Families of Sets with Intersecting Clusters

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In Memory of Professor Chao Ko

Abstract

A collection of k-subsets $A_1, A_2, \ldots, A_d$on $[n] = \{1, 2, \ldots, n\}$, not necessarily distinct, is called a (d, c)-cluster if the union $A_1 \cup A_2 \cup \cdots \cup A_d$ contains at most ck elements with $c < d$. Let $\mathcal{F}$ be a family of k-subsets of an n-element set. We show that for $k \geq 2$ and $n \geq k + 2$, if every $(k, 2)$-cluster of $\mathcal{F}$ is intersecting, then $\mathcal{F}$ contains no $(k-1)$-dimensional simplices. This leads to an affirmative answer to Mubayi's conjecture for $d = k$ based on Chvatal's simplex theorem. We also show that for any d with $3 \leq d \leq k$ and $n \geq \frac{dk}{d-1}$, if every $(d, \frac{d+1}{2})$-cluster is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star. This result contains Frankl's theorem for $d \geq 2$ and Mubayi's theorem for $d = 3$ as special cases.

Keywords: Clusters of subsets, Chvatal's simplex theorem, d-simplex, Erdös-Ko-Rado Theorem, Mubayi's conjecture

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1 Introduction

This paper is concerned with the study of families of subsets with intersecting clusters. The first result is a proof of an important case of a conjecture recently proposed by Mubayi [7] on intersecting families with the aid of Chvatal's simplex theorem. The second result is a theorem that is an extension of Frankl's theorem and Mubayi's theorem.

Let us review some notation and terminology. The set $\{1, 2, \ldots, n\}$ is usually denoted by $[n]$ and the family of all k-subsets of a finite set X is denoted by X^k or $\binom{X}{k}$. A family $\mathcal{F}$ of sets is called a *star* if $\cap_{F \in \mathcal{F}} F = \{x\}$ for some $x \in X$.

In 1961, Erdös, Ko, and Rado [3] proved the following classical result.

Theorem 1.1 (The EKR Theorem) *Let $n \geq 2k$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

In 1976, Frankl [4] obtained the following generalization of the EKR Theorem.

Theorem 1.2 (Frankl) *Let $k \geq 2$, $d \geq 2$, and $n \geq dk/(d-1)$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that every d sets of $\mathcal{F}$ have a nonempty intersection, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

In fact, the following two conjectures due to Erdös and Chvatal imply Frankl's Theorem for $d \geq 3$. Recall that a *d-dimensional simplex* or a d-simplex for short, is defined as a collection of $d+1$ sets $A_1, A_2, \ldots, A_{d+1}$ such that every d of them have a nonempty intersection, but $A_1 \cap A_2 \cap \cdots \cap A_{d+1} = \emptyset$. A 2-dimensional simplex is called a *triangle*.

The Erdös conjecture [2] is stated as follows:

Conjecture 1.3 (Erdös) *For $n \geq \frac{3k}{2}$, if $\mathcal{F} \subseteq [n]^k$ contains no triangle, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

Chvatal [1] proposed the following conjecture as a generalization of the Erdös conjecture.

Conjecture 1.4 (Chvatal's Simplex Conjecture) *Let $k \geq d+1 \geq 3$, $n \geq k(d+1)/d$, and $\mathcal{F} \subseteq [n]^k$. If $\mathcal{F}$ contains no d-dimensional simplex, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

Chvatal's simplex conjecture remains open. Nevertheless, important progress has been made on the asymptotic properties and special cases. Chvatal proved his conjecture for the case $d = k-1$, which we call Chvatal's simplex theorem [1]. Frankl and Füredi [5] have shown that Chvatal's conjecture holds for sufficiently large n.

Theorem 1.5 (Chvatal's Simplex Theorem) *For $n \geq k+2 \geq 5$, if $\mathcal{F} \subseteq [n]^k$ contains no $(k-1)$-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

Theorem 1.6 (Frankl and Füredi) *For $k \geq d+2 \geq 4$, there exists n_0 such that for $n > n_0$, if $\mathcal{F} \subseteq [n]^k$ contains no d-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

As we shall see, a recent conjecture proposed by Mubayi [7] is in fact related to Chvatal's simplex theorem.

