Higher Order Log-Concavity in Euler's Difference Table

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Abstract. Let e_n^k be the entries in the classical Euler's difference table. We consider the array $d_n^k = e_n^k/k!$ for $0 \leq k \leq n$, where d_n^k can be interpreted as the number of k-fixed-points-permutations of $[n]$. We show that the sequence $\{d_n^k\}_{0\leq k\leq n}$ is 2-log-concave and reverse ultra log-concave for any given n.

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1 Introduction

Euler introduced the difference table $(e_n^k)_{0\leq k\leq n}$ defined by $e_n^n = n!$ and

$$e_n^{k-1} = e_n^k - e_{n-1}^{k-1}, \tag{1.1}$$

for $1 \leq k \leq n$; see [5]. The combinatorial interpretation of the numbers e_n^k was found by Dumont and Randrianarivony [6]. Clarke, Han and Zeng [5] further gave a combinatorial interpretation of the q-analogue of Euler's difference table, and this interpretation has been extended by Faliharimalala and Zeng [8, 9] to the wreath product $C_\ell \wr S_n$ of the cyclic group with the symmetric group.

It is easily seen from the recurrence (1.1) that $k!$ divides e_n^k. Thus we can define the integers $d_n^k = e_n^k/k!$. Rakotondrajao [14] has shown that d_n^k counts the number of k-fixed-points-permutations of $[n]$, where a permutation $\pi \in \mathfrak{S}_n$ is called k-fixed-points-permutation if there are no fixed points in the last $n-k$ positions and the first k elements are in different cycles. Based on this combinatorial interpretation, Rakotondrajao [15] has found bijective proofs for the following two recurrence relations for $0 \leq k \leq n-1$,

$$d_n^k = (n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k, \tag{1.2}$$

$$d_n^k = nd_{n-1}^k - d_{n-2}^{k-1}. \tag{1.3}$$

Notice that $d_k^k = 1$. Recently, Eriksen, Freij and Wästlund [7] have generalized these formulas to fixed point λ-colored permutations. Employing (1.2) and (1.3), we can easily derive the following recurrence relation for $0 \le k \le n-1$,

$$d_n^k = d_{n-1}^{k-1} + (n-k) d_{n-1}^k. \tag{1.4}$$

Using the above recurrence relations (1.2) (1.3) and (1.4), we shall prove that the sequence $\{d_n^k\}_{0\le k\le n}$ has higher order log-concave properties. To be more specific, we shall show that this sequence is 2-log-concave and reverse ultra log-concave for any $n \ge 1$.

2 2-log-concavity

In this section, we shall show that the sequence $\{d_n^k\}_{0\le k\le n}$ is 2-log-concave for any $n \ge 1$. Recall that a sequence $\{a_k\}_{k\ge 0}$ of real numbers is said to be log-concave if $a_k^2 \ge a_{k+1}a_{k-1}$ for all $k \ge 1$; see Stanley [16] and Brenti [2]. From the recurrence relation (1.4), it is easy to prove by induction that the sequence $\{d_n^k\}_{0\le k\le n}$ is log-concave.

Theorem 2.1 *For $1 \le k \le n$, we have*

$$(d_n^k)^2 \ge d_n^{k-1} d_n^{k+1},$$

that is, the sequence $\{d_n^k\}_{0\le k\le n}$ is log-concave.

The notion of high order log-concavity was introduced by Moll [13]; see also, [10]. Given a sequence $\{a_k\}_{k\ge 0}$, define the operator $\mathfrak{L}$ as $\mathfrak{L}\{a_k\} = \{b_k\}$, where

$$b_k = a_k^2 - a_{k-1}a_{k+1}.$$

The log-concavity of $\{a_k\}$ becomes the positivity of $\mathfrak{L}\{a_k\}$. If the sequence $\mathfrak{L}\{a_k\}$ is not only positive but also log-concave, then we say that $\{a_k\}$ is 2-log-concave. In general, we say that $\{a_k\}$ is l-log-concave if $\mathfrak{L}^l\{a_k\}$ is positive, and that $\{a_k\}$ is infinite log-concave if $\mathfrak{L}^l\{a_k\}$ is positive for any $l \ge 1$. From numerical evidence, we pose the following conjecture.

Conjecture 2.2 *The sequence $\{d_n^k\}_{0\le k\le n}$ is infinitely log-concave.*

Recently, Brändén [1] and Cardon [3] have independently proved that if a polynomial has only real and nonpositive zeros, then its Taylor coefficients form an infinite log-concave sequence. However, this is not the case of the polynomials $\sum d_n^k x^k$. For example, for $n=2$, the polynomial x^2+x+1 does not have real roots. Nevertheless, we shall show that the sequence $\{d_n^k\}$ is 2-log concave in support of the general conjecture.

