The Balanced Property of the q-Derangement Numbers and the q-Catalan Numbers

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Abstract

Based on Bóna's condition for the balanced property of the number of cycles of permutations, we give a general criterion for the balanced property in terms of the generating function of a statistic. We show that the q-derangement numbers and the q-Catalan numbers satisfy the balanced property.

Keywords: balanced property, major index, derangement, Catalan word

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1 Introduction

The notion of the balanced property was introduced by Bóna [7]. Let ξ be a statistic on a set Q_n of combinatorial objects, and let $\xi(k)$ be the number of objects π in Q_n such that $\xi(\pi) = k$. A statistic ξ over the set Q_n is said to possess the *balanced property* if for any $m \geq 2$ and $0 \leq r \leq m-1$,

$$\lim_{n\to\infty} \sum_k \frac{\xi(k)}{|Q_n|} = \frac{1}{m}, \tag{1.1}$$

where the sum ranges over all k congruent to r modulo m. We assume that m_n is the maximum value of ξ. Then the balanced property (1.1) can be expressed in terms of a finite sum

$$\lim_{n\to\infty} \frac{\xi(r) + \xi(m+r) + \cdots + \xi(pm+r)}{|Q_n|} = \frac{1}{m},$$

where $p = \lfloor (m_n - r)/m \rfloor$. In other words, the balanced property means the asymptotically uniform distribution of a statistic ξ modulo m.

Bóna [7] has shown that the balanced property (1.1) holds for both the number of cycles over the set of permutations of $[n] = \{1, 2, \ldots, n\}$, and the number of cycles over the set $\mathscr{D}_n$ of derangements of $[n]$. Furthermore, he proved that the number of cycles over $\mathscr{D}_n$ with each cycle having length at least a also satisfies the balanced property. In a subsequent paper [8], Bóna proved the balanced property of the number of parts over compositions of n.

The main objective of this paper is to find more combinatorial objects that satisfy the balanced property with respect to certain statistics. We begin with a general criterion for the balanced property in terms of the generating functions of the statistics. This criterion enables us to derive the balanced property of the major index over derangements of [n] and the major index over Catalan words of length $2n$, which are counted by the q-derangement numbers and the q-Catalan numbers, respectively. We also show that the flag major index over type B_n derangements satisfies the balanced property.

2 A general criterion

In this section, we give a formulation of a general criterion for the balanced property, which can be used to derive the balanced property of the major index over derangements of [n] and the major index over Catalan words of length $2n$. The criterion will be stated in terms of a statistic over a set Q_n. But it can also be rephrased in terms of the generating function of the statistic. While our criterion is based on a general setting, the proof is essentially the same as the proof of Bóna for the special case concerning the number of cycles of permutations. The proof is included for the sake of completeness.

Theorem 2.1. *Let ξ be a statistic on a set Q_n of combinatorial objects. The balanced property (1.1) holds if for any $1 \le j \le m-1$,*

$$\lim_{n\to\infty} \frac{f_n\left(\omega^j\right)}{|Q_n|} = 0, \tag{2.1}$$

where $\omega = e^{2\pi i/m}$ and

$$f_n(q) = \sum_{\pi\in Q_n} q^{\xi(\pi)}$$

is the generating function of the statistic ξ.

Proof. Let

$$T_n = \sum_{0\le j\le m-1} f_n(\omega^j)\omega^{-jr}. \tag{2.2}$$

Recall that $\xi(k)$ denotes the number of elements π in Q_n such that $\xi(\pi) = k$. So the generating function $f_n(x)$ can be written as

$$f_n(q) = \sum_k \xi(k) q^k, \tag{2.3}$$

where the sum ranges over all k congruent to r modulo m. Substituting (2.3) into (2.2), we obtain that

$$T_n = \sum_k \xi(k) \sum_{0\le j\le m-1} \omega^{(k-r)j}. \tag{2.4}$$

Since

$$\sum_{0\le j\le m-1} \omega^{(k-r)j} = \begin{cases} m, & \text{if } m|(k-r); \\ 0, & \text{else,} \end{cases} \tag{2.5}$$

the double sum (2.4) simplifies to

$$T_n = m\sum_k \xi(k).$$

Consequently, the balanced property (1.1) can be recast as

$$\lim_{n\to\infty} \frac{T_n}{|Q_n|} = 1. \tag{2.6}$$

Now, in the expression (2.2) of T_n, the summand $f_n(\omega^j)\omega^{-jr}$ for $j = 0$ is $f_n(1) = |Q_n|$. Thus

$$T_n = |Q_n| + \sum_{1\le j\le m-1} f_n(\omega^j)\omega^{-jr}. \tag{2.7}$$

Substituting (2.7) into (2.6), we conclude that the the balanced property holds if and only if

$$\lim_{n\to\infty} \sum_{1\le j\le m-1} \frac{f_n(\omega^j)}{|Q_n|}\omega^{-jr} = 0. \tag{2.8}$$