Conjecture 1.7 (Mubayi's Conjecture) *Let $k \geq d \geq 3$ and $n \geq dk/(d-1)$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that for every $A_1, A_2, \ldots, A_d \in \mathcal{F}$ satisfying $|A_1 \cup A_2 \cup \cdots \cup A_d| \leq 2k$, we have $A_1 \cap A_2 \cap \cdots \cap A_d \neq \emptyset$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

Mubayi confirmed his conjecture for $d = 3$ in [7] and showed that it holds for $d = 4$ while n is sufficiently large in [8]. Here we introduce the terminology of clusters of subsets. A collection of k-subsets $A_1, A_2, \ldots, A_d$ of $[n]$ is said to be a (d, c)-cluster if the union $A_1 \cup A_2 \cup \cdots \cup A_d$ contains at most ck elements, where $c < d$ is a constant that may depend on d. A cluster is said to be intersecting if their intersection is nonempty. Then Mubayi's theorem can be stated as follows.

Theorem 1.8 (Mubayi) *Let $k \geq 3$ and $n \geq \frac{3k}{2}$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that every $(3, 2)$-cluster $A_1, A_2, A_3 \in \mathcal{F}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

In this paper, we study the case $d = k$ of Mubayi's conjecture in connection with Chvatal's simplex theorem. We show that for the case $d = k$, the conditions for Mubayi's conjecture ensures the nonexistence of any $(k-1)$-dimensional simplex. Therefore, applying Chvatal's simplex theorem gives a confirmation of Mubayi's conjecture for $d = k$, which serves as further evidence in support of Mubayi's conjecture.

In Section 3, we present a theorem on families of subsets with intersecting clusters. As direct consequences, it follows Frankl's Theorem (Theorem 1.2) for $d \geq 3$ and Mubayi's Theorem for $d = 3$ (Theorem 1.8).

2 Mubayi's Conjecture for $d = k$

We obtain the following theorem which implies Mubayi's conjecture for the case $k = d$ from Chvatal's simplex theorem.

Theorem 2.1 *Let $k \geq 3$ and $n \geq k+2$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a collection of subsets of $[n]$ such that every $(k, 2)$-cluster is intersecting. Then $\mathcal{F}$ contains no $(k-1)$-dimensional simplices.*

Proof. Suppose that $A_1, A_2, \ldots, A_k \in \mathcal{F}$ are such that every $k-1$ of them have nonempty intersection. We proceed to show that $A_1 \cap A_2 \cap \cdots \cap A_k \neq \emptyset$. To the contrary, assume that $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$. Then every $k-1$ sets of $A_1, A_2, \ldots, A_k$ intersect at a different element in $[n]$. For each i, $1 \leq i \leq k$, there are $k-1$ collections of $k-1$ sets containing A_i and so A_i has $k-1$ elements which are in the intersections of those $k-1$ collections.

Let us construct a bipartite graph $G = (X, Y, E)$, where $X = \cup_i A_i$, and $Y = \{A_1, A_2, \ldots, A_k\}$. There is an edge between $x \in X$ and A_i if $x \in A_i$. Clearly the degree of A_i equals k, and there total number of edges in G equals k^2. Since every $k-1$ sets of $A_1, A_2, \ldots, A_k$ intersect at a different element in $[n]$, there are k elements $x_1, x_2, \ldots x_k$ whose degrees are $k-1$. Hence there are $k(k-1)$ edges adjacent to $x_1, x_2, \ldots, x_k$. Assume that the remaining elements of X are $y_1, y_2, \ldots, y_m$. Therefore, there are $k^2 - k(k-1) = k$ edges adjacent to $y_1, y_2, \ldots, y_m$. Since the degree of y_i is at least one for each y_i, we have $m \leq k$. Thus the number of elements in X is at most $2k$. This implies that $A_1 \cap A_2 \cap \cdots \cap A_k \neq \emptyset$, contradicting the assumption that $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$. Hence $\mathcal{F}$ does not contain any $(k-1)$-dimensional simplex. ∎

3 Families of Sets with Intersecting Clusters

In this section, we study families of k-subsets with intersecting $(d, \frac{d+1}{2})$-clusters. The main result of this section is the following theorem that includes Frankl's theorem (Theorem 1.2) and Mubayi's theorem (Theorem 1.8) as special cases.

