On Han's Bijection via Permutation Codes

William Y.C. Chen[1], Neil J.Y. Fan[2], Teresa X.S. Li[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

[1]chen@nankai.edu.cn, [2]fjy@cfc.nankai.edu.cn, [3]lxs@cfc.nankai.edu.cn.

Abstract

We show that Han's bijection when restricted to permutations can be carried out in terms of the major code and inversion code. In other words, it maps a permutation π with a major code $(s_1, s_2, \ldots, s_n)$ to a permutation σ with an inversion code $(s_1, s_2, \ldots, s_n)$. We also show that the fixed points of Han's map can be characterized by the strong fixed points of Foata's second fundamental transformation. The notion of strong fixed points is related to the partial Foata maps introduced by Björner and Wachs. We further give a construction of a class of Mahonian statistics on permutations in terms of the major code.

Keywords: Foata's bijection, Mahonian statistics, major code, inversion code

AMS Subject Classifications: 05A05, 05A15, 05A19

1 Introduction

In his combinatorial proof of the fact that the Z-statistic introduced by Zeilberger and Bressoud [15] is Mahonian, Han [8] constructed a Foata-style bijection on words. Let H denote Han's bijection when restricted to permutations. Throughout this paper, by Han's bijection we always mean the map H. We shall show that the map H can be carried out exactly by the major code and inversion code. The major code of a permutation can be described in terms of cyclic intervals, a notion also introduced by Han [9] in his study of the joint distribution of the excedance number and Denert's statistics. Note that the major code in the context of this paper is different from the major index table defined by Skandera [14], which is also called the major code by Dzhumadil'daev [4] and called Mc-code by Hivert, Novelli, and Thibon [11], see also Han [10].

Using the code representation, we show that the fixed points of Han's map can be characterized by the strong fixed points of Foata's second fundamental transformation. The notion of strong fixed points is related to the partial Foata maps introduced by Björner and Wachs [1]. Based on the major code, we construct a class of 2^{n-1} Mahonian statistics on permutations on $[n] = \{1, 2, \ldots, n\}$.

Let us give an overview of the background and definitions. Let $X = \{1^{m_1}, 2^{m_2}, \cdots, k^{m_k}\}$ be a multiset with m_i i's and $m_1 + m_2 + \cdots + m_k = n$. The set of rearrangements of X

is denoted by $R(X)$. When $m_1 = m_2 = \cdots = m_k = 1$, $R(X)$ reduces to the set S_n of permutations on $[n]$. For a word $w = w_1 w_2 \cdots w_n \in R(X)$, the descent set $\mathrm{Des}(w)$, the descent number $\mathrm{des}(w)$, the major index $\mathrm{maj}(w)$, the inversion number $\mathrm{inv}(w)$ and the Z-statistic $\mathrm{Z}(w)$ are defined by

$$\begin{aligned}
\mathrm{Des}(w) &= \{i | 1 \le i \le n-1, w_i > w_{i+1}\}, \\
\mathrm{des}(w) &= \# \, \mathrm{Des}(w), \\
\mathrm{maj}(w) &= \sum_{i \in \mathrm{Des}(w)} i, \\
\mathrm{inv}(w) &= \#\{(i,j) | 1 \le i < j \le n, w_i > w_j\}, \\
\mathrm{Z}(w) &= \sum_{i<j} \mathrm{maj}(w_{ij}),
\end{aligned}$$

where w_{ij} is obtained from w by deleting all elements except i and j. For example, let $w = 211324314 \in R(1^3, 2^2, 3^2, 4^2)$. We have $\mathrm{Des}(w) = \{1,4,6,7\}, \mathrm{des}(w) = 4, \mathrm{maj}(w) = 18, \mathrm{inv}(w) = 9$, and $\mathrm{Z}(w)$ equals

$$\mathrm{maj}(21121) + \mathrm{maj}(11331) + \mathrm{maj}(11414) + \mathrm{maj}(2323) + \mathrm{maj}(2244) + \mathrm{maj}(3434) = 16.$$

A statistic is said to be Mahonian on $R(X)$ if it has the same distribution as the major index on $R(X)$. MacMahon [12, 13] introduced the major index and proved that the major index is equidistributed with the inversion number for $R(X)$. Foata [5] found a combinatorial proof of this classical result by constructing a bijection Φ, called the second fundamental transformation, which maps the major index to the inversion number, namely,

$$\mathrm{maj}(w) = \mathrm{inv}(\Phi(w)) \quad \text{for any } w \in R(X).$$

For completeness, we give a brief description of Foata's bijection [5], see also Haglund [7]. Let $w = w_1 w_2 \cdots w_n$ be a word on a multiset X as defined above. Let x be an element in X. If $w_n \le x$, the x-factorization of w is defined as $w = v_1 b_1 \cdots v_p b_p$, where each b_i is less than or equal to x, and every element in v_i is greater than x. Note that v_i is allowed to be empty. Similarly, when $w_n > x$, the x-factorization of w is defined as $w = v_1 b_1 \cdots v_p b_p$, where each b_i is greater than x, and every element in v_i is less than or equal to x. In each case, let $\gamma_x(w) = b_1 v_1 \cdots b_p v_p$, and let $w' = w_1 w_2 \cdots w_{n-1}$. Then the second fundamental transformation Φ can be defined recursively by setting $\Phi(a) = a$ for each $a \in X$ and setting

$$\Phi(w) = \gamma_{w_n}(\Phi(w')) \cdot w_n$$

if w contains more than one element.

To extend the theorem of MacMohan, Björner and Wachs [1] considered the problem of finding subsets U of S_n for which the major index and inversion number are equidistributed. They introduced the k-th partial Foata bijection $\phi_k \colon S_n \longrightarrow S_n$ for $1 \leq k \leq n$ as follows. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. Define $\phi_1(\sigma) = \sigma$ and for $k > 1$ define

$$\phi_k(\sigma) = \gamma_{\sigma_k}(\sigma_1\sigma_2\cdots\sigma_{k-1}) \cdot \sigma_k\sigma_{k+1}\cdots\sigma_n.$$

It is easily seen that

$$\Phi = \phi_n \circ \phi_{n-1} \cdots \circ \phi_1.$$

A subset U of S_n is said to be a strong Foata class if $\phi_k(U) = U$ for $1 \leq k \leq n$. A permutation σ is said to be a strong fixed point of Foata's map if $\phi_k(\sigma) = \sigma$ for $1 \leq k \leq n$. As will be seen, the strong fixed points of Foata's map is closely related to the fixed points of Han's map.

