Anti-lecture Hall Compositions and Overpartitions

William Y.C. Chen[1], Doris D.M. Sang[2], and Diane Y.H. Shi[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

[1]chen@nankai.edu.cn, [2]sdm@cfc.nankai.edu.cn, [3]yahuishi@gmail.com

Abstract. We show that the number of anti-lecture hall compositions of n with the first entry not exceeding $k-2$ equals the number of overpartitions of n with non-overlined parts not congruent to $0, \pm 1$ modulo k. This identity can be considered as a refined version of the anti-lecture hall theorem of Corteel and Savage. To prove this result, we find two Rogers-Ramanujan type identities for overpartition which are analogous to the Rogers-Ramanujan type identities due to Andrews. When k is odd, we give an alternative proof by using a generalized Rogers-Ramanujan identity due to Andrews, a bijection of Corteel and Savage and a refined version of a bijection also due to Corteel and Savage.

Keywords. Anti-lecture hall composition, Rogers-Ramanujan identity, overpartition, Durfee dissection

AMS Subject Classification. 05A17, 11P84

1 Introduction

The objective of this paper is to establish a connection between anti-lecture hall compositions with an upper bound on the first entry and overpartitions under a congruence condition on non-overlined parts.

In [5], Corteel and Savage introduced the notion of anti-lecture hall compositions and obtained a formula for the generating function by constructing a bijection. An anti-lecture hall composition of length k is defined to be an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_{k-1}}{k-1} \geq \frac{\lambda_k}{k} \geq 0.$$

The set of anti-lecture hall compositions of length k is denoted by A_k. Corteel and Savage have shown that

$$\sum_{\lambda \in A_k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1+q^i}{1-q^{i+1}}. \tag{1.1}$$

Let A denote the set of anti-lecture hall compositions. Since any anti-lecture hall composition can be written as an infinite vector ending with zeros, we have $A = A_\infty$ and

$$\sum_{\lambda \in A} q^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1+q^i}{1-q^{i+1}}. \tag{1.2}$$

In view of the above generating function, one sees that anti-lecture hall compositions are related to overpartitions. An overpartition of n is defined by a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined, see, Corteel and Lovejoy [6]. In the language of overpartitions, the right side of (1.2) is the generating function for overpartitions of n with the non-overlined parts larger than 1.

The main result of this paper is the following refinement of the anti-lecture hall theorem of Corteel and Savage [5]:

Theorem 1.1 *For $k \geq 3$,*

$$\sum_{\lambda_1 \leq k-2, \lambda \in A} q^{|\lambda|} = \frac{(-q;q)_\infty}{(q;q)_\infty} (q;q^k)_\infty (q^{k-1};q^k)_\infty (q^k;q^k)_\infty. \tag{1.3}$$

We shall make a connection between anti-lecture hall compositions and the overpartions with congruence restrictions. Let $F_k(n)$ be the set of anti-lecture hall compositions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n such that $\lambda_1 \leq k$. Let $H_k(n)$ be the set of overparitions of n for which the non-overlined parts are not congruent to $0, \pm 1$ modulo k. Therefore, Theorem 1.1 can be restated as the following equivalent form.

Theorem 1.2 *For $k \geq 3$ and any positive integer n, we have*

$$|F_{k-2}(n)| = |H_k(n)|. \tag{1.4}$$

To prove the main result, we need to compute the generating functions of the anti-lecture hall compositions λ with $\lambda_1 \leq k$, depending on the parity of k. Then we shall show that these two generating functions of the anti-lecture hall compositions in $F_{2k-2}(n)$ and $F_{2k-3}(n)$ are equal to the generating functions of overpartitions in $H_{2k}(n)$ and $H_{2k-1}(n)$ respectively. To this end, we establish two Rogers-Ramanujan type identities (2.9) and (2.12) for overpartitions which are analogous to the following Rogers-Ramanujan type identity obtained by Andrews [1, 2]:

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_a+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \frac{(q^a;q^{2k+1})_\infty (q^{2k+1-a};q^{2k+1})_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty} \tag{1.5}$$

where $n_i = N_i - N_{i+1}$ and $1 \leq a \leq k$. For $k = 2$ and $a = 1, 2$, (1.5) implies the classical Rogers-Ramanujan identities [8]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} (1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1} \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} (1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}. \tag{1.7}$$

It is worth mentioning that Andrews' multiple series transformation [2] can be employed to derive the overpartition analogues of (1.5).

When the upper bound k is even, the weighted counting anti-lecture hall compositions leads to the left hand side of the first Rogers-Ramanujan type identity (2.9), whereas the generating function for the number of overpartitions equals the right hand side of the first Rogers-Ramanujan type identity (2.9). The case when k is odd can be dealt with in the same way.

When k is odd, we provide an alternative proof based on a refined version of a bijection of Corteel and Savage [5], a bijection of Corteel and Savage in the original form for the anti-lecture hall theorem, and a generalized Rogers-Ramanujan identity (1.5) of Andrews.

This paper is organized as follows: In Section 2, we give two Rogers-Ramanujan type identities for overpartitions. Section 3 is concerned with the case of an even upper bound k. Two proofs for the case of an odd upper bound will be presented in Section 4.

2 Rogers-Ramanujan type identities for overpartitions

In this section, we give two Rogers-Ramanujan type identities (2.9) and (2.12) for overpartitions. It can be seen that the right side of (2.9) is the generating function for overpartitions in $H_{2k}(n)$. In the next section we shall show that the left side of (2.9) equals the generating function for anti-lecture hall compositions in $F_{2k-2}(n)$. Similarly, the right side of (2.12) equals the generating function for overpartitions in $H_{2k-1}(n)$. In Section 4 we shall show that the left side of (2.12) equals the generating function for anti-lecture hall compositions in $F_{2k-3}(n)$.

