Families of Sets with Intersecting Clusters

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In Memory of Professor Chao Ko

Abstract

A collection of k-subsets $A_1, A_2, \ldots, A_d$ on $[n] = \{1, 2, \ldots, n\}$, not necessarily distinct, is called a (d, c)-cluster if the union $A_1 \cup A_2 \cup \cdots \cup A_d$ contains at most ck elements with $c < d$. Let $\mathcal{F}$ be a family of k-subsets of an n-element set. We show that for $k \geq 2$ and $n \geq k + 2$, if every $(k, 2)$-cluster of $\mathcal{F}$ is intersecting, then $\mathcal{F}$ contains no $(k-1)$-dimensional simplices. This leads to an affirmative answer to Mubayi's conjecture for $d = k$ based on Chvatal's simplex theorem. We also show that for any d satisfying $3 \leq d \leq k$ and $n \geq \frac{dk}{d-1}$, if every $(d, \frac{d+1}{2})$-cluster is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star. This result contains both Frankl's theorem and Mubayi's theorem as special cases.

Keywords: Clusters of subsets, Chvatal's simplex theorem, d-simplex, Erdös-Ko-Rado Theorem, Mubayi's conjecture

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1 Introduction

This paper is concerned with the study of families of subsets with intersecting clusters. The first result is a proof of an important case of a conjecture recently proposed by Mubayi [7] on intersecting families with the aid of Chvatal's simplex theorem. The second result is a theorem that is an extension of both Frankl's theorem and Mubayi's theorem.

Let us review some notation and terminology. The set $\{1, 2, \ldots, n\}$ is usually denoted by $[n]$ and the family of all k-subsets of a finite set X is denoted by X^k or $\binom{X}{k}$. A family $\mathcal{F}$ of sets is *intersecting* if every pair of two sets in $\mathcal{F}$ has a nonempty intersection. A family $\mathcal{F}$ of sets in X^k is called a *complete star* if $\mathcal{F}$ consists of all k-subsets containing x for some $x \in X$.

In 1961, Erdös, Ko, and Rado [3] published the following classical result.

Theorem 1.1 (The EKR Theorem) *Let $n \geq 2k$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star when $n > 2k$.*

In 1976, Frankl [4] obtained a generalization of the EKR Theorem.

Theorem 1.2 (Frankl) *Let $k \geq 2$, $d \geq 2$, and $n \geq dk/(d-1)$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that every d sets of $\mathcal{F}$ have a nonempty intersection, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

In fact, the following two conjectures due to Erdös and Chvatal imply Frankl's Theorem for $d \geq 3$. Recall that a *d-dimensional simplex* or a d-simplex for short, is defined as a collection of $d+1$ sets $A_1, A_2, \ldots, A_{d+1}$ such that every d of them have a nonempty intersection, but $A_1 \cap A_2 \cap \cdots \cap A_{d+1} = \emptyset$. A 2-dimensional simplex is called a *triangle*.

The Erdös conjecture [2] is stated as follows:

Conjecture 1.3 (Erdös) *For $n \geq \frac{3k}{2}$, if $\mathcal{F} \subseteq [n]^k$ contains no triangle, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

Chvatal [1] proposed a generalization of the Erdös conjecture.

Conjecture 1.4 (Chvatal's Simplex Conjecture) *Let $k \geq d+1 \geq 3$, $n \geq k(d+1)/d$, and $\mathcal{F} \subseteq [n]^k$. If $\mathcal{F}$ contains no d-dimensional simplex, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

Chvatal's simplex conjecture remains open. Chvatal has shown that it is true for $d = k-1$, which we call Chvatal's simplex theorem [1]. Frankl and Füredi [5] have shown that Chvatal's conjecture holds for sufficiently large n.

Theorem 1.5 (Chvatal's Simplex Theorem) *For $n \geq k+2 \geq 5$, if $\mathcal{F} \subseteq [n]^k$ contains no $(k-1)$-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

Theorem 1.6 (Frankl and Füredi) *For $k \geq d+2 \geq 4$, there exists n_0 such that for $n > n_0$, if $\mathcal{F} \subseteq [n]^k$ contains no d-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

As we shall see, a recent conjecture proposed by Mubayi [7] is related to Chvatal's simplex theorem. Here we introduce the terminology of clusters of subsets. A collection of k-subsets $A_1, A_2, \ldots, A_d$ of $[n]$ is called a (d,c)-*cluster* if $|A_1 \cup A_2 \cup \cdots \cup A_d| \leq ck$, where $c < d$ is a constant that may depend on d. A cluster is said to be *intersecting* if their intersection is nonempty.