The paper is organized as follows. In Section 2, we recall the construction of Han's map, and give a description of the major code. Then we give a reformulation of Han's map in terms of the major code and inversion code. In Section 3, we show that a permutation is fixed by H if and only if it is a strong fixed points of Foata's map Φ. Section 4 is devoted to the construction of a class of Mahonian statistics based on the major code.

2 Han's bijection via permutation codes

In this section, we are concerned with a reformulation of Han's bijection for permutations in terms of the major code and the inversion code. For completeness, let us give an overview of the map H.

Let $x \in [n]$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_{n-1}$ be a permutation on $\{1, 2, \cdots, x-1, x+1, \cdots, n\}$. Define $C^x(\sigma)$ as $\tau_1\tau_2\cdots\tau_{n-1}$, where $\tau_i = \sigma_i - x (\text{mod } n)$, i.e.,

$$\tau_i = \begin{cases} \sigma_i - x + n, & \text{if } \sigma_i < x; \\ \sigma_i - x, & \text{if } \sigma_i > x, \end{cases}$$

and define $C_x(w)$ as the standardization of σ, i.e., $C_x(w) = \nu_1\nu_2\cdots\nu_{n-1} \in S_{n-1}$ with

$$\nu_i = \begin{cases} \sigma_i, & \text{if } \sigma_i < x; \\ \sigma_i - 1, & \text{if } \sigma_i > x. \end{cases}$$

Evidently, both C^x and C_x are bijections between permutations on $\{1, 2, \cdots, x-1, x+1, \cdots, n\}$ and S_{n-1}. So $(C^x)^{-1}$ and $(C_x)^{-1}$ are well defined. Han's bijection H can be defined by $H(1) = 1$ and

$$H(\sigma) = C_{\sigma_n}^{-1}(H(C^{\sigma_n}(\sigma'))) \cdot \sigma_n,$$

where $\sigma \in S_n$ with $n > 1$ and $\sigma' = \sigma_1\sigma_2 \cdots \sigma_{n-1}$.

We proceed to give the definition of cyclic intervals. Let $X = \{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\}$ be a multiset. For $x, y \in X$, the cyclic interval $]\!]x, y]\!]$ is defined by Han [9] as

$$]\!]x, y]\!] = \begin{cases} \{z|z \in [k], x < z \leq y\}, & \text{if } x \leq y; \\ \{z|z \in [k], z > x \quad or \quad z \leq y\}, & \text{otherwise.} \end{cases}$$

Set $]\!]x, \infty]\!] = \{z|z \in [k], z > x\}$.

For any word $w = w_1w_2 \cdots w_n$ on X and $1 \leq i \leq n$, define

$$t_i(w) = \#\{j|1 \leq j \leq i-1, w_j \in]\!]w_i, \infty]\!]\},$$

and

$$s_i(w) = \#\{j|1 \leq j \leq i-1, w_j \in]\!]w_i, w_{i+1}]\!]\},$$

where $w_{n+1} = \infty$. For example, if $w = 312432143$, then

$$(t_1(w), t_2(w), \cdots, t_9(w)) = (0, 1, 1, 0, 1, 3, 5, 0, 2),$$

and

$$(s_1(w), s_2(w), \cdots, s_9(w)) = (0, 0, 1, 3, 3, 4, 5, 6, 2).$$

The notion of cyclic intervals plays an important role in the proof of the fact that the bi-statistic (exc, Den) is equidistributed with (des, maj) on $R(X)$, where exc is the excedance number and Den is the Denert's statistic, see Denert [3], Foata and Zeilberger [6], and Han [9].

We continue to give the definition of the major code in terms of cyclic intervals. Meanwhile, the inversion code can also be described this way. Let

$$E_n = \{(a_1, a_2, \cdots, a_n) \in Z^n | 0 \leq a_i \leq i-1, i = 1, 2, \cdots, n\}.$$

Keep in mind that the above definitions of $t_i(\sigma)$ and $s_i(\sigma)$ apply to permutations. It is well known that the map $I\colon S_n \longrightarrow E_n$ defined by

$$\sigma \longmapsto (t_1(\sigma), t_2(\sigma), \cdots, t_n(\sigma))$$

is a bijection known as the Lehmer code or the inversion code such that

$$\sum_{i=1}^{n} t_i(\sigma) = \text{inv}(\sigma).$$

On the other hand, it is easy to see that the map $M\colon S_n \longrightarrow E_n$ defined by

$$\sigma \longmapsto (s_1(\sigma), s_2(\sigma), \cdots, s_n(\sigma))$$

is also a bijection. We call $M(\sigma)$ the major code of σ. To recover σ from its major code $(s_1, s_2, \ldots, s_n)$, first let $\sigma_n = n - s_n$. Suppose that $\sigma_{k+1}, \ldots, \sigma_n$ have been determined by $s_{k+1}, \ldots, s_n$. Then delete the elements in the sequence

$$\sigma_{k+1}, \sigma_{k+1} - 1, \ldots, 1, n, (n-1), \ldots, (\sigma_{k+1}) + 1$$

that are equal to σ_j for some $j \geq k+1$ and set σ_k to be the $(s_k + 1)$-st element in the resulting sequence. It has been shown by Han [9] that

$$\sum_{i=1}^{n} s_i(\sigma) = \mathrm{maj}(\sigma).$$

For example, $I(38516427) = (0, 0, 1, 3, 1, 3, 5, 1)$ and $M(38516427) = (0, 1, 1, 2, 3, 4, 4, 1)$.

The following theorem states that Han's bijection H can be carried out in terms of the major code and inversion code.