Let us recall Andrews' multiple series transformation [2]:

$$\begin{aligned}
&{}_{2k+4}\phi_{2k+3}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, \ldots, b_k, c_k, q^{-N}; q, \frac{a^k q^{k+N}}{b_1\cdots b_k c_1 \cdots c_k} \\ \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, aq/b_2, aq/c_2, \ldots, aq/b_k, aq/c_k, aq^{N+1} \end{matrix}\right] \\
&= \frac{(aq)_N (aq/b_k c_k)_N}{(aq/b_k)_N (aq/c_k)_N} \sum_{m_1,\ldots,m_{k-1}\geq 0} \frac{(aq/b_1c_1)_{m_1}(aq/b_2c_2)_{m_2}\cdots(aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}} \\
&\quad \cdot \frac{(b_2)_{m_1}(c_2)_{m_1}(b_3)_{m_1+m_2}(c_3)_{m_1+m_2}\cdots(b_k)_{m_1+\cdots+m_{k-1}}}{(aq/b_1)_{m_1}(aq/c_1)_{m_1}(aq/b_2)_{m_1+m_2}(aq/c_2)_{m_1+m_2}\cdots(aq/b_{k-1})_{m_1+\cdots+m_{k-1}}} \\
&\quad \cdot \frac{(c_k)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1})_{m_1+\cdots+m_{k-1}}} \cdot \frac{(q^{-N})_{m_1+m_2+\cdots+m_{k-1}}}{(b_kc_kq^{-N}/a)_{m_1+m_2+\cdots+m_{k-1}}} \\
&\quad \cdot \frac{(aq)^{m_{k-2}+2m_{k-3}+\cdots+(k-2)m_1} q^{m_1+m_2+\cdots+m_{k-1}}}{(b_2c_2)^{m_1}(b_3c_3)^{m_1+m_2}\cdots(b_{k-1}c_{k-1})^{m_1+m_2+\cdots+m_{k-2}}}.
\end{aligned} \tag{2.8}$$

The following summation formula can be derived from the above transformation formula of Andrews. It can be considered as a Rogers-Ramanujan type identity for overpartitions.

Theorem 2.1 *For $k \geq 2$, we have*

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q;q)_{N_1}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q;q)_{N_{k-1}}}$$
$$= \frac{(-q;q)_\infty (q;q^{2k})_\infty (q^{2k-1};q^{2k})_\infty (q^{2k};q^{2k})_\infty}{(q;q)_\infty}. \tag{2.9}$$

Proof. Applying the above transformation formula of Andrews by setting all variables to infinity except for c_k, a and q, we get

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{(c_k)_{N_1} a^{N_1+\cdots+N_{k-1}} q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-c_k)^{N_1}}$$
$$= \frac{(aq/c_k;q)_\infty}{(a,q)_\infty} \sum_{n \geq 0} \frac{(1-aq^{2n})(a,c_k;q)_n a^{kn} q^{kn^2}}{(q,aq/c_k;q)_n c_k^n}.$$

Setting $a = q$ and $c_k = -q$, we find that

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n \geq 0} (-1)^n (1-q^{2n+1}) q^{kn^2+(k-1)n}. \tag{2.10}$$

Using Jacobi's triple product identity, we get

$$(q;q^{2k})_\infty (q^{2k-1};q^{2k})_\infty (q^{2k};q^{2k})_\infty$$
$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+(k-1)n}$$
$$= \sum_{n=0}^{\infty} (-1)^n (1-q^{2n+1}) q^{kn^2+(k-1)n}. \tag{2.11}$$

In view of (2.10) and (2.11), we obtain (2.9). This completes the proof. ∎

Our second Rogers-Ramanujan type identity for overpartitions is stated as follows.

Theorem 2.2 *For $k \geq 2$, we have*

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q;q)_{N_1}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q;q)_{N_{k-1}}(-q;q)_{N_{k-1}}}$$
$$= \frac{(-q;q)_\infty (q;q^{2k-1})_\infty (q^{2k-2};q^{2k-1})_\infty (q^{2k-1};q^{2k-1})_\infty}{(q;q)_\infty}. \tag{2.12}$$

Proof. Applying Andrews' transformation formula by setting all variables except for c_1, c_k, a and q to infinity, we find

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{(c_k)_{N_1} a^{N_1+\cdots+N_{k-1}} q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-c_k)^{N_1}(aq/c_1)_{N_{k-1}}}$$
$$= \frac{(aq/c_k;q)_\infty}{(a,q)_\infty} \sum_{n\geq 0} \frac{(-1)^n(1-aq^{2n})(a,c_k;q)_n(c_1)_n a^{kn} q^{kn^2-(n-1)n/2}}{(q,aq/c_k;q)_n(aq/c_1)_n c_1^n c_k^n}.$$

Moreover, setting $a=q$, $c_k=-q$ and $c_1=-q$ yields

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_2+\cdots+N_{k-1}} q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}}$$
$$= \frac{(-q;q)_\infty}{(q,q)_\infty} \sum_{n\geq 0}(-1)^n(1-q^{2n+1})q^{kn^2+kn-n^2/2-3n/2}. \qquad (2.13)$$

Using Jacobi's triple product identity, we have

$$(q;q^{2k-1})_\infty(q^{2k-2};q^{2k-1})_\infty(q^{2k-1};q^{2k-1})_\infty$$
$$= \sum_{n=-\infty}^{\infty}(-1)^n q^{kn^2+kn-n^2/2-3n/2}$$
$$= \sum_{n=0}^{\infty}(-1)^n(1-q^{2n+1})q^{kn^2+kn-n^2/2-3n/2}. \qquad (2.14)$$

Combining (2.13) and (2.14), we deduce (2.12). This complete the proof. ∎

3 The case when k is even

In this section, we shall give a proof of Theorem 1.2 for an even upper bound $2k-2$. More precisely, this case can be stated as follows.