Conjecture 1.7 (Mubayi's Conjecture) *Let $k \geq d \geq 3$ and $n \geq dk/(d-1)$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that every $(d,2)$-cluster of $\mathcal{F}$ is intersecting i.e., for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$, $|A_1 \cup A_2 \cup \cdots \cup A_d| \leq 2k$ implies $A_1 \cap A_2 \cap \cdots \cap A_d \neq \emptyset$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

Mubayi proved that this conjecture holds for $d = 3$ (Theorem 1.8) [7]. He also showed that his conjecture holds for $d = 4$ when n is sufficiently large [8].

Theorem 1.8 (Mubayi) *Let $k \geq 3$ and $n \geq \frac{3k}{2}$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a family such that every $(3,2)$-cluster $A_1, A_2, A_3 \in \mathcal{F}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

In this paper, we study the case $d = k$ of Mubayi's conjecture in connection with Chvatal's simplex theorem. We show that for the case $d = k$, the conditions for Mubayi's conjecture ensures the nonexistence of any $(k-1)$-dimensional simplex. Therefore, Chvatal's simplex theorem leads to Mubayi's conjecture for $d = k$. As the main results of this paper, we present a theorem on families of subsets with intersecting clusters which contains both Frankl's Theorem (Theorem 1.2) and Mubayi's Theorem (Theorem 1.8).

2 Subset Families with Intersecting Clusters

In this section, we first consider Mubayi's conjecture in the case $k = d$. We show that this case is related to Chvatal's simplex theorem (Theorem 1.5). Then we study families of k-subsets with intersecting $(d, \frac{d+1}{2})$-clusters and obtain a theorem that contains Frankl's theorem (Theorem 1.2) and Mubayi's theorem (Theorem 1.8) as special cases.

Theorem 2.1 *Let $k \geq 3$ and $n \geq k+2$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a collection of subsets of $[n]$ such that every $(k,2)$-cluster is intersecting. Then $\mathcal{F}$ contains no $(k-1)$-dimensional simplices.*

Proof. Suppose that $A_1, A_2, \ldots, A_k \in \mathcal{F}$ are such that every $k-1$ of them have nonempty intersection. We proceed to show that $A_1 \cap A_2 \cap \cdots \cap A_k \neq \emptyset$. To the contrary, assume that $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$. Then every $k-1$ sets of $A_1, A_2, \ldots, A_k$ intersect at a different element in $[n]$. For each i, $1 \leq i \leq k$, there are $k-1$ collections of $k-1$ sets containing A_i and so A_i has $k-1$ elements which are in the intersections of those $k-1$ collections.

Let us construct a bipartite graph $G = (X, Y, E)$, where $X = \cup_i A_i$, and $Y = \{A_1, A_2, \ldots, A_k\}$. There is an edge between $x \in X$ and A_i if $x \in A_i$. Clearly the degree of A_i equals k, and there total number of edges in G equals k^2. Since every $k-1$ sets of $A_1, A_2, \ldots, A_k$ intersect at a different element in $[n]$, there are k elements $x_1, x_2, \ldots, x_k$ whose degrees are $k-1$. Hence there are $k(k-1)$ edges adjacent to $x_1, x_2, \ldots, x_k$. Assume that the remaining elements of X are $y_1, y_2, \ldots, y_m$. Therefore, there are $k^2 - k(k-1) = k$ edges adjacent to $y_1, y_2, \ldots, y_m$. Since the degree of y_i is at least one for each y_i, we have $m \leq k$. Thus the number of elements in X is at most $2k$. This implies that $A_1 \cap A_2 \cap \cdots \cap A_k \neq \emptyset$, contradicting the assumption that $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$. Hence $\mathcal{F}$ does not contain any $(k-1)$-dimensional simplex. ∎

The following theorem is the main result of this paper.

Theorem 2.2 *Let $k \geq d \geq 3$ and $n \geq \frac{dk}{d-1}$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a family of subsets of $[n]$ such that every $(d, \frac{d+1}{2})$-cluster is intersecting (i.e., for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$, $|A_1 \cup A_2 \cup \cdots \cup A_d| \leq \frac{d+1}{2}k$ implies that $\cap_{i=1}^d A_i \neq \emptyset$). Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

The following lemma gives an upper bound on the number of edges in a graph with intersecting clusters, and it will be used in the proof of Theorem 3.1.

