Lattice Paths and the Flagged Cauchy Determinant

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Abstract. We obtain a flagged form of the Cauchy determinant and establish a correspondence between this determinant and nonintersecting lattice paths, from which it follows that Cauchy identity on Schur functions. By choosing different origins and destinations for the lattice paths, we are led to an identity of Gessel on the Cauchy sum of Schur functions in terms of the complete symmetric functions in the full variable sets. The algebraic proof of this equivalence involves the Cauchy-Binet formula and mutli-Schur functions based on the complete super symmetric function. We also present an evaluation of the Cauchy determinant by the Jacobi symmetrizer.

Keywords: Divided difference, Cauchy theorem, flagged Cauchy determinant, multi-Schur function, lattice paths, Jacobi symmetrizer.

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1. Introduction

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two sets of variables, and $s_\lambda(X)$ and $s_\lambda(Y)$ be the Schur functions indexed by a partition λ. Then the classical Cauchy identity on Schur functions is stated as follows:

Theorem 1.1 *For $n \geq 1$, we have*

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(X) s_\lambda(Y), \tag{1.1}$$

where the sum ranges over all partitions with length $\leq n$.

The classical treatments of Theorem 1.1 include the Robinson-Schensted-Kunth correspondence and the Cauchy-Binet formula [18, 19]. There is also a derivation based on matrix product involving the elementary symmetric functions as given in Macdonald [18, p. 67]. This paper aims to establish a connection between the Cauchy identity and the lattice path method due to Gessel-Viennot [8, 9]. The key ingredient in our lattice path construction is a flagged form of the Cauchy determinant with respect to the variable sets. Recall the Cauchy determinant on X and Y:

$$\left|\frac{1}{1-x_iy_j}\right|_{n\times n}.$$

We follow the common notation for the Vandermonde determinant:

$$\Delta(X) = \left|x_i^{n-j}\right|_{n\times n} = \prod_{1\leq i<j\leq n}(x_i - x_j).$$

Symmetrizing operators have been used for the construction of symmetric functions. In this paper, we show that the Jacobi symmetrizer [16], also called the total symmetrizer in a slightly different version by Lascoux and Pragacz [15], can be efficiently used to compute the Cauchy determinant.

Definition 1.2 (Jacobi Symmetrizer) *For any polynomial $f(X)$, the Jacobi symmetrizer ∂ is given by*

$$f(X)\partial = \frac{1}{\Delta(X)}\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)f^{\sigma}(X),$$

where S_n is the set of permutations on $\{1,2,\ldots,n\}$, and $\mathrm{sign}(\sigma)$ *is the sign of σ.*

The operator ∂ can also be defined by the divided difference operators. Let ∂_{x_i,x_j} be the divided difference operator given by

$$f(x_1,\ldots,x_n)\partial_{x_i,x_j} = \frac{f(x_1,\ldots,x_n) - f(\ldots,x_j,\ldots,x_i,\ldots)}{x_i - x_j}.$$

We follow the notation of Lascoux [16] to write the operators to the right of a function. Usually we denote $\partial_{x_i,x_{i+1}}$ by ∂_i. Then the Jacobi symmetrizer ∂ (see Lascoux [16]) can be expressed as:

$$\partial = (\partial_{n-1})\cdot(\partial_{n-2}\partial_{n-1})\cdots(\partial_1\partial_2\cdots\partial_{n-1}).$$

Notice that the operators are applied from left to right.

The equivalence of these two definitions of the Jacobi symmetrizer may be verified in several ways. For example, it follows from the basis theorem of the module of polynomials in X over symmetric functions. The Jacobi symmetrizer can also be represented as an integral as given by Bernstein-Gelfand-Gelfand [3] and Demazure [5], and restated by Tamvakis [21].

This paper contains the following results:

1. We obtain the flagged Cauchy determinant:

$$F(X,Y) = \left| \sum_k h_{k-n+i}(x_i, \ldots, x_n) h_{k-n+j}(y_j, \ldots, y_n) \right|_{n\times n}, \tag{1.2}$$

   where h_i are the complete symmetric functions.