Theorem 3.1 *For $k\geq 2$ and $n\geq 1$, we have*

$$|F_{2k-2}(n)| = |H_{2k}(n)|. \qquad (3.15)$$

Recall that the generating function for overpartitions in $H_{2k}(n)$ equals

$$\sum_{n\geq 0}|H_{2k}(n)|q^n = \frac{(-q;q)_\infty(q;q^{2k})_\infty(q^{2k-1};q^{2k})_\infty(q^{2k};q^{2k})_\infty}{(q;q)_\infty}. \qquad (3.16)$$

In view of (2.9), in order to prove Theorem 3.1 we only need to show that the generating function of anti-lecture hall compositions in $F_{2k-2}(n)$ equals the left hand side of (2.9), as stated below.

Theorem 3.2 *The generating function of anti-lecture hall compositions in $F_{2k-2}(n)$ is given by*

$$\sum_{n=0}^{\infty} |F_{2k-2}(n)| q^n = \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2 + N_2^2 + \cdots + N_{k-1}^2 + N_2 + \cdots + N_{k-1}} (-q;q)_{N_1}}{(q;q)_{N_1 - N_2} \cdots (q;q)_{N_{k-2} - N_{k-1}} (q;q)_{N_{k-1}}}. \quad (3.17)$$

In order to prove Theorem 3.2, we introduce a triangular representation $T(\lambda) = (t_{ij})_{1 \leqslant i \leqslant j}$ of an anti-lecture hall composition λ which is similar to a T-triangles introduced by Bousquet-Mélou [4].

It should be noted that Corteel and Savage [5] used a representation of a composition λ as a pair of vectors $(l, r) = ((l_1, l_2, \ldots), (r_1, r_2, \ldots))$, where $\lambda_i = il_i + r_i$, with $0 \leq r_i \leq i - 1$. Then $l = \lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \ldots)$. It can be checked that a composition λ is an anti-lecture hall composition if and only if

(1) $l_1 \geq l_2 \geq \cdots \geq 0$, and

(2) $r_i \geq r_{i+1}$ whenever $l_i = l_{i+1}$.

Definition 3.3 *The A-triangular representation $T(\lambda) = (t_{i,j})_{1 \leqslant i \leqslant j}$ of an anti-lecture hall composition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is defined to be a triangular array $(t_{i,j})_{1 \leqslant i \leqslant j}$ of nonnegative integers satisfying the following conditions:*

(1) A diagonal entry $t_{j,j}$ in $T(\lambda)$ equals $l_j = \lfloor \lambda_j / j \rfloor$.

(2) The first r_j entries of the j-th column are equal to $t_{j,j} + 1$, while the other entries in the j-th column are equal to $t_{j,j}$.

The sum of all entries of $T(\lambda)$ is equal to $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. It can be verified that the A-triangular representation $T(\lambda)$ of an anti-lecture hall composition possesses the following properties:

(1) The diagonal entries of T are weakly decreasing, that is, $t_{1,1} \geq t_{2,2} \geq \cdots \geq 0$.

(2) The entries in the j-th column are non-increasing, and they are equal to either the $t_{j,j}$ or $t_{j,j} + 1$.

(3) If $t_{j,j} = t_{j+1,j+1}$, then $t_{i,j} \geq t_{i,j+1}$.

Conversely, a triangular array satisfying the above conditions must be the A-triangular representation of an anti-lecture hall composition.

For example, let $\lambda = (4, 8, 11, 14, 16, 15, 11, 10, 5, 2)$. The A-triangular representation $T(\lambda)$ of λ is illustrated as follows.

$$
\begin{array}{cccccccccc}
4 & 4 & 4 & 4 & 4 & 3 & 2 & 2 & 1 & 1 \\
 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
 & & 3 & 3 & 3 & 3 & 2 & 1 & 1 & 0 \\
 & & & 3 & 3 & 2 & 2 & 1 & 1 & 0 \\
 & & & & 3 & 2 & 1 & 1 & 1 & 0 \\
 & & & & & 2 & 1 & 1 & 0 & 0 \\
 & & & & & & 1 & 1 & 0 & 0 \\
 & & & & & & & 1 & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{array}
$$

Now we are ready to give a proof of Theorem 3.2 by using the A-triangular representation of an anti-lecture hall composition.

Proof of Theorem 3.2. Let λ be an anti-lecture hall composition with $\lambda_1 \leq 2k-2$. Let us consider the A-triangular representation $T(\lambda)$ of λ. We use N_i to denote the number of diagonal entries $t_{j,j}$ in $T(\lambda)$ which are greater than or equal to $2i-1$ for $1 \leq i \leq k-1$. Then we have $N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0$. Let $F_{2k-2}(N_1, \ldots, N_{k-1}; n)$ denote the set of anti-lecture hall compositions λ such that there are N_i diagonal entries in $T(\lambda)$ that are greater than or equal to $2i-1$ and $\lambda_1 \leq 2k-2$. We aim to compute the generating function of anti-lecture hall composition in $F_{2k-2}(N_1, \ldots, N_{k-1}; n)$, which can be summed up to yield the generating function of the anti-lecture hall compositions in $F_{2k-2}(n)$.

