1, 1, 1), \emptyset)$, we can transform $((1), (3))$ to $(\emptyset, (3, 1))$ and transform $(\emptyset, (4))$ to $((1, 1, 1, 1), \emptyset)$.

3   A partition identity for $\phi(-q)$

In this section, we shall prove the partition identity for $\phi(-q)$ as stated in Theorem 1.2. Let us begin with an interpretation of $\phi(-q)$ given by Fine [17, p.49]. Let DO denote the set of partitions with distinct odd parts. Fine showed that

$$\phi(-q) = 1 + \sum_{\lambda \in DO} (-1)^{\frac{\lambda_1 + 1}{2}} q^{|\lambda|}. \tag{3.22}$$

Note that Choi and Kim found another interpretation of $\phi(q)$ in terms of n-color partitions [14, Theorem 3.1].

Hence we have

$$(-q;q)_\infty \phi(-q) = \sum_{\mu \in D} q^{|\mu|} + \sum_{(\lambda,\mu) \in DO \times D} (-1)^{\frac{\lambda_1+1}{2}} q^{|\lambda|+|\mu|}. \tag{3.23}$$

In order to deal with the sum $\sum_{k \geq 1}(-1)^k q^{k^2}$ on the right hand side of (1.15), special attention has to be paid to certain bipartitions $Q_k = (2k-1, 2k-3, \ldots, 1)$, which is a partition of k^2.

Let $s_o(\mu)$ $(s_e(\mu))$ denote the smallest odd (even) part of μ, and let W denote the set of bipartitions (λ, μ) of n in $DO \times D$ except for those of the form $(Q_k, \emptyset)$ and those bipartitions $(\lambda, \mu) = ((1), \mu)$ with $s_o(\mu) + 1 < s_e(\mu)$. Obviously, there is a cancellation between the set of bipartitions $(\lambda, \mu) = ((1), \mu)$ with $s_o(\mu) + 1 < s_e(\mu)$ and the set of partitions into distinct part with the smallest part being even in the first summand of (3.23). Consequently, the remaining partitions for the first summand in (3.23) give the sum $\sum_{n=1}^\infty p_{do}(n)q^n$ in (1.15).

Finally, to prove Theorem 1.2, we are required to construct an involution on W, denoted by Φ, which changes the parity of $(\lambda_1 + 1)/2$.

The involution Φ. For a partition $\lambda \in DO$, let $c(\lambda)$ denote the maximum number of consecutive odd parts of λ starting with the first part. For example, let $\lambda = (11, 9, 7, 3)$, then $c(\lambda) = 3$. The involution Φ consists of two parts.

Part I of Φ. If $2c(\lambda) \geq s_e(\mu)$, then remove the smallest even part from μ, and add 2 to each of the first $s_e(\mu)/2$ parts $\lambda_1, \lambda_2, \cdots, \lambda_{s_e(\mu)/2}$ of λ.

If $2c(\lambda) < s_e(\mu)$ and $\lambda_{c(\lambda)} > 1$, then subtract 2 from each of the first $c(\lambda)$ parts $\lambda_1, \lambda_2, \ldots, \lambda_{c(\lambda)}$ of λ, and add a part of size $2c(\lambda)$ to μ.

It is easy to see the above process is well defined and bipartitions which can not be paired by the involution are those bipartitions (Q_k, μ) where $s_e(\mu) > 2k$. This is the task of the second part of the involution.

Part II of Φ. If $s_e(\mu) > s_o(\mu) + (2k-1)$, then delete the smallest odd part of μ and delete the part $2k-1$ from Q_k. Then add an even part of size $s_o(\mu) + (2k-1)$ to μ.

If $s_e(\mu) \leq s_o(\mu) + (2k-1)$, then split the smallest even part of μ into two parts, one is of size $2k+1$ and the other is of size $s_e(\mu) - (2k+1)$. Observe that $s_e(\mu) - (2k+1)$ is smaller than $s_o(\mu)$. Then add a part of size $2k+1$ to Q_k and add a part of size $s_e(\mu) - (2k+1)$ to μ.

So we have obtained an involution Φ. It is readily seen that this involution changes the parity of $(\lambda_1 + 1)/2$.

For example, when $n = 7$, there are 6 bipartitions $(\lambda, \mu) \in DO \times D$ with $\lambda_1 \equiv 1 \bmod 4$,

$$((5), (2)), ((5,1), (1)), ((1), (6)), ((1), (4,2)), ((1), (3,2,1)), ((1), (5,1)).$$

On the other hand, there are 5 bipartitions $(\lambda, \mu) \in DO \times D$ with $\lambda_1 \equiv 3 \bmod 4$, that is,

$$((7), \emptyset), ((3,1), (2,1)), ((3,1), (3)), ((3), (4)), ((3), (3,1)),$$

and there is only one partition of 7 into distinct parts with the smallest part being even, namely, $(5, 2)$.