2. Using the Jacobi symmetrizer, we derive the classical formula on Cauchy determinant:

$$\left| \frac{1}{1-x_iy_j} \right|_{n\times n} = \Delta(X)\,\Delta(Y) \prod_{i,j=1}^{n} \frac{1}{1-x_iy_j}. \tag{1.3}$$

3. We obtain a lattice path evaluation of the flagged Cauchy determinant and a correspondence with Young tableaux, leading to the Cauchy theorem.

4. Choosing different origins and destinations, we obtain the equivalence of the flagged Cauchy determinant and a determinant in the complete symmetric functions in the full variable sets X and Y:

$$\left| \sum_k h_{k-n+i}(X) h_{k-n+j}(Y) \right|_{n\times n}.$$

   Notice that the above formula has the same indices as in the flagged formula (1.2). This leads to the following identity of Gessel [7]:

   Theorem 1.3 *We have*

$$\left| \sum_k h_{k-n+i}(X) h_{k-n+j}(Y) \right|_{n\times n} = \sum_\lambda s_\lambda(X) s_\lambda(Y). \tag{1.4}$$

To conclude this section, we note that the idea of flagged Schur functions and mutli-Schur functions has proved to be very efficient in the study of Schubert polynomials in connection with divided difference operators (see Lascoux [16] and Wachs [22]). Flagged determinants with respect to the variable sets can also be used to give simple character formulas for the symplectic groups and the orthogonal groups, see Chen-Li-Louck [4] and Hamel-King [14].

2. The Jacobi Symmetrizer

In this section, we give an evaluation of the Cauchy determinant by using the Jacobi symmetrizer. Roughly speaking, the Jacobi symmetrizer plays an analogous role as the Cauchy-Binet formula as far as the computation is concerned. First we observe the action of the Jacobi symmetrizer on a monomial:

$$x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}\,\partial=\frac{1}{\Delta(X)}\left|x_j^{k_i}\right|_{n\times n}. \tag{2.5}$$

In particular, the Schur function $s_\lambda(X)$ can be written as

$$x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\cdots x_n^{\lambda_n}\partial.$$

The Jacobi symmetrizer has the vanishing property: $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}\,\partial=0$ if any two exponents k_i and k_j are equal for $i\neq j$. Moreover, if $[k_1,k_2,\ldots,k_n]$ is a permutation of $[n-1,n-2,\ldots,0]$, then $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}\,\partial$ equals the sign of $[k_1,k_2,\ldots,k_n]$ with respect to the permutation $[n-1,n-2,\ldots,1,0]$.

We are now ready to present a proof of (1.3) by using the Jacobi symmetrizer. We use $X^{(i)}$ to denote the variable set $\{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\}$. Then we have

$$\left|\frac{1}{1-x_iy_j}\right|_{n\times n}=\prod_{i,j=1}^{n}\frac{1}{1-x_iy_j}\left|\prod_{k\neq i}(1-x_ky_j)\right|_{n\times n}.$$

The determinant on the right hand side of the above equation can be expanded as:

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k e_k(X^{(i)})y_j^k\right|_{n\times n}\\&=\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\prod_{i=1}^{n}\left(\sum_{k_i=0}^{n-1}(-1)^{k_i}e_{k_i}(X^{(i)})y_{\sigma_i}^{k_i}\right),\end{aligned}\tag{2.6}$$

where σ ranges over all permutations on $\{1,2,\ldots,n\}$. Using the Jacobi symmetrizer ∂_Y acting on the variable set Y , we may rewrite (2.6) as

$$\begin{aligned}&\prod_{i=1}^{n}\left(\sum_{k_i=0}^{n-1}(-1)^{k_i}e_{k_i}(X^{(i)})y_i^{k_i}\right)\partial_Y\,\Delta(Y)\\&=\sum_{0\leq k_1,k_2,\ldots,k_n\leq n-1}\left(\prod_{i=1}^{n}(-1)^{k_i}e_{k_i}(X^{(i)})y_i^{k_i}\right)\partial_Y\,\Delta(Y).\end{aligned}\tag{2.7}$$