Let λ be an anti-lecture hall composition in $F_{2k-2}(N_1, \ldots, N_{k-1}; n)$, and let $\lambda^{(1)} = (\lambda_1, \ldots, \lambda_{N_1})$, $\lambda^{(2)} = (\lambda_{N_1+1}, \ldots, \lambda_l)$. Since $\lfloor \lambda_{N_1+1}/(N_1+1) \rfloor = \cdots = \lfloor \lambda_l / l \rfloor = 0$, we see that $\lambda_l \leq \cdots \leq \lambda_{N_1+1} \leq N_1$. Evidently $\lambda^{(2)}$ is a partition whose first part is less than N_1+1, and the generating function for possible choices of $\lambda^{(2)}$ equals $1/(q;q)_{N_1}$.

Let us examine the composition $\lambda^{(1)}$ and its A-triangular representation $T(\lambda^{(1)})$. The triangular array $T(\lambda^{(1)})$ can be split into k triangular arrays and we can compute the generating function for possible choices of $\lambda^{(1)}$.

Step 1. Let $T^{(1)} = T(\lambda^{(1)})$. Extract 1 from each entry in the first N_1 columns of $T^{(1)}$ to form a triangular array of size N_1 with all the entries equal to 1, denoted by $R(N_1, 1)$.

Step 2. For $2 \leq i \leq k-1$, extract 2 from each entry in the first N_i columns of the remaining triangular array $T^{(1)}$ to generate a triangular array of size N_i with all the entries equal to 2, denoted by $R(N_i, 2)$.

Step 3. Let S denote the remaining triangular array $T^{(1)}$.

After the above operations, $T(\lambda^{(1)})$ is decomposed into k triangular arrays, including an A-triangle $R(N_1, 1)$ of size N_1 with entries 1, $k-2$ A-triangular arrays $R(N_i, 2)$ of sizes $N_2, \ldots, N_{k-1}$ respectively with entries 2 where $i = 2, \ldots, k-1$, and a triangular array $S = (s_{i,j})_{1 \leq i \leq j \leq N_1}$ of size N_1. It is easy to see that the generating function for triangular arrays in $R(N_1, 1)$ is $q^{(N_1+1)N_1/2}$ and the generating function of triangular arrays in $R(N_i, 2)$ is $q^{N_i^2+N_i}$.

It can be verified that S possesses the following properties by the definition of the A-triangular representation of an anti-lecture hall composition:

(1) All the entries in the diagonals of S are equal to 1 or 0. Note that S has N_1 diagonal elements $s_{1,1}, s_{2,2}, \ldots, s_{N_1,N_1}$. These diagonal elements can be divided into $k-1$ segments

such that the first segment contains $n_1 = N_1 - N_2$ elements $s_{N_2+1,N_2+1}, \ldots, s_{N_1,N_1}$, the second segment contains $n_2 = N_2 - N_3$ elements $s_{N_3+1,N_3+1}, \ldots, s_{N_2,N_2}$, and so on, while the last segment contains $n_{k-1} = N_{k-1}$ elements $s_{1,1}, \ldots, s_{N_{k-1},N_{k-1}}$. Moreover, the i-th segment contains m_i 1's followed by 0's.

(2) The entries in the j-th column are non-increasing, and they are equal to either the $t_{j,j}$ or $t_{j,j} + 1$.

(3) If $s_{j,j} = s_{j+1,j+1}$, then $s_{i,j} \geq s_{i,j+1}$.

We denote the set of triangular arrays possessing the above three properties by $S(N_1, N_2, \ldots, N_{k-1})$. Now we are in a position to compute the generating function of triangular arrays in $S(N_1, N_2, \ldots, N_{k-1})$.

We may partition a triangular array $S \in S(N_1, N_2, \ldots, N_{k-1})$ into $k-1$ blocks of columns, where the i-th block consists of the $(N_{i+1}+1)$-th column to the N_i-th column of S. We denote the i-th block by S_i. According to the above three properties, we deduce that the first m_i diagonal entries of S_i must be 1 and the entries in the first m_i columns of S_i are either 1 or 2.

We shall split S_i into three trapezoidal arrays $S_i^{(1)}$, $S_i^{(2)}$ and $S_i^{(3)}$. First, we may form a trapezoidal array $S_i^{(1)}$ of the same size as S_i and with the entries in the first m_i columns equal to 1 and the other entries equal to 0. Let $S_i^{'}$ denote the trapezoidal array obtained from S_i by subtracting 1 from every entry in the first m_i columns. Observe that every entry in $S_i^{'}$ is either 1 or 0, and $S_i^{(1)}$ can be regarded as the Ferrers diagram of the conjugate of the partition

$$\alpha^{(1)} = (N_{i+1} + m_i, N_{i+1} + m_i - 1, \ldots, N_{i+1} + 1).$$

Furthermore, $S_i^{'}$ satisfies the following conditions:

(1) All entries in $S_i^{'}$ are equal to 0 or 1, but the diagonal entries must be 0.

(2) The entries in the same column must be non-increasing.

(3) The first m_i entries in the j-th row must be non-increasing, and the remaining entries in the j-th row are also non-increasing.