Lemma 2.3 *Let $n > d \geq 3$. Suppose that $\mathcal{F} \subseteq [n]^2$ is a family of 2-subsets of $[n]$ such that every $(d, \frac{d+1}{2})$-cluster is intersecting. Then $|\mathcal{F}| \leq n-1$ with equality only when $\mathcal{F}$ is a complete star.*

Proof. Since $\mathcal{F}$ is a family of 2-subsets, we may consider it as a graph G with vertex set $[n]$. The conditions in the lemma imply that any d edges A_1, A_2, …, A_d of G either intersect at a common vertex or cover at least $d+2$ vertices.

We now proceed by induction on n. For $n = d+1$, since any d edges cover at most $n = d+1$ vertices, any d edges of G must intersect at a common vertex and thus form a star. This implies that $|\mathcal{F}| = |E(G)| \leq d = n-1$ with equality only when $\mathcal{F}$ (or G) is a complete star.

Assume that $n \geq d+2$ and that the lemma holds for $n-1$. We first claim that G must contain a vertex of degree one. Otherwise, every vertex of G has degree at least two which implies that every connected component C of G satisfies

$$|V(C)| \leq |E(C)|. \tag{2.1}$$

Let C_1, C_2, $\ldots$, C_m be the connected components of G ordered by the relation

$$|E(C_1)| \geq |E(C_2)| \geq \cdots \geq |E(C_m)|.$$

We proceed to find d edges that form a non-intersecting $(d, \frac{d+1}{2})$-cluster to reach a contradiction. Let us consider two cases.

Case 1: $|C_1| \geq d$. Since C_1 is not a star, it contains a path P with three edges. Since $d \geq 3$, we can add $d-3$ edges to P to obtained a connected subgraph H of C_1. Let $A_1, A_2, \ldots, A_d$ be d edges of H. Then we have

$$|A_1 \cup A_2 \ldots \cup A_d| = |V(H)| \leq |E(H)| + 1 = d + 1.$$

Since H is not a star, we obtain $A_1 \cap A_2 \ldots \cap A_d = \emptyset$.

Case 2: $|C_1| < d$. Let $r \geq 1$ be the integer such that

$$b = \sum_{i=1}^{r} |E(C_i)| < d \quad \text{and} \quad \sum_{i=1}^{r+1} |E(C_i)| \geq d.$$

It is clear that C_{r+1} has at least $d-b$ edges. We now take any connected subgraph H of C_{r+1} with $d-b$ edges. Since H is connected, we have the relation

$$|E(H)| \geq |V(H)| - 1. \tag{2.2}$$

Let $A_1, A_2, \ldots, A_d$ be the d edges in $C_1, C_2, \ldots, C_r, H$. From (2.1) and (2.2) it follows that

$$\begin{aligned}
&|A_1 \cup A_2 \cdots \cup A_d| \\
&\quad = \quad |V(C_1)| + |V(C_2)| + \cdots + |V(C_r)| + |V(H)| \\
&\quad \leq \quad |E(C_1)| + |E(C_2)| + \cdots + |E(C_r)| + |E(H)| + 1 \\
&\quad = \quad d + 1.
\end{aligned}$$

Noting that $C_1, C_2, \ldots, C_r$ and H are disjoint, we have $A_1 \cap A_2 \cdots \cap A_d = \emptyset$.

In summary, we have reached the conclusion that G has a vertex with degree one. Let v be a vertex of degree one in G and let G' be the induced graph obtained from G by deleting the vertex v. Clearly, G' is a graph with $n-1$ vertices in which every d edges A_1, A_2, $\ldots$, A_d either intersect at a common vertex or cover at least $d+2$ vertices. By the inductive hypothesis, we have $|E(G')| \leq n-2$ with equality only if G' is a complete star. Hence

$$|\mathcal{F}| = |E(G)| = |E(C)| + 1 \leq n - 1$$

with equality only if $\mathcal{F}$ (or G) is a complete star. ∎

The following lemma is an extension of Lemma 3 of Mubayi [7]. While the proof of Mubayi relies on the EKR theorem, our proof is based on the above Lemma 2.3 and Frankl's theorem (Theorem 1.2).