The involution Φ gives the following pairs of bipartitions:

$$((5), (2)) \leftrightarrow ((7), \emptyset), \qquad ((5,1), (1)) \leftrightarrow ((3,1), (2,1)), \quad ((1), (6)) \leftrightarrow ((3,1), (3)),$$

$$((1), (4,2)) \leftrightarrow ((3), (4)), \quad ((1), (3,2,1)) \leftrightarrow ((3), (3,1)).$$

For the remaining bipartition $((1), (5,1))$, we can construct a partition into distinct part with smallest part being even, that is, $(5, 2)$.

4 A partition identity for $\psi(-q)$

The aim of this section is to prove the partition identity (1.16) for $\psi(-q)$. There is also a combinatorial interpretation of $\psi(-q)$ given by Fine [17, p.49]. Let OC denote the set of partitions of n into odd parts without gaps. Fine [17, p.49] showed that

$$\psi(-q) = \sum_{\lambda \in OC} (-1)^{l(\lambda)} q^{|\lambda|}. \tag{4.24}$$

Note that Agarwal [1, 2] found two combinatorial interpretations for $\psi(q)$ by using q-difference equations.

Let D^0 denote the set of partitions with distinct parts where the zero part is allowed. So the number of partitions of n in the set D^0 is twice the number of partitions of n in the set D. Let R denote the set of bipartitions $(\lambda, \mu) \in OC \times D^0$ except for those of the form $(Q_k, \emptyset)$ and those bipartitions $(\lambda, \mu) = ((1), \mu)$ with $s_e(\mu) + 1 < s_o(\mu)$. It is clear that these excluded bipartitions correspond to the right hand side of (1.16). In order to prove (1.16), it suffices to construct an involution Ψ on the set R.

The involution Ψ. Let $r_p(\lambda)$ be the largest part of λ which occurs at least twice in λ, where we let $r_p(\lambda) = \infty$ if λ has no repeated parts. The involution Ψ consists of three parts.

Part I of Ψ. If $s_o(\mu) > r_p(\lambda)$, then delete one part of size $r_p(\lambda)$ from λ and add it as a part to μ. On the other hand, if $s_o(\mu) \leq r_p(\lambda)$, then move a part of size $s_o(\mu)$ from μ to λ.

The above process is well defined and the bipartitions not covered by this case are those bipartitions (Q_k, μ) for which $s_o(\mu) > 2k - 1$. We continue to describe the second part of Ψ.

Part II of Ψ. Assume that (Q_k, μ) is a bipartition such that $s_o(\mu) > 2k - 1$. If μ has a zero part, then we get a bipartition (Q_{k-1}, μ^*) where μ^* is obtained from μ by adding a part of size $2k - 1$ and deleting the zero part.

If μ has no zero part and $s_o(\mu) = 2k + 1$, then we get a bipartition (Q_{k+1}, μ^*) where μ^* is obtained from μ by removing a part of size $2k + 1$ and adding a zero part.

It can be seen that the above mapping is well defined except for those bipartitions (Q_k, μ) such that μ has no zero part and $s_o(\mu) > 2k + 1$. Indeed, it is the object of the third part of Ψ to deal with these remaining bipartitions.

Part III of Ψ. If $s_o(\mu) > s_e(\mu) + (2k - 1)$, then delete the smallest even part of μ and delete the part $2k - 1$ from Q_k, and add an odd part of size $s_e(\mu) + (2k - 1)$ to μ.

If $s_o(\mu) \leq s_e(\mu) + (2k - 1)$, then split the smallest odd part of μ into two parts, one is of size $2k + 1$ and the other is of size $s_o(\mu) - (2k + 1)$, which is less than $s_e(\mu)$, add a part of size $2k + 1$ to Q_k and add a part of size $s_o(\mu) - (2k + 1)$ to μ.

It is routine to check that the map Ψ is an involution and it changes the parity of the length of λ.