We continue to consider the trapezoidal array formed by the first m_i columns of $S_i^{'}$, and denote it by $S_i^{(2)}$. Similarly, we see that $S_i^{(2)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(2)}$, where

$$\alpha_1^{(2)} \leq N_{i+1}, \quad \text{and} \quad l(\alpha^{(2)}) \leq m_i.$$

Define $S_i^{(3)}$ to be the trapezoidal array formed by the (m_i+1)-th column to the $(N_i - N_{i+1})$-th column of S_i'. Again, $S^{(3)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(3)}$, where

$$\alpha_1^{(3)} \leq N_{i+1} + m_i \quad \text{and} \quad l(\alpha^{(3)}) \leq N_i - N_{i+1} - m_i.$$

So the generating function for possible choices of the i-th block S_i is given by

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{\frac{(N_{i+1}+1+N_{i+1}+m_i)m_i}{2}} \frac{(q;q)_{N_{i+1}+m_i}}{(q;q)_{m_i}(q;q)_{N_{i+1}}} \frac{(q;q)_{N_i}}{(q;q)_{N_{i+1}+m_i}(q;q)_{N_i-N_{i+1}-m_i}}. \tag{3.18}$$

which equals

$$\frac{(q;q)_{N_i}}{(q;q)_{N_{i+1}}(q;q)_{N_i-N_{i+1}}} \sum_{m_i=0}^{N_i-N_{i+1}} q^{\frac{(N_{i+1}+1+N_{i+1}+m_i)m_i}{2}} \frac{(q;q)_{N_i-N_{i+1}}}{(q;q)_{m_i}(q;q)_{N_i-N_{i+1}-m_i}}. \tag{3.19}$$

Observe that the sum

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{\frac{(N_{i+1}+1+N_{i+1}+m_i)m_i}{2}} \frac{(q;q)_{N_i-N_{i+1}}}{(q;q)_{m_i}(q;q)_{N_i-N_{i+1}-m_i}}$$

is the generating function for partitions with distinct parts between $N_{i+1}+1$ and N_i. Therefore,

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{\frac{(N_{i+1}+1+N_{i+1}+m_i)m_i}{2}} \frac{(q;q)_{N_i-N_{i+1}}}{(q;q)_{m_i}(q;q)_{N_i-N_{i+1}-m_i}} = (-q^{N_{i+1}+1};q)_{N_i-N_{i+1}}. \tag{3.20}$$

By (3.20), the generating function (3.18) can be simplified to

$$\frac{(q;q)_{N_i}}{(q;q)_{N_{i+1}}(q;q)_{N_i-N_{i+1}}}(-q^{N_{i+1}+1};q)_{N_i-N_{i+1}}. \tag{3.21}$$

Thus the generating function for triangular arrays in S can be written as

$$\prod_{i=1}^{k-1} \frac{(q;q)_{N_i}}{(q;q)_{N_{i+1}}(q;q)_{N_i-N_{i+1}}}(-q^{N_{i+1}+1};q)_{N_i-N_{i+1}} = \frac{(q)_{N_1}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}.$$

Recall that the generating function for possible choices of $T(\lambda^{(2)})$ equals $1/(q;q)_{N_1}$ and the generating functions for $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ are equal to $q^{(N_1+1)N_1/2}$, $q^{N_2^2+N_2}$,$\ldots$, $q^{N_{k-1}^2+N_{k-1}}$ respectively. We also note that the generating function for anti-lecture hall compositions in $F_{2k-2}(N_1,\ldots,N_{k-1},n)$ is the product of the generating functions for $T(\lambda^{(2)})$, $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ and S, and therefore it equals

$$\begin{aligned} &\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}}{(q)_{N_1}} \frac{(q)_{N_1}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} \\ &= \frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \end{aligned}$$

Summing up the generating functions of anti-lecture hall compositions in $F_{2k-2}(N_1,\ldots,N_{k-1},n)$, we get the generating function for anti-lecture hall compositions in $F_{2k-2}(n)$,

$$\sum_{n\geq 0}|F_{2k-2}(n)|q^n = \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \tag{3.22}$$

The proof is therefore completed. ∎

For example, let $\lambda = (4,8,11,14,16,15,11,10,5,2)$ and let $k=3$. Then $\lambda^{(2)} = (5,2)$, $N_1 = 8$, $N_2 = 5$, $m_1 = 2$ and $m_2 = 1$. The decomposition of $T(\lambda)$ is illustrated as follows:

$$
\begin{array}{cccccccccc}
4&4&4&4&4&3&2&2&1&1\\
&4&4&4&3&3&2&2&1&1\\
&&3&3&3&3&2&1&1&0\\
&&&3&3&2&2&1&1&0\\
&&&&3&2&1&1&1&0\\
&&&&&2&1&1&0&0\\
&&&&&&1&1&0&0\\
&&&&&&&1&0&0\\
&&&&&&&&0&0\\
&&&&&&&&&0
\end{array}
\longrightarrow
\begin{array}{cccccccc}
1&1&1&1&1&1&1&1\\
&1&1&1&1&1&1&1\\
&&1&1&1&1&1&1\\
&&&1&1&1&1&1\\
&&&&1&1&1&1\\
&&&&&1&1&1\\
&&&&&&1&1\\
&&&&&&&1
\end{array}
+
\begin{array}{ccccc}
2&2&2&2&2\\
&2&2&2&2\\
&&2&2&2\\
&&&2&2\\
&&&&2
\end{array}
$$