Lemma 2.4 *Let $k \geq d \geq 2$, $t \geq 2$, and $2 \leq l \leq k$. Let $S_1, S_2, \ldots, S_t$ be pairwise disjoint k-subsets and $X = S_1 \cup S_2 \cup \cdots \cup S_t$. Suppose that $\mathcal{F}$ is a family of l-subsets of X satisfying the following conditions*

1. *$S_i \in \mathcal{F}$ for all i if $l = k$.*
2. *$|\mathcal{F}| \leq d$ if $t = 2$.*
3. *For every $A_1, A_2, \ldots, A_d \in \mathcal{F}$ and $1 \leq i \leq t$, $A_1 \cap A_2 \cdots \cap A_d \cap S_i = \emptyset$ implies $|A_1 \cup A_2 \cdots \cup A_d - S_i| > \frac{dl}{2}$.*

Then we have $|\mathcal{F}| < \binom{tk-1}{l-1}$.

Proof. For $d = 2$, the above lemma reduces to Lemma 3 in [7]. So we may assume that $d \geq 3$. Let $n = |X| = tk$. We consider the following two cases.

Case 1. Assume $l = 2$. We claim that any $(d, \frac{d+1}{2})$-cluster of $\mathcal{F}$ is intersecting, namely, for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$, we have either $A_1 \cap A_2 \cdots \cap A_d \neq \emptyset$ or $|A_1 \cup A_2 \cup \cdots \cup A_d| \geq d + 2$. To this end, we assume that $A_1 \cap A_2 \cdots \cap A_d = \emptyset$. This gives $A_1 \cap A_2 \cdots \cap A_d \cap S_i = \emptyset$ for any S_i. Since $X = \cup S_i$ is the ground set of $\mathcal{F}$, there exists S_m such that $A_1 \cap S_m \neq \emptyset$. As $A_1 \cap A_2 \cdots \cap A_d \cap S_m = \emptyset$ and $l = 2$, in view of Condition 3 we get

$$|A_1 \cup A_2 \cdots \cup A_d - S_m| > d.$$

Furthermore, the condition $A_1 \cap S_m \neq \emptyset$ yields

$$|A_1 \cup A_2 \cdots \cup A_d| > d + 1.$$

So the claim holds.

Since $d \geq 3$, by Lemma 3.2, we obtain that $|\mathcal{F}| \leq n - 1$, where $n = tk$. So it remains to show that it is impossible for $|\mathcal{F}|$ to reach the upper bound $n - 1$. Assume that $|\mathcal{F}| = n - 1$. Again, by Lemma 3.2, $\mathcal{F}$ must be a complete star, i.e., $\mathcal{F}$ consists of all 2-subsets of X for some x in X. Without loss of generality, we may assume that $x \in S_1$. Let A_1 be a 2-subset from $\mathcal{F}$ such that $A_1 \subseteq S_1$. Since $d - 1 \leq k$, we may choose $d - 1$ 2-subsets $A_2, A_3, \ldots, A_d$ such that $A_i \in \mathcal{F}$ and $A_i - x \subseteq S_2$ for $2 \leq i \leq d$. Then $A_1 \cap A_2 \cdots \cap A_d \cap S_2 = \emptyset$ and

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = 2 < d,$$

contradicting Condition 3. Hence we have $|\mathcal{F}| < n - 1 = tk - 1$. So the lemma is proved for $l = 2$.

Case 2. Assume $l \geq 3$. Then $k \geq l \geq 3$. We proceed by induction on t.

We first consider the case $t = 2$, namely, $X = S_1 \cup S_2$. We will show that $A_1 \cap A_2 \cdots \cap A_d \neq \emptyset$ for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$. If this is not true, then there exist $A_1, A_2, \ldots, A_d \in \mathcal{F}$ for which

$$A_1 \cap A_2 \cdots \cap A_d = \emptyset. \tag{2.3}$$

Let $A = A_1 \cup A_2 \cup \cdots \cup A_d$. It is clear that A contains at most dl elements. Since S_1 and S_2 are disjoint, so are $A \cap S_1$ and $A \cap S_2$. Therefore, either $A \cap S_1$ or $A \cap S_2$ contains at most half of the elements in A. We may assume without loss of generality that

$$|A \cap S_1| \le \frac{dl}{2}.$$

Note that (2.3) yields $A_1 \cap A_2 \cdots \cap A_d \cap S_1 = \emptyset$. Since $X = S_1 \cup S_2$, we get