From the vanishing property of ∂_Y it follows that (2.7) can be expressed as a summation over permutations of $[n-1, n-2, \ldots, 1, 0]$:

$$\Delta(Y) \sum_{\tau} \operatorname{sign}(\tau) \prod_{i=1}^{n} (-1)^{\tau_i} e_{\tau_i}(X^{(i)}), \tag{2.8}$$

where τ ranges over all permutations of $[n-1, n-2, \ldots, 0]$.

Using the Jacobi symmetrizer ∂_X for the variable set X, we may write the summation in (2.8) as

$$\left(\prod_{i=1}^{n} (-1)^{n-i} e_{n-i}(X^{(i)}) \right) \partial_X \, \Delta(X). \tag{2.9}$$

By the vanishing property of ∂_X, in the expansion of the above product in (2.9) we only need to consider the terms $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, where $[k_1, k_2, \ldots, k_n]$ are permutations of $[n-1, n-2, \ldots, 1, 0]$. Since x_n is the only variable that appears in $e_{n-i}(X^{(i)})$ for every $i \leq n-1$, we have to choose $k_n = n-1$. Moreover, $[k_1, k_2 \ldots, k_{n-1}]$ has to be a permutation of $[n-2, n-3, \ldots, 1, 0]$ and $x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}}$ has to come from the expansion of

$$\prod_{i=1}^{n-1} (-1)^{n-i-1} e_{n-i-1}(X^{(i,n)}), \tag{2.10}$$

where $X^{(i,n)} = X \backslash \{x_i, x_n\}$. Iterating this procedure, we reach the conclusion that the monomial $x_2 x_3^2 \cdots x_n^{n-1}$ is obtained by choosing $x_2 x_3 \cdots x_n$ from $e_{n-1}(X^{(1)})$, $x_3 x_4 \cdots x_n$ from $e_{n-2}(X^{(2)}$, ..., and x_n from $e_1(X^{(n-1)})$. Since $x_2 x_3^2 \cdots x_n^{n-1} \partial_X = \prod_{i=1}^{n} (-1)^{n-i}$, we obtain

$$\prod_{i=1}^{n} (-1)^{n-i} e_{n-i}(X^{(i)}) \partial_X = 1.$$

Keeping track of the computation, we get the desired formula. ∎

We remark that in the above proof we only used the definition of the Jacobi symmetrizer without resort to divided difference operators.

3. The Flagged Cauchy Determinant

Let $h_k(x_i, x_{i+1}, \ldots, x_n)$ be the complete symmetric function in $x_i, x_{i+1}, \ldots, x_n$. Then we may transform the Cauchy determinant into a flagged form with respect to the variable sets X and Y.

Theorem 3.1 *We have*

$$\left|\frac{1}{1-x_iy_j}\right|_{n\times n} = \Delta(X)\cdot\Delta(Y)\cdot F(X,Y), \tag{3.11}$$

where $F(X,Y)$ denotes the determinant as in (1.2), namely,

$$F(X,Y) = \left|\sum_k h_{k-n+i}(x_i,\ldots,x_n)h_{k-n+j}(y_j,\ldots,y_n)\right|_{n\times n}.$$

Proof. First, we express the (i,j)-entry in the Cauchy determinant as

$$\frac{1}{1-x_iy_j} = \sum_{k\geq 0}(x_iy_j)^k = \sum_{k\geq 0} h_k(x_i)h_k(y_j).$$

We recall the divided difference property of the complete symmetric functions:

$$\frac{h_k(x_i,\ldots,x_j) - h_k(x_{i+1},\ldots,x_{j+1})}{x_i - x_{j+1}} = h_{k-1}(x_i,\ldots,x_{j+1}).$$