Theorem 2.1 *For each $n \geq 1$, we have*

$$H = I^{-1} \circ M.$$

In other words, H is a bijection on S_n such that

$$M(\sigma) = I(H(\sigma)).$$

Proof. We proceed by induction on n. For $n = 1$, the theorem is obvious. Suppose $n > 1$ and let $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$ with $M(\sigma) = (s_1(\sigma), s_2(\sigma), \cdots, s_n(\sigma))$. By definition, $s_n(\sigma) = \#\{\sigma_n + 1, \sigma_n + 2, \cdots, n\} = n - \sigma_n$. By the construction of H, we have

$$H(\sigma) = C_{\sigma_n}^{-1}[H(C^{\sigma_n}(\sigma')] \cdot \sigma_n,$$

which implies $t_n(H(\sigma)) = n - \sigma_n$. Since the standardization of a permutation preserves the relative order, we find that

$$I(C_{\sigma_n}(\sigma_1\sigma_2 \cdots \sigma_{n-1})) = (t_1(\sigma), t_2(\sigma), \cdots, t_{n-1}(\sigma)).$$

By induction, it suffices to show that

$$M(C^{\sigma_n}(\sigma_1\sigma_2 \cdots \sigma_{n-1})) = (s_1(\sigma), s_2(\sigma), \cdots, s_{n-1}(\sigma)). \tag{2.1}$$

Suppose $C^{\sigma_n}(\sigma_1\sigma_2 \cdots \sigma_{n-1}) = \tau_1\tau_2 \cdots \tau_{n-1}$. Let $\tau_n = \infty$. For $1 \leq i \leq n-1$ and $1 \leq k \leq i-1$, we claim that $\sigma_k \in]\!]\sigma_i, \sigma_{i+1}]\!]$ if and only if $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$. If it is true, then (2.1) follows immediately. This claim can be verified as follows.

(1) If $i \neq n-1$, there are two cases each of which has three subcases, namely,

$$(1a) \quad \sigma_n < \sigma_i < \sigma_{i+1};$$

$$(1b)\quad \sigma_i < \sigma_n < \sigma_{i+1};$$

$$(1c)\quad \sigma_i < \sigma_{i+1} < \sigma_n;$$

$$(2a)\quad \sigma_n > \sigma_i > \sigma_{i+1};$$

$$(2b)\quad \sigma_i > \sigma_n > \sigma_{i+1};$$

$$(2c)\quad \sigma_i > \sigma_{i+1} > \sigma_n.$$

We only give the proof of case (1b), the other cases can be proved in the same manner. Let us assume that $\sigma_i < \sigma_n < \sigma_{i+1}$. By definition, $\tau_i = n + \sigma_i - \sigma_n, \tau_{i+1} = \sigma_{i+1} - \sigma_n$, so we have $\tau_{i+1} < \tau_i$. Suppose $\sigma_k \in]\!]\sigma_i, \sigma_{i+1}]\!]$. Then we see that $\sigma_i < \sigma_k < \sigma_{i+1}$ and

$$\tau_k = \begin{cases} \sigma_k - \sigma_n + n, & \text{if } \sigma_k < \sigma_n < \sigma_{i+1}; \\ \sigma_k - \sigma_n, & \text{if } \sigma_i < \sigma_n < \sigma_k. \end{cases}$$

If $\sigma_k < \sigma_n < \sigma_{i+1}$, then $\tau_k = \sigma_k - \sigma_n + n > \sigma_i - \sigma_n + n = \tau_i$, it follows that $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$; if $\sigma_i < \sigma_n < \sigma_k$, then $\tau_k = \sigma_k - \sigma_n < \sigma_{i+1} - \sigma_n = \tau_{i+1}$, which implies $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$. Conversely, if $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$, then we deduce that $\tau_k > \tau_i$ or $\tau_k < \tau_{i+1}$. Assume that $\sigma_k \notin]\!]\sigma_i, \sigma_{i+1}]\!]$, then $\sigma_k < \sigma_i$ or $\sigma_k > \sigma_{i+1}$. Consequently,

$$\tau_k = \begin{cases} \sigma_k - \sigma_n + n, & \text{if } \sigma_k < \sigma_i < \sigma_n; \\ \sigma_k - \sigma_n, & \text{if } \sigma_k > \sigma_{i+1} > \sigma_n. \end{cases}$$

If $\sigma_k < \sigma_i$, then $\tau_k = \sigma_k - \sigma_n + n < \sigma_i - \sigma_n + n = \tau_i$. However, $\tau_k = \sigma_k + n - \sigma_n > \sigma_{i+1} - \sigma_n = \tau_{i+1}$, which is a contradiction. If $\sigma_k > \sigma_{i+1}$, then $\tau_k = \sigma_k - \sigma_n > \sigma_{i+1} - \sigma_n = \tau_{i+1}$, but now $\tau_k = \sigma_k - \sigma_n < \sigma_i - \sigma_n + n = \tau_i$, a contradiction too. So we reach the conclusion that $\sigma_k \in]\!]\sigma_i, \sigma_{i+1}]\!]$.

(2) If $i = n - 1$, there are two cases, namely $\sigma_{n-1} > \sigma_n$ or $\sigma_{n-1} < \sigma_n$. For the first case, by definition we have $\tau_{n-1} = \sigma_{n-1} - \sigma_n$. It follows that

$$\begin{aligned} \sigma_k \in]\!]\sigma_{n-1}, \sigma_n]\!] &\Rightarrow \sigma_k > \sigma_{n-1} \quad \text{or} \quad \sigma_k < \sigma_n \\ &\Rightarrow \tau_k = \begin{cases} \sigma_k - \sigma_n, & \text{if } \sigma_k > \sigma_{n-1}; \\ \sigma_k - \sigma_n + n, & \text{if } \sigma_k < \sigma_n. \end{cases} \\ &\Rightarrow \tau_k > \tau_{n-1} \\ &\Rightarrow \tau_k \in]\!]\tau_{n-1}, \infty]\!]. \end{aligned}$$

Conversely, assume that $\tau_k \in]\!]\tau_{n-1}, \infty]\!]$, i.e., $\tau_k > \tau_{n-1} = \sigma_{n-1} - \sigma_n$. Suppose that $\sigma_k \notin]\!]\sigma_{n-1}, \sigma_n]\!]$, namely, $\sigma_n < \sigma_k < \sigma_{n-1}$. Then we have

$$\tau_k = \sigma_k - \sigma_n < \sigma_{n-1} - \sigma_n = \tau_{n-1},$$

a contradiction. So we have $\sigma_k \in]\!]\sigma_{n-1}, \sigma_n]\!]$. Similarly, one can verify the case $\sigma_{n-1} < \sigma_n$. This completes the proof. ∎

The following corollary provides an alternative way to compute the major index, which will be useful for the construction of Mahonian statistics.