$$|A - S_2| = |A \cap S_1| \le \frac{dl}{2},$$

contradicting Condition 3. Thus, we shown that $A_1 \cap A_2 \cdots \cap A_d \neq \emptyset$ for any A_1, A_2, ..., $A_d \in \mathcal{F}$. By Frankl's Theorem (Theorem 1.2) we obtain

$$|\mathcal{F}| \le \binom{2k-1}{l-1}. \tag{2.4}$$

Next we prove that equality in (2.4) can never be reached. Let us assume that

$$|\mathcal{F}| = \binom{2k-1}{l-1}. \tag{2.5}$$

By Frankl's theorem and $d \ge 3$, $\mathcal{F}$ is a complete star, i.e., $\mathcal{F}$ consists of all the l-subsets of $[2k]$ which contain the element x for some x in $[2k]$. Without loss of generality, we may assume that $x \in S_1$. Then any subset $A_i \in \mathcal{F}$ is either of the form $B \cup \{x\}$ for $B \in [S_1 - x]^{l-1}$ or of the form $C \cup \{x\}$ for $C \in [S_2]^{l-1}$. Since $d \le k$ and $3 \le l \le k$, we have

$$d - 1 \le k \le \binom{k}{l-1}.$$

Now we may choose $A_1 \in \mathcal{F}$ with $A_1 \subseteq S_1$ and $d-1$ sets $A_2, A_3, \ldots, A_d \in \mathcal{F}$ with $A_i - x \subseteq S_2$ for each $i \ge 2$. Since $A_1 \cap S_2 = \emptyset$, we have $A_1 \cap A_2 \cdots \cap A_d \cap S_2 = \emptyset$. Moreover, since $A_i - x \subseteq S_2$ for $i = 2, 3, \ldots, d$, we have

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = |A_1| = l < \frac{dl}{2},$$

contradicting Condition 3. Thus, we have derived that $|\mathcal{F}| < \binom{2k-1}{l-1}$ and the lemma is valid for $t = 2$.

Next suppose $t \ge 3$ and the result holds for $t - 1$. We first show that there exists at most one set S_m such that

$$|\mathcal{F} \cap [S_m]^l| \ge \frac{d}{2}.$$

Suppose, to the contrary, that there exist two sets, say S_1 and S_2 , such that

$$|\mathcal{F} \cap [S_i]^l| \ge \frac{d}{2},$$

for $i = 1, 2$. Then

$$|\mathcal{F} \cap [S_1]^l| + |\mathcal{F} \cap [S_2]^l| \geq d.$$

Hence we are able to choose d sets $A_1, A_2, \ldots, A_d$ from $(\mathcal{F} \cap [S_1]^l) \cup (\mathcal{F} \cap [S_2]^l)$ such that $A_1 \subseteq S_1$ and $A_2 \subseteq S_2$. Since $|(A_1 \cup A_2 \cup \cdots \cup A_d)| \leq dl$ and $S_1 \cap S_2 = \emptyset$, we have either

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_1| \leq \frac{dl}{2} \tag{2.6}$$

or

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_2| \leq \frac{dl}{2}. \tag{2.7}$$

Without loss of generality, assuming that (2.6) is valid. Then we have

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = |(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_1| \leq \frac{dl}{2}.$$

However, the choices of $A_1, A_2, \ldots, A_d$ ensure that $A_1 \cap A_2 \cdots \cap A_d \cap S_2 = \emptyset$, contradicting Condition 3. Thus we have shown that there exists at most one set S_m such that

$$|\mathcal{F} \cap [S_m]^l| \geq \frac{d}{2}.$$

For convenience, let us assume $m = t$. Thus we have

$$|\mathcal{F} \cap [S_i]^l| \leq \frac{d-1}{2},$$

for $i = 1, \ldots, t-1$. Set $\mathcal{H}_i = \{F \in \mathcal{F} : |F \cap S_i| = l - 1\}$ and $\deg_{\mathcal{H}_i}(B) = |\{F \in \mathcal{H}_i : B \subset F\}|$.