$$T(\lambda) \qquad\qquad R(8,1) \qquad\qquad R(5,2)$$

$$
+
\begin{array}{cccccccc}
1&1&1&1&1&2&1&1\\
&1&1&1&0&2&1&1\\
&&0&0&0&2&1&0\\
&&&0&0&1&1&0\\
&&&&0&1&0&0\\
&&&&&1&0&0\\
&&&&&&0&0\\
&&&&&&&0
\end{array}
+
\begin{array}{cccccccccc}
0&0&0&0&0&0&0&0&1&1\\
&0&0&0&0&0&0&0&1&1\\
&&0&0&0&0&0&0&1&0\\
&&&0&0&0&0&0&1&0\\
&&&&0&0&0&0&1&0\\
&&&&&0&0&0&0&0\\
&&&&&&0&0&0&0\\
&&&&&&&0&0&0\\
&&&&&&&&0&0\\
&&&&&&&&&0
\end{array}
$$

$$S \qquad\qquad T(\lambda^{(2)})$$

4 The case when k is odd

The objective of this section is to provide two proofs of the following theorem which is the case of an odd upper bound $2k-3$ of Theorem 1.2. The first is analogous to the proof of the even case. The second requires a Rogers-Ramanujan type identity of Andrews, a bijection of Corteel and Savage, and a refined version of a bijection also due to Corteel and Savage. The approach of the second proof does not seem to apply to the even case, namely, Theorem 3.1.

Theorem 4.1 *For $k \geq 2$ and a positive integer n, we have*

$$|F_{2k-3}(n)| = |H_{2k-1}(n)|. \tag{4.23}$$

The first proof relies on the following generating function formula for anti-lecture hall compositions in $F_{2k-3}(n)$. The proof of this formula is analogous to that of Theorem 3.2.

Theorem 4.2 *For $k \geq 2$,*

$$\sum_{n=0}^{\infty} |F_{2k-3}(n)| q^n = \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{(N_1+1)N_1/2 + N_2^2 + \cdots + N_{k-1}^2 + N_2 + \cdots + N_{k-1}} (-q;q)_{N_1}}{(q;q)_{N_1-N_2} \cdots (q;q)_{N_{k-2}-N_{k-1}} (q;q)_{N_{k-1}} (-q;q)_{N_{k-1}}}. \tag{4.24}$$

Proof of Theorem 4.2. Let λ be an anti-lecture hall composition with $\lambda_1 \leq 2k-3$. We consider the A-Triangular representation $T(\lambda)$ of λ. Let N_i be the number of diagonal entries t_{jj} in $T(\lambda)$ which are greater than or equal to $2i-1$ for $1 \leq i \leq k-1$. Then we have $N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0$. Let $F_{2k-3}(N_1, \ldots, N_{k-1}; n)$ denote the set of anti-lecture hall compositions λ for which there are N_i diagonal entries in $T(\lambda)$ that are greater than or equal to $2i-1$ and $\lambda_1 \leq 2k-3$.

Let $\lambda^{(1)} = (\lambda_1, \ldots, \lambda_{N_1})$, $\lambda^{(2)} = (\lambda_{N_1+1}, \ldots, \lambda_l)$. It is immediately verified that $\lambda^{(2)}$ is a partition whose first part does not exceed N_1. Hence the generating function for possible choices of $\lambda^{(2)}$ equals $1/(q;q)_{N_1}$.

Now consider $\lambda^{(1)}$ and its A-Triangular representation $T(\lambda^{(1)})$. We can split $T(\lambda^{(1)})$ into k triangular arrays to compute the generating function for possible choices of $\lambda^{(1)}$.

Step 1. Let $T^{(1)} = T(\lambda^{(1)})$. Extract 1 from each entry in the first N_1 columns of $T^{(1)}$ to form a triangular array of size N_1 with all entries equal to 1, denoted by $R(N_1, 1)$.

Step 2. For $i = 2, \ldots, k-1$, extract 2 from each entry in the first N_i columns of the remaining array $T^{(1)}$ to form a triangular array of size N_i with all entries equal to 2, denoted by $R(N_i, 2)$.

Step 3. Let S be the remaining triangular array $T^{(1)}$.

After the above procedures, $T(\lambda^{(1)})$ is decomposed into k triangular arrays, including an A-Triangle $R(N_1, 1)$ of size N_1 with all entries being 1, $(k-2)$ A-Triangles $R(N_i, 2)$ of sizes $N_2, \ldots, N_{k-1}$ respectively with all entries being 2 and a triangular array $S = (s_{i,j})$ of size N_1 satisfying the following conditions:

(1) All the entries in the diagonals of S are equal to 1 or 0. Note that S has N_1 diagonal elements $s_{1,1}, s_{2,2}, \ldots, s_{N_1,N_1}$. These diagonal elements can be divided into $k-1$ segments such that the first segment contains $n_1 = N_1 - N_2$ elements $s_{N_2+1,N_2+1}, \ldots, s_{N_1,N_1}$, the second segment contains $n_2 = N_2 - N_3$ elements $s_{N_3+1,N_3+1}, \ldots, s_{N_2,N_2}$, and so on, while the last segment contains $n_{k-1} = N_{k-1}$ elements $s_{1,1}, \ldots, s_{N_{k-1},N_{k-1}}$. Moreover, the i-th segment contains m_i 1's followed by 0's.

(2) The entries in the j-th column are non-increasing, and they are equal to either $t_{j,j}$ or $t_{j,j}+1$.

(3) If $s_{j,j} = s_{j+1,j+1}$, then $s_{i,j} \geq s_{i,j+1}$.