Subtracting the $(i+1)$-th row from the i-th row and dividing by $(x_i - x_{i+1})$ for $i = 1,2,\ldots,n-1$, we get the determinant

$$\begin{vmatrix} \sum_k h_{k-1}(x_1,x_2)h_k(y_1) & \sum_k h_{k-1}(x_1,x_2)h_k(y_2) & \cdots & \sum_k h_{k-1}(x_1,x_2)h_k(y_n) \\ \sum_k h_{k-1}(x_2,x_3)h_k(y_1) & \sum_k h_{k-1}(x_2,x_3)h_k(y_2) & \cdots & \sum_k h_{k-1}(x_2,x_3)h_k(y_n) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_k h_k(x_n)h_k(y_1) & \sum_k h_k(x_n)h_k(y_2) & \cdots & \sum_k h_k(x_n)h_k(y_n) \end{vmatrix}.$$

Subtracting the $(i+1)$-th row from the i-th row and dividing by $(x_i - x_{i+2})$ for $i = 1,2,\ldots,n-2$, then subtracting the $(i+1)$-th row from the i-th row and dividing by $(x_i - x_{i+3})$ for $i = 1,2,\ldots,n-3$, ... Eventually, we obtain the determinant

$$\left|\sum_k h_{k-n+i}(x_i,\ldots,x_n)h_k(y_j)\right|_{n\times n}.$$

Applying analogous operations to the columns of the above determinant: Subtracting the $(i+1)$-th column from the i-th column and dividing by $(y_i - y_{i+1})$ for $i = 1,2,\ldots,n-1$, then subtracting the $(i+1)$-th column from the i-th column and dividing by $(y_i - y_{i+2})$ for $i = 1,2,\ldots,n-2$, and so on, we get a flagged determinant with respect to the complete functions in X and Y:

$$\left|\sum_k h_{k-n+i}(x_i,\ldots,x_n)h_{k-n+j}(y_j,\ldots,y_n)\right|_{n\times n}. \tag{3.12}$$

All the division operations yield the Vandermonde determinants $\Delta(X)$ and $\Delta(Y)$. This completes the proof. ∎

4. Lattice paths and the Cauchy identity

The lattice path method introduced by Gessel and Viennot [8, 9] has been widely used as a powerful technique for the study of symmetric functions, plane partitions and many combinatorial problems (see also [1, 2, 6, 10, 11, 12, 13, 20, 23]).

We first construct the underlying (lattice) digraph D as the integer lattice $\mathbb{Z} \times \mathbb{Z}$, where the arcs (or steps) are horizontal or vertical with the following requirements: if a vertical arc lies strictly to the left of the y-axis, it must be an up step from (i, j) to $(i, j+1)$; if a vertical edge lies strictly to the right of y-axis, then it must be a down step from (i, j) to $(i, j-1)$; and there are no vertical steps on the y-axis. A path in D is also called a lattice path, and a path is always meant to be in D.

The weights of arcs in D are given below:

1. A horizontal arc has weight 1.
2. For $i < 0$, a vertical arc from (i, j) to $(i, j+1)$ has weight x_{n+i-1}.
3. For $i > 0$, a vertical arc from (i, j) to $(i, j+1)$ has weight y_{n-i+1}.

The weight of a path P, denoted by $w(P)$, is defined as the product of the weights of the arcs on the path P. Given an n-tuple $(P_1, P_2, \ldots, P_n)$ of lattice paths, its weight is defined to be the product of the weights of P_i. We now suppose that $A_1, A_2, \ldots, A_n$ are the origins and $B_1, B_2, \ldots, B_n$ are the destinations. Let $\mathcal{P}(A_i, B_j)$ be the set of lattice paths from A_i to B_j in D. Similarly, we use $\mathcal{P}(A, B)$ to denote the set of all n-tuples $(P_1, P_2, \ldots, P_n)$ of lattice paths in D where P_i starts with A_i and ends with B_i. We also follow the notation $\mathcal{P}_0(A, B)$ for the set of all n-tuples $(P_1, P_2, \ldots, P_n)$ of nonintersecting lattice paths where P_i has origin A_i and destination B_i. By $\mathrm{GF}(\mathcal{P}(A, B))$ and $\mathrm{GF}(\mathcal{P}_0(A, B))$ we mean the generating functions, or the sums of weights, of the n-tuples of lattice paths in $\mathcal{P}(A, B)$ and $\mathcal{P}_0(A, B)$ respectively.