Theorem 3.1 *Let $k \geq d \geq 3$ and $n \geq \frac{dk}{d-1}$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a family of subsets of $[n]$ such that every $(d, \frac{d+1}{2})$-cluster is intersecting. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star.*

The following lemma gives an upper bound on the number of edges in a graph with intersecting clusters, and it will be used in the proof of Theorem 3.1.

Lemma 3.2 *Let $n > d \geq 3$. Suppose that $\mathcal{F} \subseteq [n]^2$ is a family of 2-subsets of $[n]$ such that every $(d, \frac{d+1}{2})$-cluster is intersecting. Then $|\mathcal{F}| \leq n-1$ with equality only when $\mathcal{F}$ is a star.*

Proof. Since $\mathcal{F}$ is a family of 2-subsets, we may consider it as a graph G with vertex set $[n]$. The conditions in the lemma imply that any d edges A_1, A_2, …, A_d of G either intersect at a common vertex or cover at least $d+2$ vertices.

We now proceed by induction on n. For $n = d+1$, any d edges trivially form a $(d, \frac{d+1}{2})$-cluster since they cover at most $n = d+1$ vertices. Therefore, any d edges of G must intersect at a common vertex and thus form a star. This implies that any d edges of G cover $d+1=n$ vertices. It is to be shown that $|\mathcal{F}|=|E(G)|\leq d=n-1$. Otherwise, we may assume that $|\mathcal{F}|\geq n=d+1$. Let $A_1, A_2, \cdots, A_{d+1}$ be $d+1$ distinct edges of G. We claim that $A_1, A_2, \ldots, A_{d+1}$ also form a star. Since $A_1, A_2, \cdots, A_d$ form a star and $d \geq 3$, $A_1, A_2, \ldots, A_{d-1}$ form a star. Note that $A_1, A_2, \ldots, A_{d-1}, A_{d+1}$ form a star as well. Consequently, the edge A_{d+1} contains the intersecting point of the star formed by $A_1, A_2, \ldots, A_{d-1}$. It can be deduced that $A_1, A_2, \ldots, A_{d+1}$ cover $d+2=n+1$ vertices, which is contradiction to the cardinality of G. Thus, we have shown that $|\mathcal{F}|=|E(G)|\leq d=n-1$ with equality only when G is a star.

Now assume that $n \geq d+2$ and that the lemma holds for $n-1$. We claim that G must contain a vertex of degree one. Otherwise, every vertex of G has degree at least two. Now, for each connected component C of G, the following relation holds

$$|V(C)| \leq |E(C)|. \tag{3.1}$$

Moreover, C cannot be a star since the degree of any vertex is at least two. Let C_1, C_2, $\ldots$, C_m be the connected components of G ordered by the relation

$$|E(C_1)| \geq |E(C_2)| \geq \cdots \geq |E(C_m)|.$$

We proceed to find d edges that form a $(d, \frac{d+1}{2})$-cluster, but are not intersecting. Let us consider two cases.

Case 1: $|C_1| \geq d$. Since C_1 is not a star, it contains a path P with three edges. Since $d \geq 3$, we can add $d-3$ edges to P to obtained a connected subgraph H of C_1. Let $A_1, A_2, \ldots, A_d$ be d edges of H. Then we have

$$|A_1 \cup A_2 \ldots \cup A_d| = |V(H)| \leq |E(H)|+1 = d+1.$$

Since H is not a star, we obtain $A_1 \cap A_2 \ldots \cap A_d = \emptyset$.

Case 2: $|C_1| < d$. Let $r \geq 1$ be the integer such that

$$b = \sum_{i=1}^{r} |E(C_i)| < d \quad \text{and} \quad \sum_{i=1}^{r+1} |E(C_i)| \geq d.$$

It is clear that C_{r+1} has at least $d-b$ edges. We now take any connected subgraph H of C_{r+1} with $d-b$ edges. Since H is connected, we have the relation

$$|E(H)| \geq |V(H)|-1. \tag{3.2}$$

Let $A_1, A_2, \ldots, A_d$ be the d edges in $C_1, C_2, \ldots, C_r, H$. From (3.1) and (3.2) it follows that

$$\begin{aligned} & |A_1 \cup A_2 \cdots \cup A_d| \\ = \ & |V(C_1)| + |V(C_2)| + \cdots + |V(C_r)| + |V(H)| \\ \leq \ & |E(C_1)| + |E(C_2)| + \cdots + |E(C_r)| + |E(H)| + 1 \\ = \ & d+1. \end{aligned}$$

Noting that $C_1, C_2, \ldots, C_r$ and H are disjoint, we have $A_1 \cap A_2 \cdots \cap A_d = \emptyset$.