Evidently, the condition (2.1) implies (2.8) since $\left|\omega^{-jr}\right| = 1$. This completes the proof. ∎

3 The q-derangement numbers

In this section, we shall show that the major index over derangements of $[n]$ satisfies the balanced property. Let S_n be the set of permutations of the set $[n]$. The major index of a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ is defined to be the sum of the indices i such that $\pi_i > \pi_{i+1}$, that is,

$$\mathrm{maj}(\pi) = \sum_{\pi_i > \pi_{i+1}} i.$$

Denote by $\mathrm{maj}(k)$ the number of permutations of $[n]$ with major index k.

The problem on the balanced property of the major index over permutations has been considered by Gordon [14] and Roselle [15]. For any coprime numbers $k, l \le n$, the number of permutations π of $[n]$ such that $\mathrm{maj}(\pi)$ is congruent to i modulo k and $\mathrm{maj}(\pi^{-1})$ is congruent to j modulo l equals $n!/(kl)$, which is independent of i and j. To be more specific,

$$\left|\{\pi \mid \mathrm{maj}(\pi) \equiv i \pmod{k},\ \mathrm{maj}\left(\pi^{-1}\right) \equiv j \pmod{l}\}\right| = \frac{n!}{kl}. \tag{3.1}$$

Taking $l = 1$, the formula (3.1) specializes to the fact that the number of permutations of $[n]$ with major index i modulo k equals $n!/k$. In other words, for any $n \ge m$ and any $0 \le i \le m-1$, we have

$$\sum_k \frac{\mathrm{maj}(k)}{n!} = \frac{1}{m}, \tag{3.2}$$

where the sum ranges over all k congruent i modulo m. As noted in [5], the relation (3.1) was implicit in Gordon [14] and has been made explicit by Roselle [15]. When l divides $n-1$ and k divides n, a combinatorial proof has been given by Barcelo, Maule and Sundaram [5]. A more detailed description of the background on the relation (3.1) can also be found in [5]. The identity (3.2) can be viewed as an exact balanced property in comparison with the balanced property in the asymptotic sense. Moreover, (3.1) can be considered as a bivariate version of the exact balanced property.

Using representations of the symmetric group, Barcelo and Sundaram [6, Theorem 2.6] have obtained (3.2) for the special case $m = n$. They also gave a bijective proof in this case. Recently, Barcelo, Sagan, and Sundaram [9] gave a combinatorial interpretation of (3.1) in the general case by using shuffles of permutations.

We note that our proof of Theorem (2.1) easily applies to the exact balanced property (3.2). In general, we say a statistic ξ processes the *exact balanced property* if for any $0 \le r \le m-1$ and $n \ge N$,

$$\sum_k \frac{\xi(k)}{|Q_n|} = \frac{1}{m},$$

where the sum ranges over all k congruent to r modulo m, and N depends only on m. Inspecting the derivation of the formula (2.8), we see that the exact balanced property holds if and only if

$$\sum_{1\le j\le m-1} \frac{f_n\left(\omega^j\right)}{|Q_n|}\omega^{-jr} = 0. \tag{3.3}$$

In the usual notation $[0]_q! = 1$, $[n]_q! = [1]_q[2]_q\cdots[n]_q$, where $[n]_q = 1 + q + \cdots + q^{n-1}$ for $n \ge 1$, the generating function for the major index of permutations of $[n]$ is known to be

$$f_n(q) = \sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)} = [n]_q!,$$

see Andrews [2]. Since for $n \ge m$, $f_n(q)$ contains the factor $[m]_q$, $f_n\left(\omega^j\right)$ contains the factor $[m]_{\omega^j}$. It follows that $f_n\left(\omega^j\right)$ equals zero for any $1 \le j \le m-1$. Therefore the relation (3.3) holds, which implies the exact balanced property (3.2).