Theorem 2.3 *The sequence $\{d_n^k\}_{0\le k\le n}$ is 2-log-concave. In other words, for $n \ge 4$ and $2 \le k \le n-2$, we have*

$$\left((d_n^k)^2 - d_n^{k-1}d_n^{k+1}\right)^2 - \left((d_n^{k-1})^2 - d_n^{k-2}d_n^k\right)\left((d_n^{k+1})^2 - d_n^k d_n^{k+2}\right) \ge 0. \tag{2.1}$$

The idea to prove Theorem 2.3 may be described as follows. As the first step, we reformulate the left hand side of the above inequality (2.1) a cubic function f on $\frac{d_{n+1}^k}{d_n^k}$ by applying the recurrence relations (1.2), (1.3), (1.4) and the recurrence relation presented in the following Lemma 2.4. Then Theorem 2.3 is equivalent to the assertion that $f \geq 0$ on the interval

$$I = [n + \frac{n-k}{n}, n + \frac{n-k}{n} + \frac{n-k}{n^2}],$$

since it can be verified that for $n \geq 4$ and $2 \leq k \leq n-2$,

$$\frac{n-k}{n} \leq \frac{d_{n+1}^k}{d_n^k} \leq n + \frac{n-k}{n} + \frac{n-k}{n^2}. \tag{2.2}$$

Moreover, when $f(x)$ is considered as a continuous function on x, we will be able to show that $f'(x) < 0$ for $x \in I$ and

$$f\left(n + \frac{n-k}{n} + \frac{n-k}{n^2}\right) \geq 0.$$

Hence we deduce that $f > 0$ on the interval I so that Theorem 2.3 is immediate.

As mentioned above, the following recurrence relation will be needed in the proof of Theorem 2.3.

Lemma 2.4 *For* $1 \leq k \leq n$*, we have*

$$d_n^{k-1} = (k+1)(n-k)d_n^{k+1} - (n-2k+1)d_n^k. \tag{2.3}$$

Proof. First, it is easy to establish the following recurrence relation for $1 \leq k \leq n$,

$$d_n^{k-1} = kd_n^k - d_{n-1}^{k-1}. \tag{2.4}$$

By (1.2) and (1.4), we have

$$\begin{aligned} d_n^{k-1} &= d_{n+1}^k - (n-k+1)d_n^k \\ &= (n+1)d_n^k - d_{n-1}^{k-1} - (n-k+1)d_n^k \\ &= kd_n^k - d_{n-1}^{k-1}, \end{aligned}$$

as claimed. By (1.4), (2.4), for $1 \leq k \leq n$, we find

$$\begin{aligned} d_n^k &= (k+1)d_n^{k+1} - d_{n-1}^k \\ &= (k+1)d_n^{k+1} - \left(\frac{1}{n-k}d_n^k - \frac{1}{n-k}d_{n-1}^{k-1}\right) \\ &= (k+1)d_n^{k+1} - \frac{1}{n-k}d_n^k + \frac{1}{n-k}\left(kd_n^k - d_n^{k-1}\right) \\ &= (k+1)d_n^{k+1} - \frac{k-1}{n-k}d_n^k - \frac{1}{n-k}d_n^{k-1}. \end{aligned}$$

Consequently,

$$d_n^{k-1} = (k+1)(n-k)d_n^{k+1} - (n-2k+1)d_n^k,$$

as desired. ∎

In order to prove (2.2), we first give a lower bound for d_{n+1}^k/d_n^k.

Lemma 2.5 *For $n \geq 1$ and $1 \leq k \leq n-1$, we have*

$$\frac{d_{n+1}^k}{d_n^k} \geq n + \frac{n-k}{n}. \tag{2.5}$$

Proof. We proceed by induction on n. It is clear that (2.5) holds for $n = 1$ and $n = 2$. We now assume that (2.5) holds for positive integers less than n. By the recurrence (1.2), we have

$$\begin{aligned}\frac{d_{n+1}^k}{d_n^k} &= \frac{nd_n^k + (n-k)d_{n-1}^k}{d_n^k} \\ &= n + (n-k)\frac{d_{n-1}^k}{d_n^k} \\ &= n + (n-k)\frac{d_{n-1}^k}{(n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k}.\end{aligned}$$

Thus (2.5) can be recast as

$$(n-1) + (n-k-1)\frac{d_{n-2}^k}{d_{n-1}^k} \leq n.$$

So it suffices to check that

$$\frac{d_{n-1}^k}{d_{n-2}^k} \geq n-k-1.$$

Since $n \geq 3$, by the inductive hypothesis, we have

$$\begin{aligned}\frac{d_{n-1}^k}{d_{n-2}^k} &\geq n-2+\frac{n-2-k}{n-2} \\ &= n-1-\frac{k}{n-2} \\ &\geq n-k-1.\end{aligned}$$

as required. ∎

Next we give an upper bound for d_{n+1}^k/d_n^k.

