On the Hook Length Formula for Binary Trees

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Abstract. We present a simple combinatorial proof of Postnikov's hook length formula for binary trees.

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Let $[n] = \{1, 2, \ldots, n\}$. It is well known that the number of labeled trees on $[n]$ equals n^{n-2}, and the number of rooted trees on $[n]$ equals n^{n-1} [1, 4]. Recently, Postnikov [2] derived the following identity on binary trees and asked for a combinatorial proof [2]. We adopt the terminology of Postnikov [2]. Given a binary tree T and a vertex v of T, we use $h(v)$ to denote the "hook-length" of v, namely, the number of descendants of v (including v itself). Postnikov [2] obtained the following identity.

Theorem 1 *For $n \geq 1$, we have*

$$(n+1)^{n-1} = \sum_T \frac{n!}{2^n} \prod_{v \in T} \left(1 + \frac{1}{h(v)}\right), \tag{1}$$

where the sum ranges over all binary trees T with n vertices.

Our combinatorial proof is based on the following equivalent formulation of (1) in terms of rooted trees:

$$(n+1)^n = \sum_T \frac{(n+1)!}{2^n} \prod_{v \in T} \left(1 + \frac{1}{h(v)}\right), \tag{2}$$

Proof. Let F_n denote the quantity on the right hand side of (2). For any unlabeled binary tree T with n vertices, the hook length of the root is always n. Let us consider binary trees T such that the left subtree T_1 has k vertices and the right subtree T_2 has $n-k-1$ vertices. From the relation

$$\frac{(n+1)!}{2^n}\left(1+\frac{1}{n}\right) = \frac{n+1}{2n}\binom{n+1}{k+1}\frac{(k+1)!}{2^k}\,\frac{(n-k)!}{2^{n-k-1}},$$

we have

$$F_n = \frac{n+1}{2n}\sum_{k=0}^{n-1}\binom{n+1}{k+1}\sum_{T_1}\frac{(k+1)!}{2^k}\prod_{v\in T_1}\left(1+\frac{1}{h(v)}\right)\sum_{T_2}\frac{(n-k)!}{2^{n-k-1}}\prod_{v\in T_2}\left(1+\frac{1}{h(v)}\right),$$

where T_1 (or T_2) ranges over all binary trees on k (or $n-k-1$) vertices. Hence F_n satisfies the following recurrence relation:

$$F_n = \frac{n+1}{2n}\sum_{k=0}^{n-1}\binom{n+1}{k+1}F_k F_{n-k-1}. \tag{3}$$

It is known that the number $T_n = n^{n-2}$ of labeled trees with n vertices has the same recurrence relation:

$$2nT_{n+1} = \sum_{k=0}^{n-1} \binom{n+1}{k+1} (k+1)T_{k+1}(n-k)T_{n-k}. \tag{4}$$

Let $R_n = nT_n$ denote the number of rooted tree on n vertices. Then the above recurrence (4) can be recast as

$$R_{n+1} = \frac{n+1}{2n} \sum_{k=0}^{n-1} \binom{n+1}{k+1} R_{k+1}R_{n-k}, \tag{5}$$

A combinatorial interpretation of (4) is given by Moon[1]: The left hand side of (4) equals the number of labeled trees on $[n+1]$ with a distinguished edge and a direction on this distinguished edge. Let T be such a tree, we may decompose it into an ordered pair of rooted trees by cutting off the distinguished edge.

Combining the recurrence (3) of F_n with the recurrence (5) of R_n, we arrive at the conclusion that $(n+1)T_{n+1} = F_n$. Hence we obtain (2). ∎

S. Seo [3] also found combinatorial proof of the identity (1).

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