Moreover, it is not difficult to derive the balanced property and the exact balanced property of the flag major index over permutations of type B_n, which is introduced by Adin and Roichman [4]. Denote by S_n^B the set of B_n-permutations. Then the generating function for the flag major index over S_n^B is given by

$$f_n^B(q) = \sum_{\pi\in S_n^B} q^{\mathrm{fmaj}(\pi)} = [2]_q[4]_q\cdots[2n]_q,$$

see Chow [11]. For $n \ge 2m-1$, $f_n^B(q)$ contains the factor $[2m]_q$. It follows from (2.5) that $f_n^B\left(\omega^j\right) = 0$. This yields (3.3), leading to the exact balanced property of the flag major index over B_n-permutations.

A derangement of $[n]$ is a permutation $\pi_1\pi_2\cdots\pi_n$ of $[n]$ such that $\pi_i \ne i$ for all $1 \le i \le n$. The following counting formula of derangements with respect to the major index was given by Gessel and published later in [13],

$$d_n(q) = \sum_{\pi\in\mathscr{D}_n} q^{\mathrm{maj}(\pi)} = \sum_{0\le k\le n} (-1)^k q^{\binom{k}{2}} \prod_{k+1\le t\le n} [t]_q, \tag{3.4}$$

where $[0]_q! = 1$. Combinatorial proofs for (3.4) has been found by Wachs [16], and Chen and Xu [10].

Theorem 3.1. *The major index over derangements of* $[n]$ *satisfies the balanced property. In other words, for any* $0 \le r \le m-1$, *we have*

$$\lim_{n\to\infty} \frac{\mathrm{maj}(r) + \mathrm{maj}(m+r) + \cdots + \mathrm{maj}(pm+r)}{|\mathscr{D}_n|} = \frac{1}{m},$$

where

$$p = \left\lfloor \frac{\binom{n}{2} - r}{m} \right\rfloor.$$

Proof. Consider the values of $d_n(q)$ evaluated at $q = \omega^j$, namely,

$$d_n\left(\omega^j\right) = \sum_{0\le k\le n} (-1)^k \omega^{j\binom{k}{2}} \prod_{k+1\le t\le n} [t]_{\omega^j}. \tag{3.5}$$

Note that among the m consecutive integers $n, n-1, \ldots, n-m+1$, there exists an integer a which can be divided by m. For such a choice of a, we have $[a]_{\omega^j} = 0$. So any summand in (3.5) containing the factor $[a]_{\omega^j}$ can be ignored. Consequently, the formula (3.5) reduces to the following form

$$d_n\left(\omega^j\right) = \sum_{n-m+2\le k\le n} (-1)^k \omega^{j\binom{k}{2}} \prod_{k+1\le t\le n} [t]_{\omega^j}. \tag{3.6}$$

In order to estimate $[t]_{\omega^j}$, let us compute $[t]_{\omega^j}^2$. For any $t \in \{k+1, k+2, \ldots, n\}$, we have

$$|[t]_{\omega^j}|^2 = \left|\frac{1-\omega^{tj}}{1-\omega^j}\right|^2 = \frac{1-\cos(2\pi tj/m)}{1-\cos(2\pi j/m)}, \tag{3.7}$$

which is clearly bounded by $2/c$, where

$$c = \min\{1 - \cos(2\pi j/m) \,|\, 1 \le j \le m-1\}.$$

It is clear that $2/c \ge 1$. Hence,

$$\prod_{k+1\le t\le n} |[t]_{\omega^j}|^2 \le \left(\frac{2}{c}\right)^{n-k} \le \left(\frac{2}{c}\right)^{m-2}.$$

Observe that the above estimate is independent of j. It follows from (3.6) that

$$\left|d_n\left(\omega^j\right)\right| \le \sum_{n-m+2\le k\le n} \left|(-1)^k \omega^{j\binom{k}{2}} \prod_{k+1\le t\le n} [t]_{\omega^j}\right| \le (m-1)\left(\frac{2}{c}\right)^{(m-2)/2}.$$

Thus $d_n\left(\omega^j\right)$ is bounded by a constant. By Theorem 2.1, the major index over $\mathscr{D}_n$ satisfies the balanced property. This completes the proof. ∎

For the type B case, denote by $\mathscr{D}_n^B$ the set of derangements of type B_n. Adin and Roichman [4] have shown that the generating function of the flag major index over $\mathscr{D}_n^B$ equals

$$\sum_{\pi\in\mathscr{D}_n^B} q^{\mathrm{fmaj}(\pi)} = \sum_{0\le k\le n} (-1)^k q^{k(k-1)} [2n]_q [2n-2]_q \cdots [2k+2]_q,$$

see also Chow [11], and Adin, Brenti and Roichman [3]. By an argument very similar to the proof of Theorem 3.1, we can derive the balanced property of the flag major index over B_n-derangements.