Summing up, we reach the claim that G has a vertex with degree one. This allows us to assume that v is a vertex of degree one in G. Let G' be the induced graph obtained from G by deleting the vertex v. Clearly, G' is a graph with $n-1$ vertices in which every d edges A_1, A_2, $\ldots$, A_d either intersect at a common vertex or cover at least $d+2$ vertices. By the inductive hypothesis, we have $|E(G')| \leq n-2$. Hence

$$|\mathcal{F}| = |E(G)| = |E(C)| + 1 \leq n-1.$$

Finally, it is necessary to show that $|\mathcal{F}| = n-1$ only when G is a star. Let $A_1, A_2, \ldots, A_{n-1}$ be the $n-1$ edges of G. Clearly,

$$|A_1 \cup A_2 \ldots \cup A_{n-2}| = n-1.$$

By the inductive hypothesis, $A_1, A_2, \ldots, A_{n-2}$ form a star. So there is a vertex of A_1 with degree one. Since $n \geq 5$, we may repeat this procedure to conclude that $A_2, \ldots, A_{n-1}$ form a star. Then the intersecting vertex of $A_2, \ldots, A_{n-2}$ belongs to A_1 and A_{n-1}. So we infer that $A_1, \ldots, A_{n-1}$ form a star. ∎

The following lemma is an extension of Lemma 3 of Mubayi [7]. While the proof of Mubayi relies on the EKR theorem, our proof is based on the above Lemma 3.2 and Frankl's theorem (Theorem 1.2).

Lemma 3.3 *Let $k+1 \geq d \geq 2$, $t \geq 2$, and $2 \leq l \leq k$. Let S_1, S_2, $\ldots$, S_t be pairwise disjoint k-subsets and $X = S_1 \cup S_2 \cup \cdots \cup S_t$. Suppose that $\mathcal{F}$ is a family of l-subsets of X satisfying the following conditions*

1. *$S_i \in \mathcal{F}$ for all i if $l = k$.*
2. *$|\mathcal{F}| \leq d$ if $t = 2$.*
3. *For every A_1, A_2, $\ldots$, $A_d \in \mathcal{F}$ and $1 \leq i \leq t$, $A_1 \cap A_2 \cdots \cap A_d \cap S_i = \emptyset$ implies $|A_1 \cup A_2 \cdots \cup A_d - S_i| > \frac{dl}{2}$.*

Then we have $|\mathcal{F}| < \binom{tk-1}{l-1}$.

Proof. For $d = 2$, the above lemma reduces to Lemma 3 in [7]. So we may assume that $d \geq 3$. We begin with the case $l = 2$. We claim that any $(d, \frac{d+1}{2})$-cluster of $\mathcal{F}$ is intersecting, namely, for any A_1, A_2, $\ldots$, $A_d \in \mathcal{F}$, we have either $A_1 \cap A_2 \cdots \cap A_d \neq \emptyset$ or $|A_1 \cup A_2 \cup \cdots \cup A_d| \geq d+2$. To this end, we assume that $A_1 \cap A_2 \cdots \cap A_d = \emptyset$. This gives $A_1 \cap A_2 \cdots \cap A_d \cap S_i = \emptyset$ for any S_i. Since $X = \cup S_i$ is the ground set of $\mathcal{F}$, there exists S_m such that $A_1 \cap S_m \neq \emptyset$. Since $A_1 \cap A_2 \cdots \cap A_d \cap S_m = \emptyset$ and $l = 2$, from Condition 3 we get

$$|A_1 \cup A_2 \cdots \cup A_d - S_m| > d.$$

Furthermore, the condition $A_1 \cap S_m \neq \emptyset$ yields

$$|A_1 \cup A_2 \cdots \cup A_d| > d + 1.$$

This concludes the proof of the claim.