Claim A. There exists at least one set S_i ($i \in \{1, \ldots, t\}$) such that

$$|\mathcal{H}_i| \leq \binom{k}{l-1} \quad \text{and} \quad |\mathcal{F} \cap [S_i]^l| \leq \frac{d-1}{2}.$$

Suppose that Claim A is not true. Then

$$|\mathcal{H}_i| \geq \binom{k}{l-1} + 1, \tag{2.8}$$

for $i = 1, \cdots, t-1$. Moreover, if $|\mathcal{F} \cap [S_t]^l| \leq \frac{d-1}{2}$, then

$$|\mathcal{H}_t| \geq \binom{k}{l-1} + 1.$$

By (2.8), there exists a $(l-1)$-subset B of S_1 such that

$$\deg_{\mathcal{H}_1}(B) \geq 2. \tag{2.9}$$

Assume that $A_1, A_2 \in \mathcal{H}_1$ are chosen subject to the conditions $B \subset A_1$ and $B \subset A_2$. Since

$$|\mathcal{H}_2| \geq \binom{k}{l-1} + 1 > d - 2,$$

we can choose $A_3, \ldots A_d$ from $\mathcal{H}_2$. Since $A_1 \cap A_2 = B \subseteq S_1$,

$$A_1 \cap \cdots \cap A_d \cap S_2 = \emptyset$$

and

$$|A_1 \cup \cdots \cup A_d - S_2| \leq (l+1) + (d-2) = l + d - 1 \leq \frac{dl}{2}$$

for $d \geq 4$ and $l \geq 3$. So we have reached a contradiction to Condition 3 when $d \geq 4$. Assume $d = 3$. Let $\{x_i\} = A_i - B$ for $i = 1, 2$. Then $x_i \notin S_1$ and let $x_1 \in S_{i_0}$ for some $i_0 \geq 2$. Choose A_3 to be either in $\mathcal{H}_{i_0}$ or $\mathcal{F} \cap [S_{i_0}]^l$. We have

$$A_1 \cap A_2 \cap A_3 \cap S_{i_0} = \emptyset$$

and

$$|A_1 \cup A_2 \cup A_3 - S_{i_0}| \leq (l-1) + 1 + 1 = l + 1 \leq \frac{dl}{2}$$

for $l \geq 3$ and $d = 3$, contradicting Condition 3 again. Thus Claim A holds.

Without loss of generality, we assume that

$$|\{F \in \mathcal{F} : |F \cap S_1| = l - 1\}| = |\mathcal{H}_1| \leq \binom{k}{l-1} \quad \text{and} \quad |F \cap [S_1]^l| \leq \frac{d-1}{2}.$$

Now consider any $F \in \mathcal{F}$. We may express F as $F_1 \cup F_2$, where $F_1 = F \cap S_1$ and $F_2 = F - F_1$. For a fixed F_1 of size $l - r$ $(1 \leq r \leq l)$, let $\mathcal{F}_r$ be the family of all r-sets $F_2 \subset S_2 \cup S_3 \cup \cdots \cup S_t$ such that $F_1 \cup F_2 \in \mathcal{F}$.

We claim that $\mathcal{F}_r$ satisfies the conditions of the lemma. For otherwise, we may assume that there exist $A_1, A_2, \ldots, A_d \in \mathcal{F}_r$ and $i \in \{2, \cdots, t\}$ such that $A_1 \cap A_2 \cap \cdots \cap A_d \cap S_i = \emptyset$ and

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_i| \leq \frac{d}{2} r.$$

Now, let $A'_j = A_j \cup F_1$ for $1 \leq j \leq d$. Then $A'_1, A'_2, \ldots, A'_d \in \mathcal{F}$ and

$A'_1 \cap A'_2 \cap \cdots \cap A'_d \cap S_i = \emptyset$. Recalling that $l \geq r$, we get

$$|(A'_1 \cup A'_2 \cup \cdots \cup A'_d) - S_i| = |F_1| + |(A_1 \cup A_2 \cup \cdots \cup A_d) - S_i|$$

$$\leq l - r + \frac{dr}{2} = l + \frac{d-2}{2} r \leq l + \frac{d-2}{2} l = \frac{dl}{2},$$

contradicting Condition 3. Thus we have shown that $\mathcal{F}_r$ satisfies the conditions of the lemma. For $r \geq 2$, by the inductive hypothesis, we see that

$$|\mathcal{F}_r| < \binom{(t-1)k - 1}{r - 1}.$$

Since $l \geq 3$ and $d \leq k$, it is easy to check that

$$\sum_{r=2}^{l} \binom{k}{l-r} - d \geq 0.$$

Hence $|\mathcal{F}|$ can be bounded as follows:

$$\begin{aligned}
|\mathcal{F}| &\leq \sum_{r=2}^{l}\binom{k}{l-r}|\mathcal{F}_r| + |\{F\in\mathcal{F} : |F\cap S_1| = l-1\}| + |\mathcal{F}\cap[S_1]^l| \\
&\leq \sum_{r=1}^{l}\binom{k}{l-r}\binom{(t-1)k-1}{r-1} - \sum_{r=1}^{l}\binom{k}{l-r} + \binom{k}{l-1} + \frac{d-1}{2} \\
&< \binom{tk-1}{l-1} - \sum_{r=2}^{l}\binom{k}{l-r} + d \leq \binom{tk-1}{l-1}.
\end{aligned}$$