Corollary 2.2 *For any permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, define*

$$\mathscr{C}(\sigma) = C^{\sigma_n}(\sigma_1\sigma_2\cdots\sigma_{n-1})$$

and define $L(\sigma) = \sigma_n$. Then we have

$$s_i(\sigma) = i - L(\mathscr{C}^{n-i}(\sigma)),$$

for $1 \le i \le n$, where $\mathscr{C}^0(\sigma) = \sigma$ and $\mathscr{C}^k(\sigma) = \mathscr{C}(\mathscr{C}^{k-1}(\sigma))$.

Proof. It is clear that $\mathscr{C}^{n-i}(\sigma) \in S_i$. By the definition of $s_n(\sigma)$, we have

$$s_n(\sigma) = \#\{\sigma_n + 1, \cdots, n\} = n - \sigma_n = n - L(\sigma) = n - L(\mathscr{C}^0(\sigma)).$$

By the proof of Theorem 1.1, we see that

$$M(\mathscr{C}^{n-i}(\sigma)) = (s_1(\sigma), \cdots, s_i(\sigma)),$$

which implies that $s_i(\sigma) = i - L(\mathscr{C}^{n-i}(\sigma))$ for $i = 1, 2, \cdots, n$. ∎

The sequence

$$L(\mathscr{C}^n(\sigma)), L(\mathscr{C}^{n-1}(\sigma)), \ldots, L(\mathscr{C}^0(\sigma))$$

gives an alternative way to compute the major code. It also facilitates the computation of $H(\sigma)$. For example, let $\sigma = 392648517$. We have

$$M(\sigma) = (0, 0, 1, 3, 1, 4, 3, 5, 2), \quad H(\sigma) = 496182537,$$

see Table 1.

The following corollary shows that Han's bijection H commutes with the complementation operator c. For a permutation $\sigma \in S_n$, we define $c\sigma$ as $\tau_1\tau_2\cdots\tau_n$, where $\tau_i = n + 1 - \sigma_i$. For a code $a = (a_1, a_2, \ldots, a_n) \in E_n$, we define $ca = (b_1, b_2, \ldots, b_n)$, where $b_i = i - 1 - a_i$.

Corollary 2.3 *For $\sigma \in S_n$ and $s = (s_1, s_2, \cdots, s_n) \in E_n$. We have*

$$H(c\sigma) = cH(\sigma).$$

The above corolalry can be easily verified by induction on n. It also follows from Theorem 2.1 and the relations

$$M(c\sigma) = c(M(\sigma)),$$

$$I(c\sigma) = c(I(\sigma)).$$

$\sigma = 392648517$	$I(H(\sigma)) = (0,0,1,3,1,4,3,5,2)$
$\downarrow$	$\Uparrow$
$\mathscr{C}^0(\sigma) = 39264851\mathbf{7}$	$C_7^{-1}(48617253) \cdot 7 = 496182537$
$\downarrow$	$\uparrow$
$\mathscr{C}^1(\sigma) = 5248617\mathbf{3}$	$C_3^{-1}(3751624) \cdot 3 = 48617253$
$\downarrow$	$\uparrow$
$\mathscr{C}^2(\sigma) = 271536\mathbf{4}$	$C_4^{-1}(364152) \cdot 4 = 3751624$
$\downarrow$	$\uparrow$
$\mathscr{C}^3(\sigma) = 53416\mathbf{2}$	$C_2^{-1}(25314) \cdot 2 = 364152$
$\downarrow$	$\uparrow$
$\mathscr{C}^4(\sigma) = 3125\mathbf{4}$	$C_4^{-1}(2431) \cdot 4 = 25314$
$\downarrow$	$\uparrow$
$\mathscr{C}^5(\sigma) = 423\mathbf{1}$	$C_1^{-1}(132) \cdot 1 = 2431$
$\downarrow$	$\uparrow$
$\mathscr{C}^6(\sigma) = 31\mathbf{2}$	$C_2^{-1}(12) \cdot 2 = 132$
$\downarrow$	$\uparrow$
$\mathscr{C}^7(\sigma) = 1\mathbf{2}$	$C_2^{-1}(1) \cdot 2 = 12$
$\downarrow$	$\uparrow$
$\mathscr{C}^8(\sigma) = \mathbf{1}$	$C_1^{-1}(\emptyset) \cdot 1 = 1$
$\downarrow$	$\uparrow$
(1,2,2,1,4,2,4,3,7)	$\emptyset$
$\Downarrow$	
$M(\sigma)$=(0,0,1,3,1,4,3,5,2)	the construction of $H(\sigma)$

Table 1: The procedures to compute $H(\sigma)$ and $M(\sigma)$

3 A characterization of fixed points

In this section, we give a characterization of the fixed points of Han's map H. As will be seen, the fixed points of Han's map are related to the strong fixed points of Foata's second fundamental transformation which are easier to describe.

The notion of strong fixed points of Foata's map is related to the strong Foata classes introduced by Björner and Wachs [1]. A labeling w of a poset P is called recursive if every principal order ideal of P is labeled by a set of consecutive numbers. In particular, if P is a chain with n elements and $w : P \longrightarrow [n]$ is a labeling of P. Reading the labels from bottom to top, the labels form a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. It is easily seen that w is a recursive labeling of P if and only if for each $i \in [n]$, $\{\sigma_1, \sigma_2, \cdots, \sigma_i\}$ forms a set of consecutive numbers. By the Theorem 4.2 in [1], we deduce that a permutation $\sigma \in S_n$ is a strong fixed point of Foata's map if and only if for each $i \in [n]$, $\{\sigma_1, \sigma_2, \cdots, \sigma_i\}$ forms a set of consecutive numbers. For example, $\sigma = 45367281 \in S_8$ is a strong fixed point of Foata's map, while $\pi = 34125678$ is not, since $\{\pi_1, \pi_2, \pi_3\} = \{1, 3, 4\}$ is not a set of consecutive numbers.