For example, when $n = 4$ there are six bipartitions $(\lambda, \mu) \in OC \times D^0$ such that $l(\lambda)$ is odd,

$$((1), (3)), ((1,1,1), (1)), ((1,1,1), (1,0)), ((1), (2,1)), ((1), (2,1,0)), ((1), (3,0)).$$

In the other case, there are five bipartitions $(\lambda, \mu) \in OC \times D^0$ such that $l(\lambda)$ is even,

$$((3,1), (0)), ((1,1,1,1), \emptyset), ((1,1,1,1), (0)), ((1,1), (2)), ((1,1), (2,0)),$$

and there is one partition of 4 with distinct parts such that the smallest part is odd, i.e., $(3,1)$.

The involution Ψ is illustrated below:

$$((1), (3)) \leftrightarrow ((3,1), (0)), \qquad ((1,1,1), (1)) \leftrightarrow ((1,1,1,1), \emptyset), \quad ((1), (2,1)) \leftrightarrow ((1,1), (2))$$

$$((1), (2,1,0)) \leftrightarrow ((1,1), (2,0)), \quad ((1,1,1), (1,0)) \leftrightarrow ((1,1,1,1), (0)).$$

For the remaining bipartition $((1), (3,0))$, we can form a partition into distinct parts with smallest part being odd, that is, $(3,1)$.

As another example, the involution Ψ also gives the following correspondence:

$$((9,7,5,3,1), (16,15,8,6,2)) \leftrightarrow ((7,5,3,1), (16,15,11,8,6)).$$

5   Andrews' generalizations

This section is devoted to proofs of Andrews' identities (1.11) and (1.12). First, we give combinatorial interpretations for $f(\alpha q; q), \phi(-\alpha q; -q)$, and $\psi(-\alpha q; -q)$ by extending the arguments of Fine. More precisely, we have the following partition theoretic interpretations.

Theorem 5.1 *We have*

$$f(\alpha q; q) = 1 + \sum_{\lambda \in P} (-1)^{r(\lambda)} \alpha^{\lambda_1} q^{|\lambda|}, \tag{5.25}$$

$$\phi(-\alpha q; -q) = 1 + \sum_{\lambda \in DO} (-1)^{\frac{\lambda_1 + 1}{2}} \alpha^{\frac{\lambda_1 + 1}{2} - l(\lambda)} q^{|\lambda|}, \tag{5.26}$$

$$\psi(-\alpha q; -q) = \sum_{\lambda \in OC} (-1)^{l(\lambda)} \alpha^{l(\lambda) - \frac{\lambda_1 + 1}{2}} q^{|\lambda|}. \tag{5.27}$$

Proof. Recall that

$$f(\alpha q; q) = \sum_{n=0}^{\infty} \frac{q^{n^2} \alpha^n}{(-q; q)_n (-\alpha q; q)_n},$$

it is easy to check that (5.25) follows from the Durfee square dissection of a partition $\lambda \in P$, see Figure 1.

From the definition of $\phi(\alpha; q)$, we see that

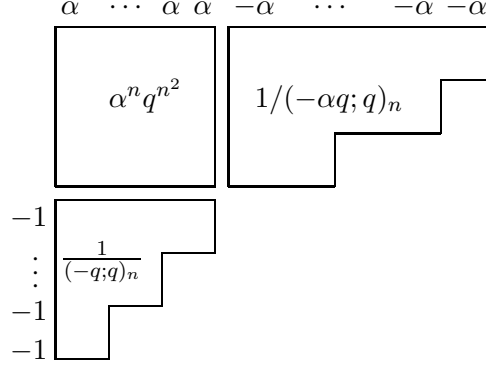$$\phi(-\alpha q; -q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-\alpha q^2; q^2)_n}.$$

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Figure 1: The Durfee square dissection.

The term $(-1)^n q^{n^2}$ corresponds to a partition π of the form $Q_n = (2n - 1, 2n - 3, \ldots, 3, 1)$, which has weight $(-1)^{(\pi_1+1)/2}$. Moreover, $1/(-\alpha q^2; q^2)_n$ is the generating function for partitions σ with at most n even parts and with no odd parts. The weight of such a partition is endowed with weight $(-\alpha)^{\sigma_1/2}$.

Define $\lambda = \pi + \sigma = (\pi_1 + \sigma_1, \pi_2 + \sigma_2, \ldots)$. We see that $\lambda \in DO$, namely, λ is a partition into distinct odd parts. Now, the weight of λ equals $(-1)^{(\lambda_1+1)/2}\alpha^{(\lambda_1+1)/2-l(\lambda)}$. So (5.26) has been verified.