4 The q-Catalan numbers

In this section, we shall derive the balanced property of the major index over Catalan words of length $2n$. The q-Catalan numbers are defined by

$$C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \prod_{2 \le t \le n} \frac{[t+n]_q}{[t]_q}, \tag{4.1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

see, for example, Andrews [1], and Fürlinger and Hofbauer [12]. A combinatorial interpretation of $C_n(q)$ in term of the major index of the Catalan words of length $2n$ has been given in [12]. A *Catalan word* w of length $2n$ is a sequence consisting of n 0's and n 1's such that no prefix contains more 1's than 0's. Denote the set of Catalan words of length $2n$ by $\mathscr{C}_n$. The *major index* for a Catalan word $w = w_1 w_2 \cdots w_{2n} \in \mathscr{C}_n$ is defined by

$$\operatorname{maj}(w) = \sum_{w_i > w_{i+1}} i.$$

Let $\operatorname{maj}(k)$ be the number of Catalan words of length $2n$ with major index k. Fürlinger and Hofbauer have shown that

$$C_n(q) = \sum_{w \in \mathscr{C}_n} q^{\operatorname{maj}(w)}.$$

The number of Catalan words of length $2n$ is given by the Catalan number

$$C_n(1) = \frac{1}{n+1} \binom{2n}{n}. \tag{4.2}$$

Theorem 4.1. *The major index over Catalan words of length $2n$ satisfies the balanced property. In other words, for any $0 \le r \le m-1$, we have*

$$\lim_{n \to \infty} \frac{\operatorname{maj}(r) + \operatorname{maj}(m+r) + \cdots + \operatorname{maj}(pm+r)}{C_n(1)} = \frac{1}{m},$$

where

$$p = \left\lfloor \frac{n(2n-1) - r}{m} \right\rfloor.$$

Proof. By Theorem 2.1, it suffices to show that for any $0 \le j \le m-1$,

$$\lim_{n \to \infty} \frac{C_n\left(\omega^j\right)}{C_n(1)} = 0.$$

Let $1 \le j \le m-1$. Suppose that $j/m = u/v$, where u and v are coprime positive integers with $2 \le v \le m$. Write $n = lv + s$, where $0 \le s \le v-1$. Denote the denominator of (4.1) by $D_n(q)$, namely,

$$D_n(q) = \prod_{2 \le t \le n} [t]_q = [n]_q!.$$

It should not be overlooked that the denominator $D_n(q)$ vanishes when q is set to ω^j. In fact, since $\omega^{jv} = e^{2\pi i jv/m} = e^{2\pi i u} = 1$, we have $1 - \omega^{jt} = 0$ for any $v|t$. More precisely, $D_n(q)$ contains the factor

$$F_n(q) = \prod_{1 \le k \le l} [kv]_q = [v]_q^l \cdot [l]_{q^v}!,$$

in which the factor $[v]_q^l$ causes $D_n(q)$ to vanish when evaluated at $q = \omega^j$. We proceed to represent $C_n(q)$ as a quotient whose denominator does not vanish for $q = \omega^j$.

If $0 \le s \le v-2$, let

$$A_n(q) = \frac{\prod_{l+1 \le k \le 2l} [kv]_q}{F_n(q)}.$$

Canceling the common factor $[v]_q^l$ in the numerator and the denominator, we can reduce it to the form

$$A_n(q) = \begin{bmatrix} 2l \\ l \end{bmatrix}_{q^v}.$$

Denote the quotient $C_n(q)/A_n(q)$ by $B_n(q)$. It can be checked that

$$B_n(q) = \left(\prod_{\substack{2 \le t \le n \\ t \notin \{v-s, 2v-s, \ldots, lv-s\}}} [t+n]_q \right) \left(\prod_{\substack{2 \le t \le n \\ v \nmid t}} [t]_q \right)^{-1}. \tag{4.3}$$

Clearly, the denominator of $B_n(q)$ does not vanish for $q = \omega^j$.

For the case $s = v-1$, let

$$U_n(q) = \frac{\prod_{l+1 \le k \le 2l+1} [kv]_q}{F_n(q)}.$$

Similarly, $U_n(q)$ can be reduced to the following form

$$U_n(q) = \begin{bmatrix} 2l+1 \\ l \end{bmatrix}_{q^v}, \tag{4.4}$$

where the denominator does not vanish for $q = \omega^j$. Moreover, we see that $V_n(q) = C_n(q)/U_n(q)$ has the following representation

$$V_n(q) = \left(\prod_{\substack{2 \le t \le n \\ t \notin \{2v-s, \ldots, lv-s, (l+1)v-s\}}} [t+n]_q \right) \left(\prod_{\substack{2 \le t \le n \\ v \nmid t}} [t]_q \right)^{-1}.$$

Again, the denominator of $V_n(q)$ is nonzero when $q = \omega^j$.

