On Singletons and Adjacencies of Set Partitions of Type B

William Y.C. Chen and David G.L. Wang

Center for Combinatorics, LPMC-TJKLC

Nankai University, Tianjin 300071, P.R. China

chen@nankai.edu.cn, wgl@cfc.nankai.edu.cn

Abstract

We show that the joint distribution of the number of singleton pairs and the number of adjacency pairs is symmetric over the set partitions of type B_n without zero-block, in analogy with the result of Callan for ordinary partitions.

Keywords: set partition of type B, singleton, adjacency, symmetric distribution

AMS Classification: 05A15, 05A18, 20F55

1 Introduction

The main objective of this paper is to give a type B analogue of an elegant property of set partitions discovered by Bernhart [1], that is, the number s_n of partitions of $[n] = \{1, 2, \ldots, n\}$ without singletons is equal to the number a_n of partitions of $[n]$ for which no block contains two adjacent elements i and $i + 1$ modulo n. In fact, it is easy to show that s_n and a_n have the same formula by the principle of inclusion-exclusion. Bernhart gave a recursive proof of the fact that $s_n = a_n$ by showing that $s_n + s_{n+1} = B_n$ and $a_n + a_{n+1} = B_n$, where B_n denotes the Bell number, namely, the number of partitions of $[n]$. As noted by Bernhart, there may be no simple way to bring the set of partitions of $[n]$ without singletons and the set of partitions of $[n]$ without adjacencies into a one-to-one correspondence.

From a different perspective, Callan [3] found a bijection in terms of an algorithm that interchanges singletons and adjacencies. In fact, Callan has established a stronger statement that the joint distribution of the number of singletons and the number of adjacencies is symmetric over the set of partitions of $[n]$. While the proof of Callan is purely combinatorial, we feel that there is still some truth in the remark of Bernhart.

The study of singletons and adjacencies of partitions goes back to Kreweras [7] for noncrossing partitions. Kreweras has shown that the number of noncrossing partitions of $[n]$ without singletons equals the number of noncrossing partitions of $[n]$ without adjacencies. Bernhart [1] found a combinatorial proof of this assertion. Deutsch and Shapiro [5] considered noncrossing partitions of $[n]$ without visible singletons and showed that such partitions are enumerated by the Fine number. Here a visible singleton of a partition means a singleton not covered by any arc in the linear representation. Canfield [4] has shown that the average number of singletons in a partition of $[n]$ is an increasing function of n. Biane [2] has derived a bivariate generating function for the number of partitions of $[n]$ containing a given number of blocks but no singletons. Knuth [6] proposed the problem of finding the generating function for the number of partitions of $[n]$ with a given number of blocks but no adjacencies. The generating function has been found by several problem solvers. The sequence of the numbers s_n is listed as the entry A000296 in Sloane [9].

It is natural to wonder whether there exist a type B analogue of Bernhart's theorem and a type B analogue of Callan's algorithm. We give the peeling and patching algorithm which implies the symmetric distribution of the number of singleton pairs and the number of adjacency pairs for type B partitions without zero-block. Moreover, we can transform the bijection to an involution. This involution is described in the last section.

2 The peeling and patching algorithm

In this section, we give a type B analogue of Callan's symmetric distribution of singletons and adjacencies. Moreover, the algorithm of Callan can be extended to the type B case. This type B algorithm will be called the peeling and patching algorithm.

A partition of type B_n is a partition π of the set $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ such that for any block B of π, $-B$ is also a block of π, and there is at most one block B, called zero-block, satisfying $B = -B$, see Reiner [8]. We call $\pm i$ a singleton pair of π if π contains a block $\{i\}$, and call $\pm(j,\, j+1)$ an adjacency pair of π if j and $j+1$ (modulo n) lie in the same block of π. Denote the number of singleton pairs (resp. adjacency pairs) of π by s_π (resp. a_π). For example, let

$$\pi = \{\pm\{1\},\ \pm\{2\},\ \pm\{3, 11, 12\},\ \pm\{4, -7, 9, 10\},\ \pm\{5, 6, -8\}\}. \tag{2.1}$$

Then have $s_\pi = 2$ and $a_\pi = 3$.

Denote by V_n the set of B_n-partitions without zero-block. The following is the main result of this paper.

Theorem 2.1 *The joint distribution of the number of singleton pairs and the number of adjacency pairs is symmetric over B_n-partitions without zero-block. In other words, let*

$$P_n(x, y) = \sum_{\pi \in V_n} x^{s_\pi} y^{a_\pi},$$

we have $P_n(x, y) = P_n(y, x)$.