This completes the proof. ∎

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. For $d = 3$, the result follows from Theorem 1.8. So we assume $d \geq 4$. Let $S_1, S_2, \ldots, S_t$ be a maximum subfamily of pairwise disjoint k-subsets from $\mathcal{F}$. We proceed by using induction on t. If $t = 1$, then $\mathcal{F}$ is intersecting and the result follows from Theorem 1.1 when $n \geq 2k$. When $\frac{dk}{d-1} \leq n < 2k$, for any $A_1, \ldots, A_d \in \mathcal{F}$, $|A_1 \cup \cdots \cup A_d| \leq n < 2k$, it follows that their intersection is nonempty from the condition of the theorem. Hence the theorem reduces to Theorem 1.2 in this case. Thus we may assume that $t \geq 2$ and the theorem holds for $t-1$. Note that $t = 1$ is the only case when $\mathcal{F}$ can be a complete star. We now proceed to prove that $|\mathcal{F}| < \binom{n-1}{k-1}$.

If $n = tk$, then we set $l = k$. The condition on $\mathcal{F}$ in Theorem 3.1 implies the condition on $\mathcal{F}$ in Lemma 3.3 with d replaced by $d-1$. In fact, suppose that there exist $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}$ for which $A_1 \cap A_2 \cdots \cap A_{d-1} \cap S_i = \emptyset$. Since every $(d, \frac{d+1}{2})$-cluster of $\mathcal{F}$ is intersecting, we see that

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} \cup S_i| > \frac{d-1}{2}k,$$

hence

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| > \frac{d+1}{2}k - k = \frac{d-1}{2}k.$$

For $t = 2$, the assumption of the theorem implies that $|\mathcal{F}| \leq d-1$. Again, by Lemma 3.3, we obtain $|\mathcal{F}| < \binom{n-1}{k-1}$.

We now assume $n > tk$ and let

$$Y = [n] - \bigcup_{i=1}^{t} S_i.$$

Given the choices of $S_1, S_2, \ldots, S_t$, Y does not contain any subset $A \in \mathcal{F}$. Put

$$\mathcal{F}' = \{F \in \mathcal{F} : |F \cap Y| = k-1\}.$$

Claim B. If $|Y| = n - tk \geq k$, then

$$|\mathcal{F}'| \leq \binom{n-tk}{k-1}. \tag{2.10}$$

Suppose that Claim B is not true, that is,

$$|\mathcal{F}'| \geq \binom{n-tk}{k-1} + 1 \geq k+1 > d.$$

Therefore, there exists a $(k-2)$-subset $B \subset Y$ such that

$$\deg_{\mathcal{F}'}(B) \geq |Y| - k + 3 = (n-tk) - k + 3. \tag{2.11}$$

Otherwise, we would have

$$|\mathcal{F}'| \leq \frac{((n-tk)-k+2)\binom{n-tk}{k-2}}{k-1} = \binom{n-tk}{k-1}.$$

Since the number of $(k-1)$-subsets containing B is equal to $|Y|-k+2$, there exists a $(k-1)$-subset C containing B such that $\deg_{\mathcal{F}'}(C) \geq 2$. Suppose that $A_1, A_2 \in \mathcal{F}'$ such that $A_1 \cap A_2 = C \subset Y$. So

$$A_1 \cap A_2 \cap S_i = \emptyset$$

for each $1 \leq i \leq t$. Let $A_3, A_4, \ldots, A_{d-1}$ be additional subsets in $\mathcal{F}'$ such that $B \subseteq A_i$ for each i if $|Y| - k + 3 \geq d-1$. We deduce that

$$A_1 \cap \cdots \cap A_{d-1} \cap S_i = \emptyset$$

for each $1 \leq i \leq t$ and

$$\begin{cases} |(A_1 \cup \cdots \cup A_{d-1})| \leq k-2+2(d-2)+1 = k+2d-5, & \text{if } |Y|-k+3 \geq d-1, \\ |(A_1 \cup \cdots \cup A_{d-1})| \leq |Y|+d-1 \leq k+2d-6, & \text{if } |Y|-k+3 < d-1. \end{cases}$$