Lemma 2.6 *For $n \geq 4$ and $2 \leq k \leq n-2$, we have*

$$\frac{d_{n+1}^k}{d_n^k} \leq n + \frac{n-k}{n} + \frac{n-k}{n^2}. \tag{2.6}$$

Proof. It follows from the recurrence (1.2) that

$$\begin{aligned}\frac{d_{n+1}^k}{d_n^k} &= n + (n-k)\frac{d_{n-1}^k}{d_n^k} \\ &= n + (n-k)\frac{d_{n-1}^k}{(n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k}.\end{aligned}$$

Thus (2.6) can be rewritten as

$$(n-1) + (n-k-1)\frac{d_{n-2}^k}{d_{n-1}^k} \geq \frac{n^2}{n+1},$$

that is,

$$\frac{d_{n-1}^k}{d_{n-2}^k} \leq (n+1)(n-k-1). \tag{2.7}$$

By recurrence (1.3) for $2 \leq k \leq n-2$, we see that

$$\frac{d_{n-1}^k}{d_{n-2}^k} \leq n-1,$$

which implies (2.7). This completes the proof. ∎

We are now ready to give the proof of Theorem 2.3.

Proof. It is easy to check that that the theorem holds for $n = 4, 5, 6$ and $2 \leq k \leq n-2$. So we may assume that $n \geq 7$.

We claim that the left hand side of (2.1) can be expressed as a cubic function f on $\frac{d_{n+1}^k}{d_n^k}$. By the recurrences (1.2), (1.3), (1.4) and (2.3), we can derive the following relations,

$$\begin{aligned}d_n^{k-2} &= (n-k+1)(n-k+3)d_n^k - (n-2k+3)d_{n+1}^k, \\ d_n^{k-1} &= d_{n+1}^k - (n-k+1)d_n^k, \\ d_n^{k+1} &= \frac{1}{(k+1)(n-k)}\left(d_{n+1}^k - kd_n^k\right), \\ d_n^{k+2} &= \frac{1}{(k+1)(k+2)(n-k-1)(n-k)}\left((n-2k-1)d_{n+1}^k + (n+k^2)d_n^k\right).\end{aligned}$$

It follows that (2.1) can be rewritten as

$$A\cdot\Big(C_3(n,k)\left(d_{n+1}^k\right)^3 + C_2(n,k)\left(d_{n+1}^k\right)^2\left(d_n^k\right) + C_1(n,k)\left(d_{n+1}^k\right)\left(d_n^k\right)^2 + C_0(n,k)\left(d_n^k\right)^3\Big) \geq 0,$$

where

$$\begin{aligned}
A &= \frac{d_n^k}{(k+1)^2(n-k)^2(k+2)(n-k-1)},\\
C_3(n,k) &= -n^2 - 5n + 6k + 6,\\
C_2(n,k) &= n^3 + n^2k + 5n^2 + 3nk - 10k^2 + n - 16k - 6,\\
C_1(n,k) &= n^2 - 2n + 14k + 14k^2 + n^3 + 10nk^2 - 10n^2k - n^3k - 3nk,\\
C_0(n,k) &= -4n^2 - 12k^2 - 12k^3 + 10nk + 18nk^2 - 9n^2k + n^2k^2 - n^3k.
\end{aligned}$$

Since d_n^k are positive integers, it suffices to show that

$$C_3(n,k)\left(\frac{d_{n+1}^k}{d_n^k}\right)^3 + C_2(n,k)\left(\frac{d_{n+1}^k}{d_n^k}\right)^2 + C_1(n,k)\left(\frac{d_{n+1}^k}{d_n^k}\right) + C_0(n,k) \geq 0. \tag{2.8}$$