(4) The entries in the first N_{k-1} columns of S are equal to 0, that is, $m_{k-1} = 0$.

Let us write $\overline{S}(N_1, N_2, \cdots, N_{k-1})$ for the set of triangular arrays possessing the above four properties. We proceed to compute the generating function for the triangular arrays in $\overline{S}(N_1, N_2, \cdots, N_{k-1})$.

We may partition a triangular array $S \in \overline{S}(N_1, N_2, \ldots, N_{k-1})$ into $k-1$ blocks of columns, where the i-th block consists of the $(N_{i+1}+1)$-th column to the N_i-th column of S. We denote the i-th block by S_i. According to the above four properties, we infer that the first m_i diagonal entries of S_i must be 1 and the entries in the first m_i columns of S_i are either 1 or 2 for $i = 1, \ldots k-2$ and S_{k-1} is a triangular array of size N_{k-1} with all entries eqaul to 0.

We shall split S_i into three trapezoidal arrays $S_i^{(1)}$, $S_i^{(2)}$ and $S_i^{(3)}$ for $i = 1, \ldots k-2$. First, we may form a trapezoidal array $S_i^{(1)}$ of the same size as S_i and with the entries in the first m_i columns equal to 1 and the other entries equal to 0. Let $S_i^{'}$ denote the trapezoidal array obtained from S_i by subtracting 1 from every entry in the first m_i columns. It is seen that every entry in $S_i^{'}$ is either 1 or 0, and $S_i^{(1)}$ can be regarded as the Ferrers diagram of the conjugate of the partition

$$\alpha^{(1)} = (N_{i+1} + m_i, N_{i+1} + m_i - 1, \ldots, N_{i+1} + 1).$$

Furthermore, $S_i^{'}$ satisfies the following conditions for $i = 1, \ldots, k-2$:

(1) All the entries in $S_i^{'}$ equal 0 or 1, but the diagonal entries must be 0.

(2) The entries in the j-th column must be non-increasing.

(3) The first m_i entries in the j-th row must be non-increasing, and the remaining entries in the j-th row are also non-increasing.

We continue to consider the trapezoidal array formed by the first m_i columns of $S_i^{'}$, and denote it by $S_i^{(2)}$. Again, we see that $S_i^{(2)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(2)}$, where

$$\alpha_1^{(2)} \leq N_{i+1}, \quad \text{and} \quad l(\alpha^{(2)}) \leq m_i.$$

Notice that there are still some columns to be dealt with. Define $S_i^{(3)}$ to be the trapezoidal array formed by the (m_i+1)-th column to the $(N_i - N_{i+1})$-th column of $S_i^{'}$. Once more, $S_i^{(3)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(3)}$, where

$$\alpha_1^{(3)} \leq N_{i+1} + m_i \quad \text{and} \quad l(\alpha^{(3)}) \leq N_i - N_{i+1} - m_i.$$

As a consequence, the generating function for possible choices of the i-th block S_i for $i = 1, \ldots, k-2$ equals

$$\sum_{m_i=0}^{N_i - N_{i+1}} q^{\frac{(N_{i+1}+1+N_{i+1}+m_i)m_i}{2}} \frac{(q;q)_{N_{i+1}+m_i}}{(q;q)_{m_i}(q;q)_{N_{i+1}}} \frac{(q;q)_{N_i}}{(q;q)_{N_{i+1}+m_i}(q;q)_{N_i-N_{i+1}-m_i}}$$

which can be rewritten as

$$\frac{(q;q)_{N_i}}{(q;q)_{N_{i+1}}(q;q)_{N_i-N_{i+1}}} \sum_{m_i=0}^{N_i-N_{i+1}} q^{\frac{(N_{i+1}+1+N_{i+1}+m_i)m_i}{2}} \frac{(q;q)_{N_i-N_{i+1}}}{(q;q)_{m_i}(q;q)_{N_i-N_{i+1}-m_i}}.$$

Evidently, the sum in the above expression is the generating function for partitions with distinct parts between $N_{i+1}+1$ and N_i. So we deduce that

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{\frac{(N_{i+1}+1+N_{i+1}+m_i)m_i}{2}} \frac{(q;q)_{N_i-N_{i+1}}}{(q;q)_{m_i}(q;q)_{N_i-N_{i+1}-m_i}} = (-q^{N_{i+1}+1};q)_{N_i-N_{i+1}}.$$

Since the generating function for S_{k-1} equals 1, the generating function for possible choices of S is the product of the generating functions for S_i for $i=1,\ldots,k-2$, that is,

$$\prod_{i=1}^{k-2} \frac{(q;q)_{N_i}}{(q;q)_{N_{i+1}}(q;q)_{N_i-N_{i+1}}}(-q^{N_{i+1}+1};q)_{N_i-N_{i+1}} = \frac{(q)_{N_1}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q;q)_{N_{k-1}}}.$$

Recall that the generating function for possible choices of $T(\lambda^{(2)})$ equals $1/(q;q)_{N_1}$ and the generating functions for $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ are equal to $q^{(N_1+1)N_1/2}$, $q^{N_2^2+N_2},\ldots$ $,q^{N_{k-1}^2+N_{k-1}}$ respectively. We also observe that the generating function for anti-lecture hall compositions in $F_{2k-2}(N_1,\ldots,N_{k-1},n)$ is the product of the generating functions for $T(\lambda^{(2)})$, $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ and S. Hence it equals

$$\begin{aligned} &\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}}{(q)_{N_1}} \frac{(q)_{N_1}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q;q)_{N_{k-1}}} \\ = &\quad \frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q;q)_{N_{k-1}}}. \end{aligned}$$