Since $d \geq 3$, with the aid of Lemma 3.2 we obtain that $|\mathcal{F}| \leq tk - 1$. So it remains to show that it is impossible for $|\mathcal{F}|$ to reach the upper bound $tk - 1$. To this end, we assume that $|\mathcal{F}| = tk - 1$. Again, by Lemma 3.2, $\mathcal{F}$ must be a star, i.e., $\cap_{F \in \mathcal{F}} F = \{x\}$ for some x in $[tk]$. Without loss of generality, we may assume that $x \in S_1$ and $A_1 \subseteq S_1$. It turns out that the above assumptions are sufficient to determine the star structure of $\mathcal{F}$: any edge A_i is either of the form $\{x, y\}$ for $y \in S_1$, or of the form $\{x, z\}$ for $z \in S_j$ $(2 \leq j \leq t)$. In other words, the elements in A_1 form a star, and every element in S_j for $2 \leq j \leq t$ is connected to $x \in S_1$ while A_i is considered as an edge in a graph. Since $d - 1 \leq k$, we may choose $d - 1$ subsets $A_2, A_2, \ldots, A_d$ such that $A_i - x \subseteq S_2$ for $2 \leq i \leq d$. At this point, we have $A_1 \cap A_2 \cdots \cap A_d \cap S_2 = \emptyset$ and

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = 2 < d,$$

which is contrary to Condition 3. Hence we have $|\mathcal{F}| < tk - 1$. So the lemma is proved for $l = 2$.

We immediate encounter the case $l \geq 3$. We proceed by induction on t. We first consider the case $t = 2$, namely, $X = S_1 \cup S_2$. If $l = k$, then from Condition 2 and $k \geq 3$ we have

$$|\mathcal{F}| \leq d \leq k + 1 < \binom{2k-1}{k-1} = \binom{2k-1}{l-1},$$

which is the required inequality.

We now come to the case $l < k$, and we will show that $A_1 \cap A_2 \cdots \cap A_d \neq \emptyset$ for any A_1, A_2, $\ldots$, $A_d \in \mathcal{F}$. If this is not true, then there exist $A_1, A_2, \ldots, A_d \in \mathcal{F}$ for which

$$A_1 \cap A_2 \cdots \cap A_d = \emptyset. \tag{3.3}$$

Let $A = A_1 \cup A_2 \cup \cdots \cup A_d$. It is clear that A contains at most dl elements. Since S_1 and S_2 are disjoint, so are $A \cap S_1$ and $A \cap S_2$. Therefore, either $A \cap S_1$ or $A \cap S_2$ contains at most half of the elements in A. There is no danger to assume that

$$|A \cap S_1| \leq \frac{dl}{2}.$$

Note that (3.3) yields $A_1 \cap A_2 \cdots \cap A_d \cap S_1 = \emptyset$. Taking into account that $X = S_1 \cup S_2$, we get

$$|A - S_2| = |A \cap S_1| \leq \frac{dl}{2},$$

contradicting Condition 3. Thus, we are led to the assertion that $A_1 \cap A_2 \cdots \cap A_d \neq \emptyset$ for any A_1, A_2, $\ldots$, $A_d \in \mathcal{F}$. In view of Frankl's Theorem (Theorem 1.2) we obtain

$$|\mathcal{F}| \leq \binom{2k-1}{l-1}. \tag{3.4}$$

Next we prove that equality in (3.4) can never be reached. Let us assume that

$$|\mathcal{F}| = \binom{2k-1}{l-1}. \tag{3.5}$$

By Frankl's theorem, $\mathcal{F}$ is a star, i.e., $\cap_{F\in\mathcal{F}} F = \{x\}$ for some x in $[2k]$. Without loss of generality, we may assume that $x \in S_1$. Moreover, from the assumption that (3.5) it follows that $\mathcal{F}$ contains all the l-subsets of $[2k]$ which contain the element x. Therefore $\mathcal{F}$ can be constructed from S_1 and S_2 as follows. Let us simply assume that $x \in S_1$. Then any subset $A_i \in \mathcal{F}$ is either of the form $B \cup \{x\}$ for $B \in [S_1 - x]^{l-1}$ or of the form $C \cup \{x\}$ for $C \in [S_2]^{l-1}$.