For example, there are three B_2-partitions without zero-block:

$$\{\pm\{1\},\, \pm\{2\}\},\ \{\pm\{1,2\}\},\ \{\pm\{1,-2\}\}.$$

So $P_2(x,y) = x^2 + y^2 + 1$. Moreover,

$$P_3(x,y) = (x^3 + y^3) + 3xy + 3(x + y),$$
$$P_4(x,y) = (x^4 + y^4) + 4(x^2y + xy^2) + 8(x^2 + y^2) + 8xy + 4(x + y) + 7.$$

It should be noted that Theorem 2.1 cannot be deduced from Callan's result for ordinary partitions. The following consequence is immediate, which is a type B analogue of Bernhart's observation.

Corollary 2.2 *The number of B_n-partitions without zero-block and singleton pairs equals the number of B_n-partitions without zero-block and adjacency pairs.*

To prove Theorem 2.1, we shall provide an algorithm $\psi\colon V_n \to V_n$, called the peeling and patching algorithm, such that for any B_n-partition π without zero-block, $s_\pi = a_{\psi(\pi)}$ and $a_\pi = s_{\psi(\pi)}$.

In fact, we need a more general setting to describe the algorithm. Let $S = \{\pm t_1, \pm t_2, \ldots, \pm t_r\}$ be a subset of $[\pm n]$, where $0 < t_1 < t_2 < \cdots < t_r$. Let π be a partition of the set S. We call π a *symmetric partition* if for any block B of π, $-B$ is also a block of π. We call $\pm t_i$ a *singleton pair* of π if π contains a block $\{t_i\}$, and call $\pm(t_j, t_{j+1})$ an *adjacency pair* of π if t_j and t_{j+1} are contained in the same block. By convention we consider t_{r+1} as t_1. We call $\pm t_j$ (resp. $\pm t_{j+1}$) a *left-point-pair* (resp. *right-point-pair*) if $\pm(t_j, t_{j+1})$ is an adjacency pair. For the case $r = 1$, the partition $\pi = \{\pm\{t_1\}\}$ contains exactly one singleton pair $\{\pm t_1\}$ and one adjacency pair $\pm(t_1, t_1)$.

The peeling and patching algorithm ψ consists of the peeling procedure α and the patching procedure β. During the peeling procedure, at each step we take out the singleton pairs and left-point-pairs, until there exists neither singleton pairs nor adjacency pairs. During the patching procedure, we first interchange the roles of singleton pairs and adjacency pairs, then put the singleton pairs and left-point-pairs back to the partition. It should be emphasized that the patching procedure is not just the reverse of the peeling procedure.

The peeling procedure α. Given an input partition π, let $\pi_0 = \pi$. We extract the set S_1 of singleton pairs and the set L_1 of left-point-pairs (of adjacency pairs) from π_0. Let π_1 be the remaining partition. Now π_1 is again a type B partition without zero-block. So we can extract the set S_2 of singleton pairs and extract the set L_2 of left-point-pairs from π_1. Denote by π_2 be the remaining partition. Repeating this process, we eventually obtain a partition π_k that does not have any singleton pairs or adjacency pairs. Notice that it is possible that π_k is the empty partition.

For example, consider the partition π in (2.1), that is,

$$\pi = \{\pm\{1\},\ \pm\{2\},\ \pm\{3, 11, 12\},\ \pm\{4, -7, 9, 10\},\ \pm\{5, 6, -8\}\}.$$

The peeling procedure is illustrated by Table 2.1.

j	S_j	L_j	π_j
1	± 1, ± 2	± 5, ± 9, ± 11	$\pm\{3, 12\},\ \pm\{4, -7, 10\},\ \pm\{6, -8\}$
2	$\emptyset$	± 12	$\pm\{3\},\ \pm\{4, -7, 10\},\ \pm\{6, -8\}$
3	± 3	$\emptyset$	$\pm\{4, -7, 10\},\ \pm\{6, -8\}$
4	$\emptyset$	± 10	$\pm\{4, -7\},\ \pm\{6, -8\}$

Table 2.1: The peeling procedure.

The patching procedure β. Let $\sigma_k = \pi_k$. As the first step, we interchange the roles of the singleton-sets S_i and the adjacency-sets represented by L_i. To be precise, we patch the elements of S_i and L_i into the partition σ_i which will be obtained recursively from σ_{i+1}, so that S_i (resp. L_i) is the right-point-set (resp. singleton-set) of the resulting partition σ_{i-1}. So S_i represents the set of adjacency pairs of σ_{i-1}.

We start the patching procedure by putting the elements of S_k and L_k back to σ_k in such a way that the resulting partition σ_{k-1} contains S_k (resp. L_k) as its right-point-set (resp. singleton-set). The existence of such a partition σ_{k-1} will be confirmed later. Next, in the same manner we put the elements of S_{k-1} and L_{k-1} back into σ_{k-1} to get σ_{k-2}. Repeating this process, we finally arrive at a partition σ_0, which is defined to be the output of the patching procedure.

