Loop Deletion for the Lamp Lighting Problem

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Abstract

Given an undirected graph with loops allowed, the loop lighting problem is to find a subset of vertices such that pressing the buttons on these vertices will turn on all the lights on loop vertices while keeping other lights off. For a graph which has a loop on every vertex, the corresponding problem is called the all-ones problem. In fact, it can be seen that the loop lighting problem is equivalent to the all-ones problem. We present a graph theoretical algorithm to delete a loop vertex of a graph G and to generate a graph H such that a solution of the loop lighting problem for H determines a solution for the original graph G, and vice versa. Our algorithm is of polynomial time.

Keywords: Loop lighting problem, all-ones problem, loop deletion.

Suggested Running Title: Lamp Lighting Problem, Loop Lighting Problem.

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1 Introduction

The All-Ones Problem for a chessboard was introduced by Sutner [7]. In general, the all-ones problem can be stated for an undirected graph with loops allowed. Let G be a graph with n vertices. We assume that for each vertex there is a light and a button. If one presses a button on a vertex v, the lights on the adjacent vertices will be turned on/off depending on its status. If there is a loop on a vertex v, then pressing the button on v will also change the status of the light on v. We assume that all the lights are initially turned off. Then the all-ones problem asks whether one can turn on all the lights by pressing certain buttons in a graph G. For the case when every vertex has a loop, one can always turn on all the lights. This problem has been extensively studied by several authors, for example, Sutner [9, 10], Barua and Ramakrishnan [1] and Dodis and Winkler [3]. In the perspective of complexity, Sutner [8] shows that the the minimum all-ones problem for a general graph is NP-complete, and Chen, Li, Wang and Zhang [2] find a linear time algorithm for the minimum all-ones problem for trees.

Using the linear algebra setting, it can be seen that the all-ones problem for graphs with a loop on every vertex is equivalent to the following seeming stronger version: one can always turn on the lights on the loop vertices and keep other lights off. We call such a more general problem *the loop lighting problem*. A linear algebra argument for the above statement is presented by Lossers [6]. The aim of this paper is to derive a graph theoretical algorithm to generate a solution of the loop lighting problem. In essence, this algorithm is a graph theoretical version of the elimination process of the linear algebra approach. However, it seems to be a necessary addition to the elegant argument for the all-ones problem for a general graph given by H. Eriksson, K. Eriksson and J. Sjöstrand [4]. We note that from the point of view of complexity, the construction of Eriksson et al is exponential and our algorithm is of polynomial time.

Sutner [7] proposed the problem of finding a graph theoretical proof of the existence of a solution to the all-ones problem. For the case of trees, Garvin [5] gives a combinatorial proof. Using the loop deletion algorithm presented in this paper, one may repeatedly delete loops and finally obtain a graph without loops. Since the loop lighting problem for a graph without loops has a trivial solution, we have a graph theoretical treatment of the loop lighting problem.

2 The Loop Deletion Algorithm

Let G be an undirected graph with loops allowed. The complementary graph of G is a graph H in which there is an edge (u, v) in H if and only if (u, v) is not an edge of G. Note that a loop (v, v) is regarded as an edge in the consideration of complementary graphs. The loop deletion algorithm is stated as follows.

The Loop Deletion Algorithm. Let v be a loop vertex of G, namely, there is a loop on v. Let S be the set of vertices adjacent to v. Let H be the graph obtained from G by deleting the vertex v, and substituting the induced subgraph on S with its complementary graph. For example, see Figure 2.1.

Theorem 2.1 *Let G be an undirected graph, and v be a loop vertex. Let H be the graph obtained from G by using the deletion algorithm. Then a solution to the loop*

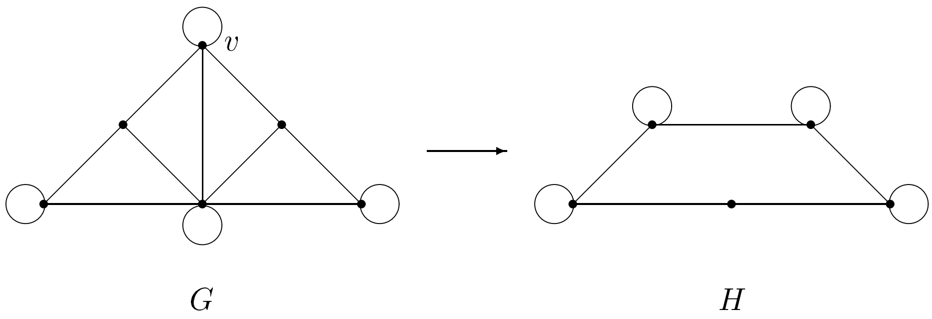

Figure 2.1: Example

lighting problem for the graph H determines a solution of the loop lighting problem for the original graph G, and vice versa.