Since $d \leq k+1$ and $2 \leq l < k$, we have

$$d - 1 \leq k \leq \binom{k}{l-1}.$$

Now we may choose $A_1 \in \mathcal{F}$ with $A_1 \subseteq S_1$ and $d-1$ sets $A_2, A_3, \ldots, A_d \in \mathcal{F}$ with $A_i - x \subseteq S_2$ for each $i \geq 2$. Since $A_1 \cap S_2 = \emptyset$, we have $A_1 \cap A_2 \cdots \cap A_d \cap S_2 = \emptyset$. Moreover, observing that $A_i - x \subseteq S_2$ for $i = 2, 3, \ldots, d$ gives

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = |A_1| = l < \frac{dl}{2}.$$

This contradicts Condition 3, which asserts that $|\mathcal{F}| < \binom{2k-1}{l-1}$. Therefore, the lemma is valid for $t = 2$, $l \geq 3$.

Up to now, we have verified the Lemma for $t = 2$. Next we deal with the case $t \geq 3$. Let us assume that the Lemma holds for $t - 1$. We first consider the case $l < k$.

We need to show that there exists $1 \leq m \leq t$ such that

$$|\mathcal{F} \cap [S_m]^l| \leq \frac{d+1}{2}. \tag{3.6}$$

To the contrary, we may assume that, for any $1 \leq i \leq t$, the following inequality holds

$$|\mathcal{F} \cap [S_i]^l| \geq \frac{d+2}{2}. \tag{3.7}$$

In particular, we may consider only S_1 and S_2. From (3.7) it follows that

$$|\mathcal{F} \cap [S_1]^l| + |\mathcal{F} \cap [S_2]^l| \geq d + 2.$$

Hence we are able to choose d sets $A_1, A_2, \ldots, A_d$ from $(\mathcal{F} \cap [S_1]^l) \cup (\mathcal{F} \cap [S_2]^l)$ such that there exist $A_i \subseteq S_1$ and $A_j \subseteq S_2$. Since $|(A_1 \cup A_2 \cup \cdots \cup A_d)| \leq dl$ and $S_1 \cap S_2 = \emptyset$, we have either

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_1| \leq \frac{dl}{2} \tag{3.8}$$

or

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_2| \leq \frac{dl}{2}. \tag{3.9}$$

There is no loss of generality in assuming that (3.8) is valid. In this case we have

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = |(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_1| \leq \frac{dl}{2}.$$

However, the choices of $A_1, A_2, \ldots, A_d$ ensure that $A_1 \cap A_2 \cdots \cap A_d \cap S_2 = \emptyset$. This is a contradiction to Condition 3. Thus (3.6) has been verified.

For notational convenience, let us take $m = 1$. Given $D_1 \subseteq S_1$ of size $l - r$ ($0 \leq r \leq l$), we construct a family of r-subsets $\mathcal{F}_r$:

$$\mathcal{F}_r = \{D_2 \mid D_2 \subseteq S_2 \cup \cdots \cup S_t,\ D_1 \cup D_2 \in \mathcal{F}\}.$$

We proceed to show that $\mathcal{F}_r$ satisfies the conditions of the lemma. Otherwise, we may assume that there exist $A_1,\ A_2,\ \ldots,\ A_d \in \mathcal{F}_r$ and $i \in \{2, \cdots, t\}$ such that $A_1 \cap A_2 \cdots \cap A_d \cap S_i = \emptyset$ and

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_i| \leq \frac{d}{2} r.$$

Now, let $A'_j = A_j \cup D_1$ for $1 \leq j \leq d$. We claim that for any $A'_1, A'_2, \ldots, A_d \in \mathcal{F}$, we have $A'_1 \cap A'_2 \cdots \cap A'_d \cap S_i = \emptyset$. Recalling that $l \geq r$, we get

$$\begin{aligned}|(A'_1 \cup A'_2 \cup \cdots \cup A'_d) - S_i| &= |D_1| + |(A_1 \cup A_2 \cup \cdots \cup A_d) - S_i| \\ &\leq l - r + \frac{dr}{2} = l + \frac{d-2}{2} r \leq l + \frac{d-2}{2} l = \frac{dl}{2},\end{aligned}$$

contradicting Condition 3. Thus $\mathcal{F}_r$ satisfies the conditions of the lemma.