For the purpose of this paper, we choose

$$A_i = (i - n - 1, -i), \quad \text{and} \quad B_i = (n - i + 1, -i), \quad i = 1, 2, \ldots, n. \tag{4.13}$$

With the above choice, we have the following lattice path interpretation of the entries in the flagged Cauchy determinant.

Lemma 4.1 *The generating function for the D-paths from A_i to B_j equals*

$$\mathrm{GF}(\mathcal{P}(A_i, B_j)) = \sum_k h_{k-n+i}(x_i, \ldots, x_n) h_{k-n+j}(y_j, \ldots, y_n). \tag{4.14}$$

Proof. We classify the D-paths P from A_i to B_j by their intersection points with the y-axis. To be more specific, assume that P intersects with the y-axis at the point Q. We define the A-height (resp. B-height) of P as the difference between the y-coordinates of A_i (resp. B_j) and Q. First we consider the case $i \geq j$ and the family of paths P with A-height k ($k \geq 0$). Since there are no arcs on the y-axis, the weights of all such paths sum to

$$h_k(x_i, \ldots, x_n) h_{k+i-j}(y_j, \ldots, y_n).$$

Summing over k, one gets the right hand side of (4.14). The case $i < j$ can be treated in a similar manner. This completes the proof. ∎

By adapting the standard Gessel-Viennot argument, we may interpret the flagged Cauchy determinant by non-intersecting lattice paths.

Theorem 4.2 *We have the following relation:*

$$F(X, Y) = \mathrm{GF}(\mathcal{P}_0(A, B)). \tag{4.15}$$

Proof. From Lemma 4.1 it follows that

$$F(X, Y) = |\mathrm{GF}(\mathcal{P}(A_i, B_j))|_{n\times n} = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} \mathrm{GF}(\mathcal{P}(A_i, B_{\sigma_i})).$$

Suppose σ is a permutation in S_n and $(P_1, P_2, \ldots, P_n)$ is an n-tuple of lattice paths such that P_i has origin A_i and destination B_{σ_i}. We now consider the situation that some paths in $\{P_1, P_2, \ldots, P_n\}$ intersect. We need a total order on the vertices of D, say the lexicographic order. With respect to this order, we may find the minimum vertex Q among all the intersection vertices. Moreover, we choose two paths P_i and P_j such that i and j are the smallest and the nearest to the smallest. Let P_i' be the path consisting of the segment of P_i from A_i to Q and the segment of P_j from Q to B_{σ_j}, and P_j' be the path consisting of the segment of P_j from A_j to Q and the segment of P_i from Q to B_{σ_i}. Meanwhile, we set σ' to be the permutation by switching the elements σ_i and σ_j. Let $P_k' = P_k$ for each $k \neq i, j$. Then $(P_1', P_2', \ldots, P_n')$ has the same weight as $(P_1, P_2, \ldots, P_n)$ and the sign of σ' is the opposite to the sign of σ. Therefore, we have obtained a sign reversing involution. Note that the paths $(P_1, P_2, \ldots, P_n)$ do not intersect with each other only when the permutation σ is the identity, namely, $\sigma_i = i$. This completes the proof. ∎

Given the sets A and B of origins and destinations, we may translate an n-tuple $(P_1, P_2, \ldots, P_n)$ of nonintersecting lattice paths into a pair of Young tableaux of the same shape on $\{1, 2, \ldots, n\}$.