We now consider the function

$$f(x) = C_3(n,k)x^3 + C_2(n,k)x^2 + C_1(n,k)x + C_0(n,k),$$

with

$$f'(x) = 3C_3(n,k)x^2 + 2C_2(n,k)x + C_1(n,k). \tag{2.9}$$

We are going to show that $f'(x) < 0$, for $2 \leq x \leq n-1$. As will be seen, the quadratic function $f'(x)$ has a zero in the interval $[-1,k]$ and a zero in the interval $[k,n]$. At the point $x = 1$, we have

$$f'(-1) = -(k+1)(n^3 + 12n^2 - 10nk + 19n - 34k - 30).$$

Since for $n \geq 7$ and $2 \leq k \leq n-2$, we find

$$\begin{aligned}
&n^3 + 12n^2 - 10nk + 19n - 34k - 30\\
&\quad \geq n^3 + 12n(k+2) + 19n - 30 - 10nk - 34k\\
&\quad \geq (n^3 - 30) + 2nk + (43n - 34k) > 0.
\end{aligned}$$

This yields that $f'(-1) < 0$. Similarly, for $x = k$, we obtain that

$$f'(k) = (k+1)(n-k)(n^2 + n + 2k - 2) > 0.$$

Moreover, for $x = n$, we have

$$f'(n) = -(n-k)(n^3 + 4n^2 - 10nk + 14k - 21n + 14). \tag{2.10}$$

To prove $f'(n) < 0$, it is sufficient to show that for $2 \leq k \leq n-2$,

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 > 0.$$

We have two cases for the ranges of k. For $2 \leq k \leq n-3$, we have

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 = n\left((n-3)^2 + 10(n-k-3)\right) + 14k + 14 > 0,$$

Meanwhile, for $k = n-2$,

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 = n(n-3)^2 + 4n - 14 > 0.$$

Thus $f'(n) < 0$ is valid for $2 \leq k \leq n-2$. Then we reach the conclusion that $f'(x)$ has a zero in the interval $[-1, k]$ and a zero in the interval$[k, n]$.

We continue to demonstrate that $f'(x) < 0$ in the interval I. By Lemma 2.5, for $k \geq 2$ we have

$$\frac{d_{n+1}^k}{d_n^k} \geq n + \frac{n-k}{n} > n,$$

which means that $f'(x)$ has no zero on the interval I. Since $n \geq k+2$, it is easily seen that

$$\begin{aligned} C_3(n,k) &= -(n^2 + 5n - 6k - 6) \\ &\leq -\left((k+2)^2 + 5(k+2) - 6k - 6\right) \\ &\leq -(k^2 + 3k + 8) < 0. \end{aligned}$$

Since $f'(n) < 0$, we see that $f'(x) < 0$ in the interval I, as expected. In other words, $f(x)$ is strictly decreasing on this interval.

Up to now, we have shown that $f(x)$ is strictly decreasing on the interval $I = [n + \frac{n-k}{n}, n + \frac{n-k}{n} + \frac{n-k}{n^2}]$. So it remains to prove that

$$f\left(n + \frac{n-k}{n} + \frac{n-k}{n^2}\right) > 0.$$

Since

$$f\left(n + \frac{n-k}{n} + \frac{n-k}{n^2}\right) = \frac{h(k)(n-k)^2}{n^6},$$

where

$$\begin{aligned} h(k) = {} & (-10n^4 - 26n^3 - 28n^2 - 18n - 6)k^2 + (-n^6 + 20n^5 + 27n^4 + 19n^3 - 7n - 6)k \\ & + (n^7 - 10n^6 - 4n^5 - 4n^4 + 9n^3 + 7n^2 + 6n). \end{aligned}$$

Clearly, the proof will be complete as long as we can show that $h(k) \geq 0$ for $n \geq 7$ and $2 \leq k \leq n-2$.

Regard $h(x)$ as a continuous function on x, that is,

$$\begin{aligned} h(x) = &(-10n^4 - 26n^3 - 28n^2 - 18n - 6)x^2 + (-n^6 + 20n^5 + 27n^4 + 19n^3 - 7n - 6)x \\ &+ (n^7 - 10n^6 - 4n^5 - 4n^4 + 9n^3 + 7n^2 + 6n). \end{aligned}$$

Since the leading coefficient $-10n^4 - 26n^3 - 28n^2 - 18n - 6$ of $h(x)$ is negative, we only need to prove that $h(2) > 0$ and $h(n-1) > 0$. For $n \geq 7$, we have

$$\begin{aligned} h(n-1) &= n(n^5 - 3n^4 + 2n^3 + 2n^2 + 2n + 1) \\ &= n\left(n^3(n-1)(n-2) + 2n^2 + 2n + 1\right) > 0, \end{aligned}$$

and

$$\begin{aligned} h(2) &= n^7 - 12n^6 + 36n^5 + 10n^4 - 57n^3 - 105n^2 - 80n - 36 \\ &= n^5(n-5)(n-7) + n^4(n-6) + 16n^3(n-7) + 55n^2(n-7) \\ &\quad + 80n(n-1) + 200n^2 - 36 > 0. \end{aligned}$$