Now let us describe the process of constructing σ_{k-1}. Suppose that the underlying set of π_{k-1} is $\{\pm t_1, \pm t_2, \ldots, \pm t_r\}$, where $0 < t_1 < t_2 < \cdots < t_r$.

Consider the case that $\sigma_k (= \pi_k)$ is the empty partition. The last step of the peeling procedure implies that π_{k-1} must be of special form, namely, either there is only one block in π_{k-1}, or every block of π_{k-1} contains exactly one element. Define σ_{k-1} to be $\{\pm\{t_1\},\ \pm\{t_2\},\ \ldots,\ \pm\{t_r\}\}$ if there is only one block in π_{k-1}; otherwise, set $\sigma_{k-1} = \{\pm\{t_1, t_2, \ldots, t_r\}\}$. When $r = 1$, it is clear to see that σ_{k-1} is well-defined.

We now assume that σ_k is not empty. We can uniquely decompose the set S_t into maximal consecutive subsets of the form

$$\{\pm t_{i+1}, \pm t_{i+2}, \ldots, \pm t_{i+h}\}. \tag{2.2}$$

The number of such subsets is at least two. By the maximality, the element t_i does not appear in S_k. On the other hand, it is clear that $t_i \notin L_k$ by the definition of L_k. Thus t_i is contained in σ_k. This observation allows us to put the elements $t_{i+1}, t_{i+2}, \ldots, t_{i+h}$ into the block of σ_k containing t_i. Accordingly, we put $-t_{i+1}, -t_{i+2}, \ldots, -t_{i+h}$ into the block containing $-t_i$. After having processed all maximal consecutive subsets of S_k, we put each element in L_k as a singleton block into the partition σ_k. The resulting partition is defined to be σ_{k-1}.

This completes the description of the step of constructing σ_{k-1}. Since $\sigma_k(=\pi_k)$ contains neither singleton pairs nor adjacency pairs, it is easy to check that L_k (resp. S_k) is the set of singleton pairs (right-point-pairs) of σ_{k-1}.

For example, Table 2.2 is an illustration of the patching procedure for partition generated in Table 2.1. In the last step, patching S_1 and L_1 to σ_1, we finally obtain

j	S_j	L_j	σ_j
4	$\emptyset$	± 10	$\pm\{4,-7\},\ \pm\{6,-8\}$
3	± 3	$\emptyset$	$\pm\{4,-7\},\ \pm\{6,-8\},\ \pm\{10\}$
2	$\emptyset$	± 12	$\pm\{4,-7\},\ \pm\{6,-8\},\ \pm\{3,10\}$
1	± 1, ± 2	± 5, ± 9, ± 11	$\pm\{4,-7\},\ \pm\{6,-8\},\ \pm\{3,10\},\pm\{12\}$

Table 2.2: The patching procedure.

$$\sigma_0 = \{\pm\{1,2,12\},\ \pm\{3,10\},\ \pm\{4,-7\},\ \pm\{5\},\ \pm\{6,-8\},\ \pm\{9\},\ \pm\{11\}\}. \qquad (2.3)$$

The peeling and patching algorithm ψ is defined by

$$\psi(\pi) = \beta(\alpha(\pi))$$

for any B_n-partition π without zero-block. Keep in mind that there is a step of interchanging the roles of singleton pairs and adjacency pairs at the beginning of the patching procedure. We are now ready to give a proof of Theorem 2.1.

Proof of Theorem 2.1. We aim to show that the peeling and patching algorithm ψ gives a bijection on B_n-partitions without zero-block, which interchanges the number of singleton pairs and the number of adjacency pairs.

It is easy to see that the inverse algorithm can be described as follows. It is in fact the composition of another peeling procedure and another patching procedure. To be precise, let σ be the input partition. Let $\sigma_0 = \sigma$. We first peel the singleton pairs and right-point-pairs at each step, until we obtain a partition σ_k which has neither singleton pairs nor adjacency pairs. Then, based on the partition $\pi_k = \sigma_k$, we recursively patch the elements that have been taken out before. Meanwhile, we also need to interchange the roles of the singleton-sets and right-point-sets at the beginning of this patching procedure. Finally, we get a partition, as the output of the inverse algorithm. Therefore, ψ is a bijection which exchanges the number of singleton pairs and the number of adjacency pairs. This completes the proof. ∎

An illustration of the peeling and patching algorithm is given by (2.1), Table 2.1, Table 2.2, and (2.3).