Theorem 4.3 *There is a one-to-one correspondence between n-tuples $(P_1, P_2, \ldots, P_n)$ of nonintersecting lattice paths with P_i going from A_i to B_i and pairs of Young tableaux of the same shape on $\{1, 2, \ldots, n\}$. Equivalently, we have*

$$\mathrm{GF}(\mathcal{P}_0(A, B)) = \sum_{\lambda, \ell(\lambda) \leq n} s_\lambda(X) s_\lambda(Y). \tag{4.16}$$

Proof. Given any n-tuple $(P_1, P_2, \ldots, P_n)$ of nonintersecting paths such that P_i is from A_i to B_i, let Q_i be the intersection point of P_i and the y-axis. We now cut each P_i into two segments U_i and V_i, where U_i goes from A_i to Q_i and V_i goes from Q_i to B_i. For the n-tuple $(U_1, U_2, \ldots, U_n)$ we can associate it with a tableau S on $\{1, 2, \ldots, n\}$. The i-th row of S is obtained from the path U_i by reading the weights on vertical steps. The column strictness of S is guaranteed by the nonintersecting property of $(U_1, U_2, \ldots, U_n)$. Similarly, the n-tuple $(V_1, V_2, \ldots, V_n)$ corresponds a tableau T on $\{1, 2, \ldots, n\}$. Thus the n-tuple $(P_1, P_2, \ldots, P_n)$ of nonintersecting lattice paths corresponds to a pair of tableaux (S, T) of the same shape. The above procedure is reversible. Hence we obtain a bijection. ∎

From the above correspondence and the evaluation of the flagged Cauchy determinant it follows the Cauchy identity (Theorem 1.1). The following Figures 4.1 and 4.2 illustrate the correspondence for n=4.

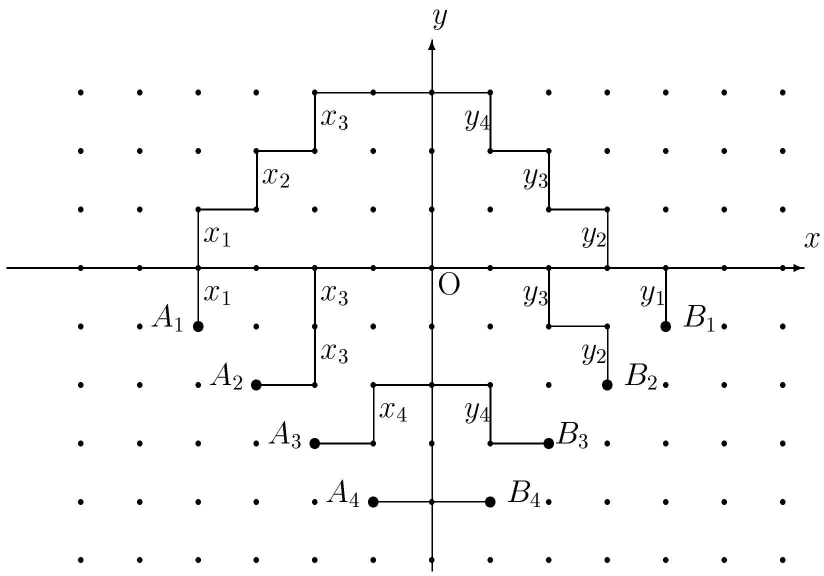

Figure 4.1 Nonintersecting paths from A_i to B_i

We now make an easy observation that enables one to write the flagged Cauchy determinant in the full variable sets X and Y. Let $A_i' = (-n, -i)$ and

$$\text{S}= \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 3&3& \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\ , \qquad \text{T}= \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 2&3& \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

Figure 4.2 A pair of tableaux of the same shape

$B'_i = (n, -i)$. It is clear that there is a one-to-one correspondence between n-tuples $(P_1, P_2, \ldots, P_n)$ of nonintersecting lattice paths with P_i going from A_i to B_i and n-tuples $(P'_1, P'_2, \ldots, P'_n)$ of nonintersecting lattice paths with P'_i going from A'_i to B'_i. Restricted by the nonintersecting property, every path P'_i must pass the points A_i and B_i; moreover, there is a unique way to extend the path P_i to the points A'_i and B'_i. The following figure shows such a correspondence with Figure 4.1.