In summary, we have confirmed that $h(k) > 0$ for $n \geq 7$ and $2 \leq k \leq n-2$. This completes the proof. ∎

3 The reverse ultra log-concavity

This section is concerned with the reverse ultra log-concavity of d_n^k. Recall that sequence $\{a_k\}_{0\leq k\leq n}$ is called ultra log-concave if $\left\{a_k/\binom{n}{k}\right\}$ is log-concave; see Liggett [12]. This condition can be restated as

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \tag{3.1}$$

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. As noticed by Liggett [12], if a sequence $\{a_k\}_{0\leq k\leq n}$ is ultra log-concave, then the sequence $\{k!a_k\}_{0\leq k\leq n}$ is log-concave.

In comparison with ultra log-concavity, a sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (3.1), that is,

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0. \tag{3.2}$$

Chen and Gu [4] have shown the Boros-Moll polynomials have this reverse ultra log-concave property. We shall show that the sequence $\{d_n^k\}_{0\leq k\leq n}$ also possesses this property.

Theorem 3.1 *For $1 \le k \le n-1$, we have*

$$\frac{d_n^{k-1}}{\binom{n}{k-1}} \cdot \frac{d_n^{k+1}}{\binom{n}{k+1}} \ge \left(\frac{d_n^k}{\binom{n}{k}} \right)^2,$$

or equivalently,

$$(n-k+1)(k+1)d_n^{k-1}d_n^{k+1} \ge k(n-k)\left(d_n^k\right)^2. \tag{3.3}$$

Proof. According to the recurrence relations (1.4) and (2.3), we find that (3.3) can be reformulated as

$$(n-k+1)\left(\frac{d_{n+1}^k}{d_n^k}\right)^2 - (n-k+1)(n+1)\left(\frac{d_{n+1}^k}{d_n^k}\right) + k(2n-2k+1) \ge 0. \tag{3.4}$$

The discriminant of the quadratic polynomial of the left side of (3.4) in d_{n+1}^k/d_n^k equals

$$\Delta = ((n-k+1)(n+1))^2 - 4k(n-k+1)(2n-2k+1).$$

We claim that $\Delta > 0$ for $1 \le k \le n-1$. Put

$$f(k) = \Delta = 8k^2 - (n^2+10n+5)k + (n^3+3n^2+3n+1).$$

Since $n \ge k+1$, we have

$$\begin{aligned} f'(k) =& 16k - (n^2+10n+5) \\ =& -(n^2+10n-16k+5) \\ \le& -\left((k+1)^2 + 10(k+1) - 16k + 5\right) \\ =& -(k-2)^2 - 12 < 0, \end{aligned}$$

which implies that $f(k)$ is monotone decreasing for $1 \le k \le n-1$. Furthermore,

$$f(n-1) = 2\left((n-2)^2 + 3\right) > 0.$$

Thus, $\Delta > 0$ for $1 \le k \le n-1$. Consequently, the quadratic function has two distinct real zeros. If we can show that for $1 \le k \le n-1$, d_{n+1}^k/d_n^k is larger than the maximal zero, then (3.4) holds since $n-k+1 > 0$. Thus we still have to show that

$$\frac{d_{n+1}^k}{d_n^k} > \frac{(n-k+1)(n+1)+\sqrt{\Delta}}{2(n-k+1)} = \frac{n+1}{2} + \frac{\sqrt{\Delta}}{2(n-k+1)} \tag{3.5}$$

In view of (2.5), we see that (3.5) can be deduced from the following inequality

$$n + \frac{n-k}{n} \ge \frac{n+1}{2} + \frac{\sqrt{\Delta}}{2(n-k+1)},$$

which is equivalent to

$$(n-k+1)(n^2+n-2k) \geq n\sqrt{\Delta}.$$

Since both sides are positive, we can transform the above relation into the following form

$$\big((n-k+1)(n^2+n-2k)\big)^2 \geq n^2\Delta.$$

Evidently,

$$\begin{aligned}
&\big((n-k+1)(n^2+n-2k)\big)^2 - n^2\Delta \\
&= (n-k+1)\left(4n^2k(2n-2k+1) - 4k(n-k+1)(n^2+n-k)\right) \\
&= 4k(n-k+1)(n-k)(n^2-n+k-1) \geq 0,
\end{aligned}$$

for $1 \leq k \leq n-1$. This completes the proof. ∎

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