For the combinatorial interpretation for $\psi(-\alpha q; -q)$, we note that

$$\psi(-\alpha q; -q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(-\alpha q; q^2)_n}.$$

The summand can be expanded as follows

$$\frac{(-1)^n q^{n^2}}{(-\alpha q; q^2)_n} = \frac{-q}{1 + \alpha q} \frac{-q^3}{1 + \alpha q^3} \cdots \frac{-q^{2n-1}}{1 + \alpha q^{2n-1}}$$
$$= (-q + \alpha q^{1+1} - \alpha^2 q^{1+1+1} \cdots)(-q^3 + \alpha q^{3+3} - \alpha^2 q^{3+3+3} + \cdots)$$
$$\cdots (-q^{2n-1} + \alpha q^{2(2n-1)} - \alpha^2 q^{3(2n-1)} \cdots).$$

It follows that the summand $(-1)^n q^{n^2}/(-\alpha q; q^2)_n$ is the generating function of partitions λ in OC with the largest part not exceeding $2n - 1$ and with weight $(-1)^{l(\lambda)}\alpha^{l(\lambda)-(\lambda_1+1)/2}$. This proves (5.27). ∎

We can extend Fine's partition theorem to Andrews' function $f(\alpha q; q)$. It can be seen that the involution Υ in Section 2 preserves the quantity $\lambda_1 + l(\mu)$. Therefore, from (5.25) we deduce the following partition theorem.

Theorem 5.2 *Let P_{do} denote the set of partitions into distinct parts with the smallest part being odd. Then*

$$(-\alpha q; q)_{\infty} f(\alpha q; q) = 1 + 2 \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|}. \tag{5.28}$$

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Next, we give a generalization of Theorem 1.2 to $\phi(-\alpha q; -q)$. By the combinatorial interpretation (5.26), we find that

$$(-\alpha q; q)_\infty \phi(-\alpha q; -q) = (-\alpha q; q)_\infty + \sum_{(\lambda,\mu)\in DO\times D} (-1)^{\frac{\lambda_1+1}{2}} \alpha^{\frac{\lambda_1+1}{2}-l(\lambda)+l(\mu)} q^{|\lambda|+|\mu|}. \qquad (5.29)$$

Moreover, we observe that the involution Φ in Section 3 preserves the quantity of

$$l(\mu) - l(\lambda) + \frac{\lambda_1+1}{2}.$$

Hence we have

$$\sum_{(\lambda,\mu)\in DO\times D} (-1)^{\frac{\lambda_1+1}{2}} \alpha^{\frac{\lambda_1+1}{2}-l(\lambda)+l(\mu)} q^{|\lambda|+|\mu|} = - \sum_{\nu\in P_{de}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}, \qquad (5.30)$$

where P_{de} denotes the set of partitions into distinct parts with the smallest part being even. On the other hand,

$$(-\alpha q; q)_\infty = 1 + \sum_{\nu\in P_{de}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{\nu\in P_{do}} \alpha^{l(\nu)} q^{|\nu|}, \qquad (5.31)$$

Therefore, from (5.30) and (5.31) we deduce the following partition identity for $\phi(-\alpha q; -q)$.

Theorem 5.3 *We have*

$$(-\alpha q; q)_\infty \phi(-\alpha q; -q) = 1 + \sum_{\nu\in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}. \qquad (5.32)$$

We now proceed to give a generalization of Theorem 1.3 to $\psi(-\alpha q; -q)$. By the combinatorial interpretation (5.27), we obtain that

$$(-\alpha; q)_\infty \psi(-\alpha q; -q) = \sum_{(\lambda,\mu)\in DO\times D^0} (-1)^{l(\lambda)} \alpha^{l(\lambda)-\frac{\lambda_1+1}{2}+l(\mu)} q^{|\lambda|+|\mu|}. \qquad (5.33)$$

It can be verified that the involution Ψ in Section 4 preserves the quantity of

$$l(\lambda) - \frac{\lambda_1+1}{2} + l(\mu)$$

and it changes the parity of $l(\lambda)$. So we get the following partition theorem.

Theorem 5.4 *We have*

$$(-\alpha; q)_\infty \psi(-\alpha q; -q) = - \sum_{\nu\in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}. \qquad (5.34)$$

Based on the above partition theorems for $f(\alpha q; q)$, $\phi(-\alpha q; -q)$ and $\psi(-\alpha q; -q)$, we can deduce Andrews' generalizations of Ramanujan's identities. More precisely, it follows from (5.32) and (5.34) that

$$(-\alpha q; q)_\infty \phi(-\alpha q; -q) - (-\alpha; q)_\infty \psi(-\alpha q; -q)$$

$$= 1 + \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}$$

$$+ \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} - \sum_{k=1}^{\infty} (-1)^k q^{k^2}$$

$$= 1 + 2 \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|}$$

$$= (-\alpha q; q)_\infty f(\alpha q; q),$$

where the last equality is a consequence of identity (5.28). Dividing both sides by $(-\alpha q; q)_\infty$ yields

$$\phi(-\alpha q; -q) - (1+\alpha)\psi(-\alpha q; -q) = f(\alpha q; q).$$

Hence we deduce the identity (1.11) by replacing αq by α.