To conclude this section, we give the generating function for the number s_n^B of B_n-partitions without zero-block and singleton pairs, that is,

$$\sum_{n\geq 0} s_n^B \frac{x^n}{n!} = \exp\left(\sinh(x)e^x - x\right). \tag{2.4}$$

By the principle of inclusion-exclusion, we obtain

$$s_n^B = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{j=0}^{k} 2^{k-j} S(k,j), \tag{2.5}$$

where $S(k,j)$ is the Stirling number of the second kind, and $2^{k-j}S(k,j)$ is the number of partitions in V_k containing exactly $2j$ blocks. The formula (2.4) can be easily derived from (2.5).

3 From bijection to involution

The bijection given in the previous section is not an involution although it interchanges the number of singleton pairs and the number of adjacency pairs. In this section, we show that the peeling and patching algorithm can be turned into an involution. Such an involution for ordinary partitions has been given by Callan [3].

For any $i \in [n]$, we define the *complement* of i to be $n+1-i$, and the complement of $-i$ as $-(n+1-i)$. This notion can be extended naturally to any symmetric partition π of $[\pm n]$ by taking the complement for each element in the partition. The complement of π is denoted by $\omega(\pi)$. It is clear that ω is an involution. Assume that σ_0 is given in (2.3). We have

$$\omega(\sigma_0) = \{\pm\{1,11,12\},\ \pm\{2\},\ \pm\{3,10\},\ \pm\{4\},\ \pm\{5,-7\},\ \pm\{6,-9\},\ \pm\{8\}\}. \tag{3.1}$$

In light of the complementation operation, we get an involution based on the peeling and patching algorithm. The proof is a straightforward verification and hence is omitted.

Theorem 3.1 *The mapping $\omega \circ \psi$ is an involution on B_n-partitions without zero-block, which interchanges the number of singleton pairs and the number of adjacency pairs.*

Let us give an example to demonstrate that $\omega \circ \psi$ is involution, that is

$$\omega(\psi(\pi)) = \psi^{-1}(\omega(\pi)). \tag{3.2}$$

Consider the partition π in (2.1). In this case, the left hand side of (3.2) is $\omega(\sigma_0)$ in (3.1). On the other hand,

$$\omega(\pi) = \{\pm\{1,\,2,\,10\},\ \pm\{3,\,4,\,-6,\,9\},\ \pm\{5,\,-7,\,-8\},\ \pm\{11\},\ \pm\{12\}\}.$$

Applying the procedure β^{-1}, we obtain the Table 3.3, where R_j (resp. S_j) denotes the set of right-point-pairs (singleton pairs). Next, by the procedure α^{-1}, we get the Table 3.4. Finally, putting R_1 and S_1 back to π_1, we arrive at the partition π_0 which is in agreement with (3.1).

j	R_j	S_j	σ_j
1	$\pm2,\,\pm4,\,\pm8$	$\pm11,\,\pm12$	$\pm\{1,10\},\ \pm\{3,-6,9\},\ \pm\{5,-7\}$
2	±1	$\emptyset$	$\pm\{10\},\ \pm\{3,-6,9\},\ \pm\{5,-7\}$
3	$\emptyset$	±10	$\pm\{3,-6,9\},\ \pm\{5,-7\}$
4	±3	$\emptyset$	$\pm\{5,-7\},\ \pm\{6,-9\}$

Table 3.3: The procedure β^{-1}.

j	R_j	S_j	π_j
4	±3	$\emptyset$	$\pm\{5,-7\},\ \pm\{6,-9\}$
3	$\emptyset$	±10	$\pm\{3\},\ \pm\{5,-7\},\ \pm\{6,-9\}$
2	±1	$\emptyset$	$\pm\{3,10\},\ \pm\{5,-7\},\ \pm\{6,-9\}$
1	$\pm2,\,\pm4,\,\pm8$	$\pm11,\,\pm12$	$\pm\{1\},\ \pm\{3,10\},\ \pm\{5,-7\},\ \pm\{6,-9\}$

Table 3.4: The procedure α^{-1}.

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] F.R. Bernhart, Catalan, Motzkin, and Riordan numbers, Discrete Math., 204 (1999) 73–112.

[2] P. Biane, Some properties of crossings and partitions, Discrete Math., 175 (1997) 41–53.

[3] D. Callan, On conjugates for set partitions and integer compositions, arXiv.math.CO/0508052.

[4] E.R. Canfield, Engel's inequality for Bell numbers, J. Combin. Theory Ser. A, 72 (1995) 184–187.

[5] E. Deutsch and L. Shapiro, A survey of the Fine numbers, Discrete Math., 241 (2001) 241–265.

[6] D. Knuth, Problem 11151, Amer. Math. Monthly, 112 (2005) 367.

[7] G. Kreweras, Sur les partitions non croisees d'un cycle, Discrete Math., 1 (1972) 333–350.

[8] V. Reiner, Non-crossing partitions for classical reflection groups, Discrete Math., 177 (1997) 195–222.

[9] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://www.research.att.com/~njas/sequences`.