Theorem 3.1 *For each $\sigma \in S_n$, σ is a fixed point of H, i.e. $H(\sigma) = \sigma$, if and only if σ is a strong fixed point of Foata's map.*

Proof. Suppose that $H(\sigma) = \sigma$. By Theorem 2.1, we see that $I(\sigma) = M(\sigma)$. In particular, we have $s_{n-1}(\sigma) = t_{n-1}(\sigma)$. If $\sigma_{n-1} > \sigma_n$, by Corollary 2.2 we have

$$s_{n-1}(\sigma) = n - 1 - L(\mathscr{C}(\sigma)) = n - 1 - (\sigma_{n-1} - \sigma_n) = n - 1 + \sigma_n - \sigma_{n-1},$$

and by definition $t_{n-1}(\sigma) = n - \sigma_{n-1}$. It follows that $\sigma_n = 1$. If $\sigma_{n-1} < \sigma_n$, then $s_{n-1}(\sigma) = \sigma_n - \sigma_{n-1} - 1$ and $t_{n-1}(\sigma) = n - \sigma_{n-1} - 1$. Hence $\sigma_n = n$. Using the relation (2.1), we get

$$M(\mathscr{C}(\sigma)) = (s_1(\sigma), s_2(\sigma), \cdots, s_{n-1}(\sigma)). \tag{3.2}$$

Moreover, when $\sigma_n = 1$ or $\sigma_n = n$, we have

$$\mathscr{C}(\sigma) = C_{\sigma_n}(\sigma_1 \cdots \sigma_{n-1}). \tag{3.3}$$

Combining (3.2), (3.3) and the fact that $I(C_{\sigma_n}(\sigma_1 \cdots \sigma_{n-1})) = (t_1(\sigma), t_2(\sigma), \cdots, t_{n-1}(\sigma))$, we obtain

$$M(\mathscr{C}(\sigma)) = I(\mathscr{C}(\sigma)).$$

By induction on n, we deduce that $\mathscr{C}(\sigma)$ is a strong fixed point of Foata's map. Consequently, by relation (3.3)

$$\{\sigma_1, \sigma_2, \cdots, \sigma_i\} = \begin{cases} \{(\mathscr{C}(\sigma))_1 + 1, (\mathscr{C}(\sigma))_2 + 1, \cdots, (\mathscr{C}(\sigma))_i + 1\}, & \text{if } \sigma_n = 1; \\ \{(\mathscr{C}(\sigma))_1, (\mathscr{C}(\sigma))_2, \cdots, (\mathscr{C}(\sigma))_i\}, & \text{if } \sigma_n = n, \end{cases}$$

which is a set of consecutive integers. Thus σ is a strong fixed point of Foata's map.

Conversely, suppose that $\sigma \in S_n$ is a strong fixed point of Foata's map. So $\{\sigma_1, \cdots, \sigma_{n-1}\}$ is a set of consecutive integers with $n-1$ numbers in $[n]$. This implies that $\sigma_n = 1$ or $\sigma_n = n$. Hence

$$C^{\sigma_n}(\sigma') = C_{\sigma_n}(\sigma') = \begin{cases} (\sigma_1 - 1)\cdots(\sigma_{n-1} - 1), & \text{if } \sigma_n = 1; \\ \sigma_1 \cdots \sigma_{n-1}, & \text{if } \sigma_n = n, \end{cases}$$

where $\sigma' = \sigma_1\sigma_2\cdots\sigma_{n-1}$. It follows that $C^{\sigma_n}(\sigma')$ is a strong fixed point of Foata's map. By induction on n we deduce that

$$H(C^{\sigma_n}(\sigma')) = C^{\sigma_n}(\sigma').$$

Consequently,

$$\begin{aligned} H(\sigma) &= C_{\sigma_n}^{-1}(H(C^{\sigma_n}(\sigma'))) \cdot \sigma_n \\ &= C_{\sigma_n}^{-1}(C^{\sigma_n}(\sigma')) \cdot \sigma_n \\ &= C_{\sigma_n}^{-1}(C_{\sigma_n}(\sigma')) \cdot \sigma_n = \sigma' \cdot \sigma_n = \sigma, \end{aligned}$$

as desired. This completes the proof. ∎

The following corollary gives another characterization of the fixed points of H in terms of codes.

Corollary 3.2 *Let $\sigma \in S_n$. The following statements are equivalent:*

(1) $M(\sigma) = I(\sigma)$, that is, σ is a fixed point of H.

(2) $I(\sigma) = (t_1(\sigma), t_2(\sigma), \cdots, t_n(\sigma))$ such that $t_i(\sigma) = 0$ or $i-1$ for each $i \in [n]$.

Proof. It is easy to check that σ satisfies the Condition (2) if σ is a strong fixed point of Foata's map. Conversely, suppose that $I(\sigma) = (t_1(\sigma), t_2(\sigma), \cdots, t_n(\sigma))$ such that $t_i(\sigma) = 0$ or $i-1$ for each $i \in [n]$. We proceed by induction on n to show that σ is a strong fixed point of Foata's map. The statement is obvious for $n = 1$. Now we may assume that the claim holds for any permutation of length $n-1$ satisfying Condition (2). It is clear that

$$I(C_{\sigma_n}(\sigma')) = (t_1(\sigma), \ldots, t_{n-1}(\sigma)).$$

The inductive hypothesis implies that $I(C_{\sigma_n}(\sigma'))$ is a strong fixed point of length $n-1$. Since $t_n = 0$ or $t_n = n-1$, we have $\sigma_n = 1$ or $\sigma_n = n$, and hence

$$C_{\sigma_n}(\sigma') = \begin{cases} (\sigma_1 - 1)\cdots(\sigma_{n-1} - 1), & \text{if } \sigma_n = 1; \\ \sigma_1 \cdots \sigma_{n-1}, & \text{if } \sigma_n = n. \end{cases}$$

It follows that σ is also a strong fixed point of Foata's map. Now the corollary is a consequence of Theorem 3.1. ∎

Corollary 3.3 *For any $n \geq 1$, Han's map H has 2^{n-1} fixed points.*

By Theorem 3.1, we see that each fixed point of H is a fixed point of Φ, but the converse is not true. For example, let $\sigma = 14235 \in S_5$, we have $\Phi(\sigma) = \sigma$. But it is not a fixed point of H.

4 A construction of Mahonian statistics

In this section, we give a construction of Mahonian statistics on permutations by using the major code. It should be mentioned that Clarke [2] gave the following construction of a class of Mahonian statistics on words based Foata's second fundamental transformation.