We are now ready to give an estimate of $\left| C_n\left(\omega^j\right) \right|$. First, consider the case $0 \le s \le v-2$. Since $\omega^{jv} = 1$, we have

$$A_n\left(\omega^j\right) = \binom{2l}{l}, \tag{4.5}$$

$$B_n\left(\omega^j\right) = \left(\prod_{\substack{2 \le t \le n \\ t \notin \{v-s, 2v-s, \ldots, lv-s\}}} [t+s]_{\omega^j} \right) \left(\prod_{\substack{2 \le t \le n \\ v \nmid t}} [t]_{\omega^j} \right)^{-1}. \tag{4.6}$$

In order to give an estimate for $\left|B_n\left(\omega^j\right)\right|$, it is necessary to reduce the above expression to a quotient such that the numbers of factors of the numerator and the denominator are both finite as n tends infinity. It is easy to verify that $B_n\left(\omega^j\right)=1$ for $s=0$ and $s=1$. By (4.6), we have

$$B_n\left(\omega^j\right)=\left([v-1]_{\omega^j}!^l \prod_{s+2\le t\le 2s}[t]_{\omega^j}\right)\left([v-1]_{\omega^j}!^l \prod_{2\le t\le s}[t]_{\omega^j}\right)^{-1}=\prod_{2\le t\le s}\frac{[t+s]_{\omega^j}}{[t]_{\omega^j}}.$$

In view of (3.7), we see that

$$\prod_{2\le t\le s}\frac{\left|[t+s]_{\omega^j}\right|^2}{\left|[t]_{\omega^j}\right|^2}=\prod_{2\le t\le s}\frac{1-\cos(2\pi(t+s)j/m)}{1-\cos(2\pi tj/m)}\le\left(\frac{2}{c}\right)^{s-1},$$

where

$$c=\min\{1-\cos(2\pi tj/m)\,|\,1\le j\le m-1,\ 2\le t\le m-2\}.$$

Consequently, $\left|B_n\left(\omega^j\right)\right|$ is bounded by a constant, say, c_1. By (4.5), for $0\le s\le v-2$, we get

$$\left|C_n\left(\omega^j\right)\right|\le c_1\binom{2l}{l}. \tag{4.7}$$

For the case $s=v-1$, substituting $q=\omega^j$ in (4.4), we obtain that

$$U_n\left(\omega^j\right)=\binom{2l+1}{l}. \tag{4.8}$$

On the other hand, by an analogous argument to the case $s\le v-2$, it can be shown that $\left|V_n\left(\omega^j\right)\right|$ is also bounded by a constant, say c_2. It follows from (4.8) that for $s=v-1$,

$$\left|C_n\left(\omega^j\right)\right|\le c_2\binom{2l+1}{l}. \tag{4.9}$$

Up to now, we have obtained the estimates for the $\left|C_n\left(\omega^j\right)\right|$ in the above two cases, namely, (4.7) for $0\le s\le v-2$, and (4.9) for $s=v-1$. By (4.2), we can derive the following general upper bound

$$\frac{\left|C_n\left(\omega^j\right)\right|}{C_n(1)}\le c_3\cdot\frac{(n+1)\binom{2l+1}{l}}{\binom{2n}{n}}, \tag{4.10}$$

where $c_3=\max(c_1,c_2)$. Based on Stirling's formula, the central binomial coefficient can be estimated as follows

$$\binom{2n}{n}\sim\frac{2^{2n}}{\sqrt{\pi n}},\qquad \text{as } n\to\infty. \tag{4.11}$$

Since $l=(n-s)/v$ tends to infinity as $n\to\infty$, we have

$$\binom{2l+1}{l}\sim 2\binom{2l}{l}\sim\frac{2^{2l+1}}{\sqrt{\pi l}},\qquad \text{as } n\to\infty. \tag{4.12}$$

Combining (4.10), (4.11) and (4.12), we find that

$$\lim_{n\to\infty}\frac{\left|C_n\left(\omega^j\right)\right|}{C_n(1)}\le\lim_{n\to\infty}c_3\cdot\sqrt{\frac{n}{l}}\cdot\frac{(n+1)}{2^{2n-2l-1}}=0.$$

Thus the balanced property follows from Theorem 2.1. This completes the proof. ∎

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