Let S_h be such that $S_h \cap A_1 \neq \emptyset$. Since $k \geq d \geq 4$, it follows that

$$|(A_1 \cup \cdots \cup A_{d-1}) \cup S_h| \leq k+2d-5+(k-1) = 2k+2d-6 \leq \frac{d+1}{2}k,$$

contradicting the assumption of the theorem. So Claim B is justified.

Given any member F in $\mathcal{F}$, we can always write F as $F_1 \cup F_2$, where $F_1 = F \cap Y$ and $F_2 = F - F_1$. Suppose that F_1 is of size $k-l$ $(1 \leq l \leq k)$. Let $\mathcal{F}_l$ be the family of all l-sets $F_2 \subset \cup_{i=1}^t S_i$ such that $F_1 \cup F_2 \in \mathcal{F}$. We claim that $\mathcal{F}_l$ satisfies the conditions in Lemma 3.3 with d replaced by $d-1$. For $l = k$, the intersecting condition on clusters for the theorem implies that (1) $|\mathcal{F}_k| \leq d-1$ for $t=2$ and (2) for every $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}_k$, if $A_1 \cap A_2 \cap \cdots \cap A_{d-1} \cap S_i = \emptyset$, then

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} \cup S_i| > \frac{d+1}{2}k$$

which implies that

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| > \frac{d-1}{2}k,$$

thus the claim is justified. Assume that $l < k$. If the claim is not true, then there exist $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}_l$ such that $A_1 \cap A_2 \cdots \cap A_{d-1} \cap S_i = \emptyset$ and

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| \leq \frac{d-1}{2} l.$$

Setting $A'_i = A_i \cup F_1$ for $i \leq d-1$, we get $A'_i \in \mathcal{F}$, $A'_1 \cap A'_2 \cdots \cap A'_{d-1} \cap S_i = \emptyset$, and

$$\begin{aligned} |(A'_1 \cup A'_2 \cup \cdots \cup A'_{d-1}) \cup S_i| &= |F_1| + |(A_1 \cup A_2 \cup \cdots \cup A_{d-1}) - S_i| + |S_i| \\ &\leq k - l + \frac{d-1}{2} l + k = 2k + \frac{d-3}{2} l \leq 2k + \frac{d-3}{2} k = \frac{d+1}{2} k, \end{aligned}$$

contradicting the assumption of the theorem. Thus we have verified the claim which shows that $\mathcal{F}_l$ satisfies the conditions in Lemma 3.3. For $l \geq 2$, it follows from Lemma 3.3 that

$$|\mathcal{F}_l| < \binom{tk-1}{l-1}.$$

Note that for $|Y| = n - tk \leq k - 2$, we have

$$|\{F \in \mathcal{F} : |F \cap Y| = k-1\}| = 0.$$

For $|Y| = k - 1$, we have

$$|\{F \in \mathcal{F} : |F \cap Y| = k-1\}| < d - 1 \leq k - 1.$$

Otherwise we can choose $d-1$ sets $A_1, \ldots, A_{d-1} \in \mathcal{F}$ together with S_1 in violation of the assumption of theorem. When $|Y| \geq k$, Claim B implies that

$$|\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \leq \binom{n-tk}{k-1}.$$

Consequently, we have

$$|\{F \in \mathcal{F} : |F \cap Y| = k-1\}| < \sum_{l=1}^{k} \binom{n-tk}{k-l}.$$

It follows that

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{l=2}^{k} \binom{|Y|}{k-l} |\mathcal{F}_l| + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \\ &\leq \sum_{l=2}^{k} \binom{|Y|}{k-l} \left[\binom{tk-1}{l-1} - 1 \right] + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \\ &= \sum_{l=1}^{k} \binom{|Y|}{k-l} \left[\binom{tk-1}{l-1} - 1 \right] + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \\ &= \sum_{l=1}^{k} \binom{n-tk}{k-l} \binom{tk-1}{l-1} - \sum_{l=1}^{k} \binom{n-tk}{k-l} + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \\ &< \binom{n-1}{k-1}, \end{aligned}$$

as required. This completes the proof. ∎

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