Theorem 4.1 *Let $X = \{1^{m_1}, 2^{m_2}, \cdots, k^{m_k}\}$ be a multiset with $m_1 + m_2 + \cdots + m_k = n$. Let $e = (e_1, e_2, \cdots, e_n) \in \{0,1\}^n$, $w \in R(X)$. For any $1 \leq i \leq n$, let*

$$u_i(w) = \begin{cases} s_i(w), & \text{if } e_i = 0; \\ t_i(w), & \text{if } e_i = 1. \end{cases}$$

Then the statistic

$$\mathrm{inmaj}_e(w) = \sum_{i=1}^{n} u_i(w)$$

is Mahonian on $R(X)$.

Since $u_1(w) = 0$ and $u_n(w) = t_n(w) = s_n(w)$, Clarke in fact constructed 2^{n-2} Mahonian statistics on $R(X)$, where

$$\begin{aligned} \mathrm{inmaj}_{(0,0,\ldots,0)}(w) &= \mathrm{maj}(w), \\ \mathrm{inmaj}_{(1,1,\ldots,1)}(w) &= \mathrm{inv}(w). \end{aligned}$$

We proceed to construct a class of Mahonian statistics $\{M_e | e \in \{0,1\}^n\}$ for permutations. The statistics M_e satisfy the following properties

$$M_{(0,0,\ldots,0)}(\sigma) = r_1(\sigma) + \cdots + r_n(\sigma) = \mathrm{maj}(H(\sigma)), \tag{4.4}$$

$$M_{(1,1,\ldots,1)}(\sigma) = s_1(\sigma) + \cdots + s_n(\sigma) = \mathrm{maj}(\sigma). \tag{4.5}$$

Definition 4.2 *Let $\sigma \in S_n$ with $M(\sigma) = (s_1(\sigma), s_2(\sigma), \cdots, s_n(\sigma))$. For any $2 \leq i \leq n$, define*

$$r_i(\sigma) = \begin{cases} i-1, & \text{if } s_{i-1}(\sigma) < s_i(\sigma); \\ 0, & \text{otherwise.} \end{cases}$$

and set $r_1(\sigma) = s_1(\sigma) = 0$. *For any vector* $e = (e_1, e_2, \cdots, e_n) \in \{0,1\}^n$, *the statistic* M_e *on* S_n *is defined by*

$$M_e(\sigma) = \sum_{i=1}^{n} u_i(\sigma),$$

where

$$u_i(\sigma) = \begin{cases} s_i(\sigma), & \text{if } e_i = 1; \\ r_i(\sigma), & \text{if } e_i = 0. \end{cases} \tag{4.6}$$

Theorem 4.3 *For any* $e = (e_1, e_2, \cdots, e_n) \in \{0,1\}^n$ *the statistic* M_e *is Mahonian on* S_n.

The following lemma shows that Theorem 4.3 holds for any $e \in \{0,1\}^n$ if it is true for vectors e with $e_n = 1$.

Lemma 4.4 *For* $1 \le k \le n$, *there exists a bijection* $H_k\colon S_n \longrightarrow S_n$ *such that*

$$M(H_k(\sigma)) = (s_1(\sigma), \cdots, s_{n-k-1}(\sigma), s_{n-k}(H_k(\sigma)), \cdots, s_n(H_k(\sigma))) \tag{4.7}$$

and

$$\sum_{j=n-k}^{n} s_j(H_k(\sigma)) = s_{n-k}(\sigma) + r_{n-k+1}(\sigma) + \cdots + r_n(\sigma). \tag{4.8}$$

In particular, $M_{e'}$ *and* $M_{e''}$ *are equidistributed on* S_n *for* $e' = (e_1, \cdots, e_{n-k-1}, 1, 0, \cdots, 0)$ *and* $e'' = (e_1, \cdots, e_{n-k-1}, 1, 1, \cdots, 1)$.

Proof. Let $\sigma \in S_n$ and $M(\sigma) = (s_1(\sigma), s_2(\sigma), \cdots, s_n(\sigma))$. For $n-k+1 \le i \le n$, set $s'_i = i - s_i(\sigma) = L(\mathscr{C}^{n-i}(\sigma))$, the last letter of $\mathscr{C}^{n-i}(\sigma)$. Define $H_k\colon S_n \longrightarrow S_n$ by

$$H_k(\sigma) = C^{-1}_{s'_n}(\cdots[C^{-1}_{s'_{n-k+1}}(\mathscr{C}^k(\sigma)) \cdot s'_{n-k+1}] \cdots) \cdot s'_n, \tag{4.9}$$

where $\mathscr{C}^n(\sigma) = \emptyset$. We first show that H_k is a bijection. Suppose that $H_k(\sigma) = H_k(\pi)$. From the definition (4.9) we see that

$$s'_j(\sigma) = s'_j(\pi) \quad \text{for} \quad j \ge n-k+1,$$

and so $\mathscr{C}^k(\sigma) = \mathscr{C}^k(\pi)$. From the definition of $\mathscr{C}$ and Corollary 2.2, we obtain

$$\mathscr{C}^{k-1}(\sigma) = (C^{s'_{n-k+1}})^{-1}(\mathscr{C}^k(\sigma)),$$

hence we deduce that $\mathscr{C}^j(\sigma) = \mathscr{C}^j(\pi)$ for $j = 0, 1, \cdots, k$. In particular, we have $\sigma = \pi$. Thus H_k is a bijection.