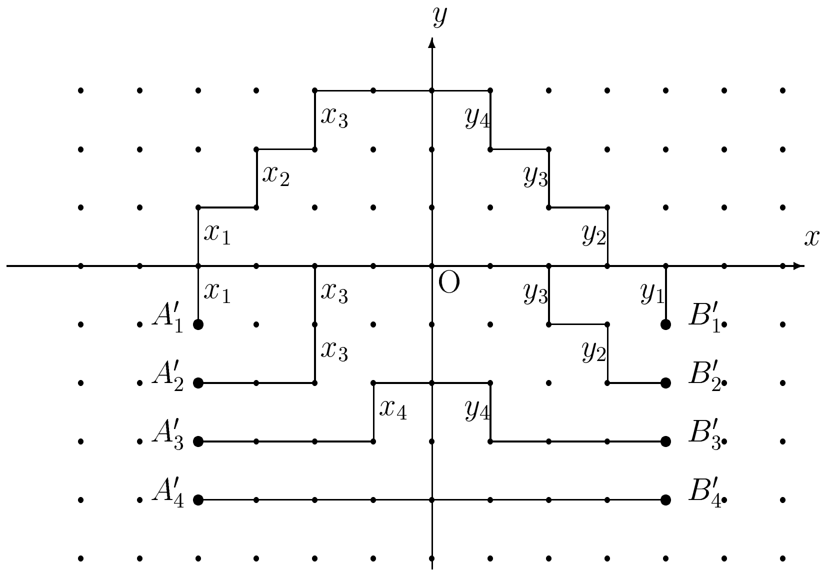

Figure 4.3 Nonintersecting paths from A'_i to B'_i

Lemma 4.4 *Let $A'_i = (-n, -i)$ and $B'_i = (n, -i)$. The generating function for the D-paths from A'_i to B'_j equals*

$$\mathrm{GF}(\mathcal{P}(A'_i, B'_j)) = \sum_k h_{k-n+i}(X) h_{k-n+j}(Y). \tag{4.17}$$

The nonintersecting lattice path argument yields Theorem 1.3 of Gessel. Notice that Theorem 1.3 and Theorem 4.2 have two different determinant forms, which implies that these two determinants are equivalent. We now present an algebraic proof of this fact. The following property of multi-Schur functions [16, 17] is needed.

Lemma 4.5 *For any family $L_0, L_1, \ldots, L_{n-1}$ of variables such that $|L_i| \le i$, we have*

$$s_\lambda(H_1, H_2, \ldots, H_n) = |h_{\lambda_j+j-i}(H_j)|_{n\times n} = |h_{\lambda_j+j-i}(H_j - L_{n-i})|_{n\times n}, \quad (4.18)$$

where $H_1, H_2, \ldots, H_n$ are sets of variables, and the complete super symmetric function $h_k(X-Y)$ is defined by the generating function

$$\sum_{k\ge 0} h_k(X-Y)t^k = \frac{\prod_{y\in Y}(1-yt)}{\prod_{x\in X}(1-xt)}.$$

Notice that matrix in equation (1.4) can be expressed as the product of two matrices:

$$\left(\sum_k h_{k-n+i}(X)h_{k-n+j}(Y)\right)_{n\times n} = \left(h_{j-i}(X)\right)_{n\times\infty} \cdot \left(h_{i-j}(Y)\right)_{\infty\times n}. \quad (4.19)$$