Proof. Let $G = (V, E)$, where V and E are the vertex set and the edge set of G. Let v be a loop vertex of G and S be the set of vertices adjacent to v. Suppose that X is a solution of the loop lighting problem for the graph H. In other words, pressing the buttons on the vertices in X will turn on the lights on loop vertices while leaving other lights off. We may partition X as $S_1 \cup T_1$, where $S_1 = X \cap S$, and $T_1 = X \backslash S_1$. Moreover, we set $S_2 = S \backslash S_1$ and $T_2 = V \backslash (S \cup T_1)$. Such a partition of V is illustrated in Figure 2.2.

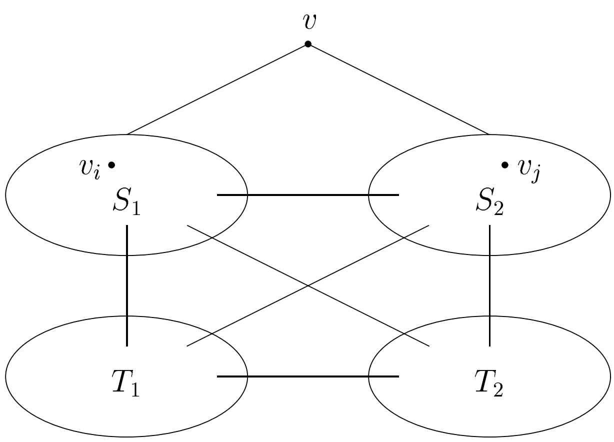

Figure 2.2: Partition of V

Let $\omega(v_i, S_1; H)$ denote the number of vertices u in S_1 that are adjacent to v_i in H. Note that a vertex v is regarded adjacent to itself if there is a loop attached to v. The parity of a vertex is said to be even if it has no loops. Since X is a solution of the loop lighting problem for the graph H, we have

- For any $v_i \in S_1$, $\omega(v_i, S_1; H) + \omega(v_i, T_1; H)$ equals the parity of v_i in H.

- For any $v_j \in S_2$, $\omega(v_j, S_1; H) + \omega(v_j, T_1; H)$ equals the parity of v_j in H.

Let us first consider the case when $|S_1|$ is even. One sees that $\omega(v_i, S_1; H)$ and $\omega(v_i, S_1; G)$ have the same parity since the complementation involves loops. Therefore, $\omega(v_i, S_1; G) + \omega(v_i, T_1; G)$ equals the parity of v_i in H, which is the opposite of the parity of v_i in G. Similarly, $\omega(v_j, S_1; H)$ and $\omega(v_j, S_1; G)$ have the same parity. Therefore, the parity of $\omega(v_j, S_1; G) + \omega(v_j, T_1; G)$ is different from the parity of v_j in G. It follows that $X \cup \{v\}$ is a solution of the loop lighting problem for the graph G.

It remains to verify the case when $|S_1|$ is odd. In this case, $\omega(v_i, S_1; H)$ and $\omega(v_i, S_1; G)$ have opposite parities. Therefore $\omega(v_i, S_1; G) + \omega(v_i, T_1; G)$ differs from the parity of v_i in H. So it equals the parity of v_i in G. For a vertex v_j in S_2, one sees that $\omega(v_j, S_1; H)$ has different parity from $\omega(v_j, S_1; G)$. It follows that X is also a solution of the loop lighting problem for the graph G.

Conversely, given a solution to the loop lighting problem of the graph G, one can reverse the above procedure to generate a solution to the problem for the graph H. This completes the proof. ∎

Roughly speaking, the reason why the deletion algorithm works lies in the fact that the change of parities of the vertices in S can be implemented either by the complementation with respect to the subset S_1 or by pressing the loop vertex v.

Using the above deletion algorithm, one can keep deleting loops till one eventually reaches a graph without loops. Since the loop lighting problem for a graph without loops has a trivial solution, namely, pressing no buttons, the deletion algorithm leads to a graph theoretical proof of the existence of a solution of the loop lighting problem. It is easily seen that our algorithm is of polynomial time. It was noted in [2] that the algorithm given by H. Eriksson, K. Eriksson and J. Sjöstrand [4] is of exponential time.

To conclude this papper, we remark that the loop lighting problem is equivalent to the all-ones problem. Given an undirected graph H with loops allowed, the loop lighting problem for H can be transformed into an all-ones problem for a graph G that has a loop on every vertex. Such a graph G can be constructed from H by adding a vertex v and reversing the procedure of the loop deletion algorithm. Let X be a solution of the loop lighting problem for the graph H. Let S_1 be the set of vertices in X that have no loops , and S_2 the set of vertices in $V(H) \backslash X$ that have no loops. Then one can reverse the procedure of the loop deletion algorithm to construct the graph G such that the all-ones problem for G is equivalent to the loop lighting problem for H.

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