We proceed to use induction to demonstrate that the bijection H_k satisfies the properties (4.7) and (4.8). For $k=1$, we have

$$H_1(\sigma) = C^{-1}_{s'_n}(\mathscr{C}(\sigma)) \cdot s'_n.$$

Since $C^{-1}_{s'_n}$ preserves the relative order, equation (4.7) is true for H_1, namely,

$$M(H_1(\sigma)) = (s_1(\sigma), \cdots, s_{n-2}(\sigma), s_{n-1}(H_1(\sigma)), s_n(H_1(\sigma))).$$

Since s'_{n-1} is defined as the last element of $\mathscr{C}(\sigma)$, we find

$$(H_1(\sigma))_{n-1} = \begin{cases} s'_{n-1}, & \text{if } s'_{n-1} < s'_n; \\ s'_{n-1}+1, & \text{if } s'_{n-1} \geq s'_n. \end{cases}$$

Again, by the definition of s'_i, we see that $s'_{n-1} < s'_n$ if and only if $s_{n-1}(\sigma) \geq s_n(\sigma)$. Therefore, from Corollary 2.2 it follows that

$$\begin{aligned} s_{n-1}(H_1(\sigma)) &= \begin{cases} (n-1) - (s'_{n-1} - s'_n + n), & \text{if } s'_{n-1} < s'_n; \\ (n-1) - (s'_{n-1} + 1 - s'_n), & \text{if } s'_{n-1} \geq s'_n. \end{cases} \\ &= \begin{cases} s_{n-1}(\sigma) - s_n(\sigma), & \text{if } s_{n-1}(\sigma) \geq s_n(\sigma); \\ s_{n-1}(\sigma) - s_n(\sigma) + n - 1, & \text{if } s_{n-1}(\sigma) < s_n(\sigma). \end{cases} \end{aligned}$$

So we deduce that

$$\begin{aligned} s_{n-1}(H_1(\sigma)) + s_n(H_1(\sigma)) &= s_{n-1}(H_1(\sigma)) + s_n(\sigma) \\ &= \begin{cases} s_{n-1}(\sigma), & \text{if } s_{n-1}(\sigma) \geq s_n(\sigma); \\ s_{n-1}(\sigma) + n - 1, & \text{if } s_{n-1}(\sigma) < s_n(\sigma). \end{cases} \\ &= s_{n-1}(\sigma) + r_n(\sigma). \end{aligned}$$

Thus the equation (4.8) holds for H_1. Assume that the relations (4.7) and (4.8) hold for H_j with $j \leq k-1$. We shall prove that H_k also satisfies the equations (4.7) and (4.8). By (4.9), we have

$$H_k(\sigma) = C^{-1}_{s'_n}(H_{k-1}(\mathscr{C}(\sigma))) \cdot s'_n$$

and by the inductive hypothesis, the major code of $H_{k-1}(\mathscr{C}(\sigma))$ satisfies the relations

$$s_i(H_{k-1}(\mathscr{C}(\sigma))) = s_i(\mathscr{C}(\sigma)) = s_i(\sigma) \quad \text{for} \quad 1 \leq i \leq n-k-1$$

and

$$\sum_{i=n-k}^{n-1} s_i(H_{k-1}(\mathscr{C}(\sigma))) = s_{n-k}(\mathscr{C}(\sigma)) + r_{n-k+1}(\mathscr{C}(\sigma)) + \cdots + r_{n-1}(\mathscr{C}(\sigma))$$

$$= s_{n-k}(\sigma) + r_{n-k+1}(\sigma) + \cdots + r_{n-1}(\sigma).$$

It is evident that the last element of $H_{k-1}(\mathscr{C}(\sigma))$ is equal to $s'_{n-1}(\sigma)$. The same argument as in the proof for the case $k = 1$ shows that

$$s_i(H_k(\sigma)) = s_i(H_{k-1}(\mathscr{C}(\sigma))) \quad \text{for} \quad i \leq n-2,$$

and

$$s_{n-1}(H_k(\sigma)) + s_n(H_k(\sigma)) = s_{n-1}(\sigma) + r_n(\sigma).$$

Consequently,

$$s_i(H_k(\sigma)) = s_i(\sigma) \quad \text{for} \quad 1 \leq i \leq n-k-1.$$

That is to say, the relation (4.7) holds for H_k. By Corollary 2.2, $s_{n-1}(H_{k-1}(\mathscr{C}(\sigma))) = n - 1 - L(H_{k-1}(\mathscr{C}(\sigma))) = s_{n-1}(\sigma)$, hence

$$\begin{aligned}
&s_{n-k}(H_k(\sigma)) + \cdots + s_{n-2}(H_k(\sigma)) + s_{n-1}(H_k(\sigma)) + s_n(H_k(\sigma)) \\
&= s_{n-k}(H_{k-1}(\mathscr{C}(\sigma))) + \cdots + s_{n-2}(H_{k-1}(\mathscr{C}(\sigma))) + s_{n-1}(\sigma) + r_n(\sigma) \\
&= s_{n-k}(H_{k-1}(\mathscr{C}(\sigma))) + \cdots + s_{n-2}(H_{k-1}(\mathscr{C}(\sigma))) + s_{n-1}(H_{k-1}(\mathscr{C}(\sigma))) + r_n(\sigma) \\
&= s_{n-k}(\sigma) + r_{n-k+1}(\sigma) + \cdots + r_{n-1}(\sigma) + r_n(\sigma).
\end{aligned}$$

So the relation (4.8) holds for H_k. This completes the proof. ∎

In view of the above lemma, we are now ready to present the proof of Theorem 4.3.

Proof of Theorem 4.3. We shall justify Theorem 4.3 by showing that for each $e \in \{0,1\}^n$, there exists a bijection Ψ_e on S_n satisfying $M_e(\sigma) = \mathrm{maj}(\Psi_e(\sigma))$ for any $\sigma \in S_n$. The argument is by induction on n. For $n = 1$, define $\Psi_{(0)} = \Psi_{(1)}$ as the identity map on S_1. Suppose for each $j \leq n-1$ and $e \in \{0,1\}^j$, we have constructed a bijection Ψ_e on S_j such that $M_e(\sigma) = \mathrm{maj}(\Psi_e(\sigma))$.

For any $e \in \{0,1\}^n$, by Lemma 4.4, we may assume the rightmost zero in e is at the position $n-k-1$, i.e., $e = (e_1, \cdots, e_{n-k-2}, 0, 1, \cdots, 1)$ with $k \geq 0$. For any $\sigma \in S_n$, we have $\mathscr{C}^{k+1}(\sigma) \in S_{n-k-1}$ and

$$M(\mathscr{C}^{k+1}(\sigma)) = (s_1(\sigma), s_2(\sigma), \cdots, s_{n-k-1}(\sigma)).$$

Let $\bar{e} = (e_1, \cdots, e_{n-k-1})$. By the inductive hypothesis, there exists a bijection $\Psi_{\bar{e}}$ on S_{n-k-1} such that $M_{\bar{e}}(\mathscr{C}^{k+1}(\sigma)) = \mathrm{maj}(\Psi_{\bar{e}}(\mathscr{C}^{k+1}(\sigma)))$. Now define

$$\Psi_e(\sigma) = (C^{s'_n})^{-1}\Big[\cdots\big((C^{s'_{n-k}})^{-1}\big[\Psi_{\bar{e}}(\mathscr{C}^{k+1}(\sigma))\big]\cdot s'_{n-k}\big)\cdots\Big]\cdot s'_n,$$

where, for $n-k \leq i \leq n$, $s'_i = i - s_i(\sigma)$ is the last element of $\mathscr{C}^{n-i}(\sigma)$.