For $r \geq 2$, by the inductive hypothesis, the inequality $|\mathcal{F}_r| < \binom{(t-1)k-1}{r-1}$ holds. For $r = 1$, we have either $|\mathcal{F}_1| = 1$ or $|\mathcal{F}_1| \geq 2$. If $|\mathcal{F}_1| \geq 2$, let

$$\mathcal{H} = \{D_1 \in [S_1]^{l-1} : D_1 \cup D_2 \in \mathcal{F}, D_2 \in \mathcal{F}_1\}.$$

Then the following inequalities hold:

$$|\mathcal{F}_1| \cdot |\mathcal{H}| \leq k < \binom{k}{l-1}.$$

If not, we can find $A_1,\ A_2,\ \ldots,\ A_d \in \mathcal{F}$ which contradict the condition of the lemma. Since $l \geq 3$ and $d \leq k + 1$, it is easy to check that

$$\sum_{r=2}^{l} \binom{k}{l-r} - \frac{d}{2} > 0.$$

Hence $|\mathcal{F}|$ can be bounded as follows:

$$\begin{aligned}|\mathcal{F}| &\leq \sum_{r=1}^{l} \binom{k}{l-r} |\mathcal{F}_r| \\ &\leq \sum_{r=1}^{l} \binom{k}{l-r} \binom{(t-1)k-1}{r-1} - \sum_{r=2}^{l} \binom{k}{l-r} + |\mathcal{F} \cap [S_1]^l| \\ &\leq \binom{tk-1}{l-1} - \sum_{r=2}^{l} \binom{k}{l-r} + \frac{d}{2} < \binom{tk-1}{l-1},\end{aligned}$$

thereby showing that the lemma is valid for $t \geq 3$ and $l < k$.

Finally, we are left with the case $t \geq 3$ and $l = k$. Since $S_1 \in \mathcal{F}$, the following relation easily holds:

$$|\mathcal{F} \cap [S_1]^k| = 1 \leq \frac{d}{2}. \tag{3.10}$$

Based on the above inequality, we may employ the same reasoning as for the case $l < k$ to reach the conclusion $|\mathcal{F}| < \binom{tk-1}{k-1}$. The details are omitted. This completes the proof of the lemma. ∎

We are now ready to present the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $S_1, S_2, \ldots, S_t$ be a maximum subfamily of pairwise disjoint k-subsets from $\mathcal{F}$. If $t = 1$, then $\mathcal{F}$ is intersecting and the result follows from Theorem 1.1. So we may assume that $t \geq 2$. If $n = tk$, then we set $l = k$. The condition on $\mathcal{F}$ in Theorem 3.1 implies the condition on $\mathcal{F}$ in Lemma 3.3 with d replaced by $d - 1$. In fact, suppose that there exist $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}$ for which $A_1 \cap A_2 \cdots \cap A_{d-1} \cap S_i = \emptyset$. Since every $(d, \frac{d+1}{2})$-cluster of $\mathcal{F}$ is intersecting, we see that

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} \cup S_i| > \frac{d-1}{2}k,$$

hence

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| > \frac{d+1}{2}k - k = \frac{d-1}{2}k.$$

For $t = 2$, the assumption states that $|\mathcal{F}| \leq d - 1$. Again, by Lemma 3.3 we obtain $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

We now consider the case $n > tk$ and let

$$Y = [n] - \cup_{i=1}^{t} S_i.$$

By the choices of $S_1, S_2, \ldots, S_t$, it can be seen that Y does not contain any subset $A \in \mathcal{F}$. Given $D_1 \subseteq Y^{k-l}$ $(1 \leq l \leq k)$, let $\mathcal{F}_l$ be the family of all sets D_2 such that

$$D_2 \subseteq S_1 \cup S_2 \cup \cdots \cup S_t,$$

and $D_1 \cup D_2 \in \mathcal{F}$. We have two cases.