According to (5.32) and (5.34), we have

$$(-\alpha q; q)_\infty \phi(\alpha q; -q) + (-\alpha; q)_\infty \psi(\alpha q; -q)$$

$$= 1 + \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}$$

$$- \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}$$

$$= 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2}$$

$$= \frac{(q; q)_\infty}{(-q; q)_\infty},$$

where the last equality follows from Gauss' identity (1.17). Dividing both sides by $(-\alpha q; q)_\infty$, we obtain

$$\phi(-\alpha q; -q) + (1+\alpha)\psi(-\alpha q; -q) = \frac{(q; q)_\infty}{(-q; q)_\infty (-\alpha q; q)_\infty},$$

we arrive at the identity (1.12) by replacing αq by α.

To conclude, we note that our approach can be viewed as combinatorial proofs of Andrews' identities in the forms multiplied by the factor $(-\alpha q; q)_\infty$.

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References

[1] A.K. Agarwal, n-Color partition theoretic interpretations of some mock theta functions, Electron. J. Combin. 11 (2004) N14.

[2] A.K. Agarwal, New combinatorial interpretations of some mock theta functions, Online J. Analytic Combin. 2 (2007), #5.

[3] G.E. Andrews, On basic hypergeometric series, mock-theta functions, and partitions (I), Quart. J. Math. (2) 17 (1966) 64–80.

[4] G.E. Andrews, On basic hypergeometric series, mock-theta functions, and partitions (II), Quart. J. Math. (2) 17 (1966) 132–143.

[5] G.E. Andrews, The Theory of Partitions, Addison-Wesley Publishing Co., 1976.

[6] G.E. Andrews, An introduction to Ramanujan's "Lost" Notebook, Amer. Math. Monthly 86 (1979) 89–108.

[7] G.E. Andrews and F.G. Garvan, Ramanujan's "lost" notebook VI: The mock theta conjectures, Adv. Math. 73 (1989) 242–255.

[8] G.E. Andrews, Mock theta functions, In Theta Functions, Bowdoin 1987, Part 2, Proc. Symp. Pure Math., vol. 49, American Mathematical Society, Providence, RI, (1989) 283–298.

[9] G.E. Andrews, Partitions with short sequences and mock theta functions, Proc. Natl. Acad. Sci. USA 102 (2005) 4666–4671.

[10] G.E. Andrews, H. Eriksson, F. Petrov and D. Romik, Integrals, partitions and MacMahon's theorem, J. Combin. Theory Ser. A 114 (2007) 545–554.

[11] B.C. Berndt, B. Kim and A.J. Yee, Ramanujan's Lost Notebook: Combinatorial proofs of identities associated with Heine's transformation or partial theta functions, J. Combin. Theory Ser. A., to appear.

[12] B.C. Bessenrodt and I. Pak, Partition congruences by involutions, European J. Combin. 25 (2004) 1139–1149.

[13] W.Y.C. Chen and E.H. Liu, A Franklin type involution for squares, Adv. Appl. Math., to appear.

[14] Y.-S. Choi and B. Kim, Partition identities from third and sixth order mock theta functions, preprint.

[15] F.J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

[16] N.J. Fine, Some new results on partitions, Proc. Nat. Acad. Sci. USA 34 (1948) 616–618.

[17] N.J. Fine, Basic Hypergeometric Series and Applications, Math. Surveys 27, AMS Providence, 1988.

[18] B. Gordon and R. McIntosh, A survey of classsical mock theta functions, preprint.

[19] K. Ono, Unearthing the visions of a master: Harmonic Maass forms and number theory, In: Proceeding of the 2008 Harvard-MIT Current Developments in Mathematics Conference, International Press, Somerville, MA, 2009, pp. 347–454.

[20] G.N. Watson, The final problem: An account of the mock theta functions, J. London Math. Soc. 11 (1936) 55–80.

[21] A.J. Yee, Bijective proofs of a theorem of Fine and related partition identities, Internat. J. Number Theory, to appear.

[22] A.J. Yee, Ramanujan's partial theta series and parity in partitions, Ramanujan J., to appear.