By the definition of Ψ_e and Corollary 2.2, we see that

$$(s_{n-k}(\Psi_e(\sigma)), \cdots, s_n(\Psi_e(\sigma))) = (s_{n-k}(\sigma), \cdots, s_n(\sigma)).$$

Moreover,

$$\begin{aligned}
s_1(\Psi_e(\sigma)) + \cdots + s_{n-k-1}(\Psi_e(\sigma)) &= \mathrm{maj}(\Psi_{\bar{e}}(\mathscr{C}^{k+1}(\sigma))) \\
&= M_{\bar{e}}(\mathscr{C}^{k+1}(\sigma)) \\
&= u_1(\sigma) + \cdots + u_{n-k-1}(\sigma).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_e(\sigma) &= u_1(\sigma) + \cdots + u_{n-k-2}(\sigma) + u_{n-k-1}(\sigma) + s_{n-k}(\sigma) + \cdots + s_n(\sigma) \\
&= M_{\bar{e}}(\mathscr{C}^{k+1}(\sigma)) + s_{n-k}(\Psi_e(\sigma)) + \cdots + s_n(\Psi_e(\sigma)) \\
&= s_1(\Psi_e(\sigma)) + \cdots + s_{n-k-1}(\Psi_e(\sigma)) + s_{n-k}(\Psi_e(\sigma)) + \cdots + s_n(\Psi_e(\sigma)) \\
&= \mathrm{maj}(\Psi_e(\sigma)),
\end{aligned}$$

where u_i is given by (4.6) for the vector e. This completes the proof. ∎

Take the same permutation $\sigma = 392648517$ as in the previous example in Table 1. The calculation of $H_4(\sigma)$ in Lemma 4.4 can be illustrated as follows. First we have

$$\begin{aligned}
(s_1(\sigma), \cdots, s_n(\sigma)) &= (0, 0, 1, 3, 1, 4, 3, 5, 2), \\
(s_1', s_2', \cdots, s_n') &= (1, 2, 2, 1, 4, 2, 4, 3, 7).
\end{aligned}$$

Then we compute

$$\begin{aligned}
H_4(\sigma) &= C_7^{-1}\{C_3^{-1}[C_4^{-1}(C_2^{-1}(31254)\cdot 2)\cdot 4]\cdot 3\}\cdot 7 \\
&= C_7^{-1}\{C_3^{-1}[C_4^{-1}(413652)\cdot 4]\cdot 3\}\cdot 7 \\
&= C_7^{-1}\{C_3^{-1}(5137624)\cdot 3\}\cdot 7 \\
&= C_7^{-1}\{61487253\}\cdot 7 \\
&= 614982537,
\end{aligned}$$

where $31254 = \mathscr{C}^4(\sigma)$, $M(614982537) = (0, 0, 1, 3, 2, 1, 5, 3, 2)$. Now one sees that Lemma 4.4 holds.

To conclude this paper, we remark that the statistic M_e coincides with the major index for the fixed points of Han's map H. To be precise, let σ be a fixed point of H. By Corollary 3.1, $s_i(\sigma) = i - 1$ or 0 for each i. Consequently, $s_{i-1} < s_i$ if and only if $s_i = i - 1$, and $s_{i-1} \geq s_i$ if and only if $s_i = 0$. Hence we have $s_i(\sigma) = r_i(\sigma)$ and $M_e(\sigma) = \mathrm{maj}(\sigma)$ for each $e \in \{0, 1\}^n$.

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] A. Björner and M.L. Wachs, Permutation statistics and linear extensions of posets, J. Combin. Theory Ser. A 58 (1991), 85–114.

[2] B. Clarke, A note on some Mahonian statistics, Sém. Lothar. Combin. 53 (2005), Article B53a.

[3] M. Denert, The genus zeta function of hereditary orders in central simple algebras over grobal fields, Math. Comp. 54 (1990), 449–465.

[4] A. Dzhumadil'daev, MacMahon's theorem for a set of permutations with given descent indices and right-maximal records, Electron. J. Combin. 17 (2010), R34.

[5] D. Foata. On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968), 236–240.

[6] D. Foata and D. Zeilberger, Denert's permutation statistics is indeed Euler-Mahonian, Studies in Applied Math. 83 (1990), 31–59.

[7] J. Haglund, The q, t-Catalan Numbers and the Space of Diagonal Harmonics, Amer. Math. Soc., Providence, Rhode Island, 2008.

[8] G.-N. Han, Une courte d'emonstration d'un résultat sur la $\mathcal{Z}$-statistique, C. R. Acad. Sci. Paris, Série I, 314 (1992), 969–971.

[9] G.-N. Han, Une transformation fondamentale sur les rearrangements de mots, Adv. Math. 105 (1994), 26–41.

[10] G.-N. Han, Euler-Mahonian triple set-valued statistics on permutations, European J. Combin. 29 (2008), 568–580.

[11] F. Hivert, J.C. Novelli, J.Y. Thibon, Multivariate generalizations of the Foata-Schützenberger equidistribution, DMTCS proc. AG, 2006, 289–300.

[12] P.A. MacMahon, The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects, Amer. J. Math. 35 (1913), 314–321.

[13] P.A. MacMahon, Two applications of general theorems in combinatory analysis, Proc. London Math. Soc. 15 (1916), 314–321

[14] M. Skandera, An Eulerian parnter for inversions, Sém. Lothar. Combin. 46 (2001), Article B46d.

[15] D. Zeilberger and D.M. Bressoud, A proof of Andrews q-Dyson conjecture, Discrete Math. 54 (1985), 201–224.