Let $X_i = \{x_1, x_2, \ldots, x_i\}$, $Y_i = \{y_1, y_2, \ldots, y_i\}$. On the left hand side of (4.19) we can substitute the pair of variable sets (X, Y) of the (i,j)-entry with $(X - X_{n-i}, Y - Y_{n-j})$. In accordance with this substitution on the left hand side, we should make the substitutions on the right hand side of (4.19) with X in the i-th row being replaced by $X - X_{n-i}$ in the first matrix and Y in j-th column being replaced by $Y - Y_{n-j}$ in the second matrix. After these substitutions, it follows that (4.19) can be rewritten as

$$\begin{aligned}&\left(\sum_k h_{k-n+i}(X-X_{n-i})h_{k-n+j}(Y-Y_{n-j})\right)_{n\times n}\\ = \;& \left(h_{j-i}(X-X_{n-i})\right)_{n\times\infty} \cdot \left(h_{i-j}(Y-Y_{n-j})\right)_{\infty\times n}. \quad (4.20)\end{aligned}$$

Applying the Cauchy-Binet formula to (4.19) we get

$$\begin{aligned}&\left|\sum_k h_{k-n+i}(X)h_{k-n+j}(Y)\right|_{n\times n}\\ = & \sum_{1\le k_1<k_2<\cdots<k_n} \left|h_{k_j-1+j-i}(X)\right|_{n\times n} \cdot \left|h_{k_i-1+i-j}(Y)\right|_{n\times n}. \quad (4.21)\end{aligned}$$

Applying the Cauchy-Binet formula to (4.20) we get

$$\begin{aligned}&\left|\sum_k h_{k-n+i}(X-X_{n-i})h_{k-n+j}(Y-Y_{n-j})\right|_{n\times n}\\ = & \sum_{k_1<\cdots<k_n} \left|h_{k_j-1+j-i}(X-X_{n-i})\right| \cdot \left|h_{k_i-1+i-j}(Y-Y_{n-j})\right|. \quad (4.22)\end{aligned}$$

From Lemma 4.5 it follows that

$$\left|h_{k_j-1+j-i}(X)\right|_{n\times n} = \left|h_{k_j-1+j-i}(X-X_{n-i})\right|_{n\times n}, \tag{4.23}$$

$$\left|h_{k_i-1+i-j}(Y)\right|_{n\times n} = \left|h_{k_i-1+i-j}(Y-Y_{n-j})\right|_{n\times n}. \tag{4.24}$$

Applying (4.23) and (4.24) to (4.21), we have

$$\begin{aligned}
&\left|\sum_k h_{k-n+i}(X)h_{k-n+j}(Y)\right|_{n\times n} \\
&= \sum_{k_1<\cdots<k_n} \left|h_{k_j-1+j-i}(X-X_{n-i})\right| \cdot \left|h_{k_i-1+i-j}(Y-Y_{n-j})\right| \\
&= \left|\sum_k h_{k-n+i}(X-X_{n-i})h_{k-n+j}(Y-Y_{n-j})\right|_{n\times n} \\
&= \left|\sum_k h_{k-n+i}(x_i,\ldots,x_n)h_{k-n+j}(y_j,\ldots,y_n)\right|_{n\times n}.
\end{aligned}$$

The last equality comes from simultaneously reversing the order of rows and columns of the determinant. Therefore, we have accomplished an algebraic proof of the equivalence of flagged Cauchy determinant (4.15) and the determinant (1.4) in the full variable sets. ∎

Furthermore, we can obtain a more general theorem:

Theorem 4.6 *For any two families $L_0, L_1, \ldots, L_{n-1}$ and $G_0, G_1, \ldots, G_{n-1}$ of variables such that $|L_i| \le i$, $|G_i| \le i$, we have*

$$\begin{aligned}
&\left|\sum_k h_{k-n+i}(X)h_{k-n+j}(Y)\right|_{n\times n} \\
&= \left|\sum_k h_{k-n+i}(X-L_{i-1})h_{k-n+j}(Y-G_{j-1})\right|_{n\times n}.
\end{aligned} \tag{4.25}$$

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