Case 1: $l < k$. We have the assertion that for every $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}_l$, $A_1 \cap A_2 \cdots \cap A_{d-1} \cap S_i = \emptyset$ implies that

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| > \frac{d-1}{2}l. \tag{3.11}$$

Otherwise, there exist $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}_l$, we have $A_1 \cap A_2 \cdots \cap A_{d-1} \cap S_i = \emptyset$ and

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| \leq \frac{d-1}{2}l.$$

Setting $A'_i = A_i \cup D_1$ for $i \leq d-1$, we get $A'_i \in \mathcal{F}$, $A'_1 \cap A'_2 \cdots \cap A'_{d-1} \cap S_i = \emptyset$, and

$$
\begin{aligned}
|(A'_1 \cup A'_2 \cup \cdots \cup A'_{d-1}) \cup S_i| &= |D_1| + |(A_1 \cup A_2 \cup \cdots \cup A_{d-1}) - S_i| + |S_i| \\
&\leq k - l + \frac{d-1}{2}l + k = 2k + \frac{d-3}{2}l \leq 2k + \frac{d-3}{2}k = \frac{d+1}{2}k,
\end{aligned}
$$

contradicting the cluster intersecting property. Thus we have (3.11), namely, the condition in Lemma 3.3.

Case 2: $l = k$. If $t = 2$, then the condition in the theorem implies that $|\mathcal{F}_k| \leq d-1$. Also, the cluster intersection condition in the theorem states that for every $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}_k \subseteq \mathcal{F}$, if $A_1 \cap A_2 \cdots \cap A_{d-1} \cap S_i = \emptyset$ then

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} \cup S_i| > \frac{d+1}{2}k,$$

which implies

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| > \frac{d+1}{2}k - k = \frac{d-1}{2}k.$$

Thus $\mathcal{F}_l$ satisfies the condition in Lemma 3.3 for $l \leq k$.

By virtue of Lemma 3.3 for $l \geq 2$, we have $|\mathcal{F}_l| < \binom{tk-1}{l-1}$. It is now necessary to consider the case $l = 1$. Let us continue to assume that $|\mathcal{F}_1| \geq 2$. Setting

$$\mathcal{H} = \{D_1 \in [Y]^{k-1} : D_1 \cup D_2 \in \mathcal{F}, D_2 \in \mathcal{F}_1\}$$

gives

$$|\mathcal{F}_1| \cdot |\mathcal{H}| \leq d - 1 \leq k - 1,$$

otherwise there exist $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}$ and S_i which violate the condition of the theorem. Clearly, for $n - tk = |Y| \geq k$, $k - 1 < \binom{n-tk}{k-1}$. Since $\mathcal{H} \subseteq [Y]^{k-1}$, $|\mathcal{H}| \leq 1$ for $|Y| = k-1$, and $|\mathcal{H}| = 0$ for $|Y| < k-1$. Thus we get either $|\mathcal{F}_1| \cdot |\mathcal{H}| = 0$ or $|Y| = n - tk \geq k - 1$. Consequently,

$$\sum_{l=1}^{k} \binom{n-tk}{k-l} > k - 1.$$

For $l = 1$ and $|\mathcal{F}_1| = 1$, we have

$$
\begin{aligned}
|\mathcal{F}| &\leq \sum_{l=1}^{k} \binom{|Y|}{k-l} |\mathcal{F}_l| \\
&< \sum_{l=1}^{k} \binom{n-tk}{k-l} \binom{tk-1}{l-1} = \binom{n-1}{k-1},
\end{aligned}
$$

and for $|\mathcal{F}_1| \geq 2$, since when $l = 1$, $\binom{tk-1}{l-1} - 1 \geq 0$, we have

$$\begin{aligned}
|\mathcal{F}| &\leq \sum_{l=1}^{k} \binom{|Y|}{k-l} |\mathcal{F}_l| \\
&\leq \sum_{l=2}^{k} \binom{|Y|}{k-l} \left[\binom{tk-1}{l-1} - 1 \right] + |\mathcal{F}_1| \cdot |\mathcal{H}| \\
&\leq \sum_{l=1}^{k} \binom{|Y|}{k-l} \left[\binom{tk-1}{l-1} - 1 \right] + |\mathcal{F}_1| \cdot |\mathcal{H}| \\
&\leq \sum_{l=1}^{k} \binom{n-tk}{k-l} \binom{tk-1}{l-1} - \sum_{l=1}^{k} \binom{n-tk}{k-l} + k - 1 < \binom{n-1}{k-1},
\end{aligned}$$

as required. ∎

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