Summing up the generating functions for anti-lecture hall compositions in $F_{2k-3}(N_1,\ldots,N_{k-1},n)$ yields the generating function for $F_{2k-3}(n)$,

$$\sum_{n\geq 0}|F_{2k-3}(n)|q^n = \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q;q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q;q)_{N_{k-1}}}. \tag{4.25}$$

This completes the proof. ∎

For example, the composition $\lambda=(5,10,14,17,18,20,18,15,12,3)$ can decomposed into the following triangular arrays

$$\begin{array}{cccccccccc} 5&5&5&5&4&4&3&2&2&1\\ &5&5&4&4&4&3&2&2&1\\ &&4&4&4&3&3&2&2&1\\ &&&4&3&3&3&2&1&0\\ &&&&3&3&2&2&1&0\\ &&&&&3&2&2&1&0\\ &&&&&&2&2&1&0\\ &&&&&&&1&1&0\\ &&&&&&&&1&0\\ &&&&&&&&&0 \end{array} \longrightarrow \begin{array}{ccccccccc} 1&1&1&1&1&1&1&1&1\\ &1&1&1&1&1&1&1&1\\ &&1&1&1&1&1&1&1\\ &&&1&1&1&1&1&1\\ &&&&1&1&1&1&1\\ &&&&&1&1&1&1\\ &&&&&&1&1&1\\ &&&&&&&1&1\\ &&&&&&&&1 \end{array} + \begin{array}{cccccc} 2&2&2&2&2&2\\ &2&2&2&2&2\\ &&2&2&2&2\\ &&&2&2&2\\ &&&&2&2\\ &&&&&2 \end{array} + \begin{array}{cc} 2&2\\ &2 \end{array}$$

$$T(\lambda) \qquad\qquad R(9,1) \qquad\qquad R(6,2) \qquad\qquad R(3,2)$$

$$
+\quad
\begin{array}{ccccccccc}
0 & 0 & 2 & 2 & 1 & 1 & 2 & 1 & 1 \\
 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\
 & & 1 & 1 & 1 & 0 & 2 & 1 & 1 \\
 & & & 1 & 0 & 0 & 2 & 1 & 0 \\
 & & & & 0 & 0 & 1 & 1 & 0 \\
 & & & & & 0 & 1 & 1 & 0 \\
 & & & & & & 1 & 1 & 0 \\
 & & & & & & & 0 & 0 \\
 & & & & & & & & 0
\end{array}
\quad + \quad
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{array}
$$

$$
\qquad\qquad S \qquad\qquad\qquad\qquad\qquad\qquad \lambda^{(2)}
$$

In virtue of (2.12), Theorem 4.1 immediately follows from Theorem 4.2, since the generating function for overpartitions in $H_{2k-1}(n)$ is given by

$$
\frac{(-q;q)_\infty (q;q^{2k-1})_\infty (q^{2k-2};q^{2k-1})_\infty (q^{2k-1};q^{2k-1})_\infty}{(q;q)_\infty}. \tag{4.26}
$$

We now come to the second proof of Theorem 4.1. In their proof of anti-lecture hall theorem, Corteel and Savage [5] established two bijections. The first is a bijection between the set $E(n)$ of anti-lecture hall compositions μ of n such that $\lfloor \mu_i/i \rfloor$ is even and the set $P(n)$ of partitions of n with each part greater than one. The second bijection is between the set $A(n)$ of anti-lecture hall compositions of n and the set $D \times E(n)$ of pairs (λ, μ) such that $|\lambda| + |\mu| = n$ and $\lambda \in D$, $\mu \in E$, where D is the set of partitions into distinct parts. Then the anti-lecture hall theorem can follows from the correspondence between $A(n)$ and $D \times P(n)$.

We shall present a bijection between a subset of $P(n)$ and a subset of $E(n)$. Together with the second bijection of Corteel and Savage, we arrive at the assertion in Theorem 4.1.

To be more specific, let $Q_k(n)$ be the subset of $E(n)$ consisting of anti-lecture hall compositions λ such that $\lambda_1 \leq k$ and let $R_k(n)$ be the subset of $P(n)$ consisting of partitions having at most $k-1$ successive $N \times (N+1)$ Durfee rectangles such that there is no part below the last Durfee rectangle. Then we have the following correspondence, which can be considered as a refined version of the first bijection of Corteel and Savage.

Theorem 4.3 *There is a bijection between the set $R_k(n)$ and the set $Q_{2k-2}(n)$.*

Proof. We proceed to give a construction of the bijection θ from $R_k(n)$ to $Q_{2k-2}(n)$. Consider the A-triangular representation $T(\mu)$ of an anti-lecture hall composition μ of n such that $\lfloor \frac{\mu_i}{i} \rfloor$ are even for all i and $\mu_1 \leq 2k-2$. By definition, all the diagonal entries of $T(\mu)$ are even and $t_{1,1} \leq 2k-2$.

Now we define the map θ from a partition λ in P with exactly $k-1$ successive Durfee rectangles to an anti-lecture hall composition μ of n.

Step 1. We break the Ferrers diagram of λ into $k-1$ blocks such that the i-th block contains the i-th Durfee rectangle and the dots on the right of the i-th Durfee rectangle.

Step 2. Change the i-th Durfee rectangle in the i-th block into a triangular array with all entries being 2, and the rest dots in the i-th block into entries equal to 1. Then these $k-1$

blocks become $k-1$ A-triangles with all the diagonal entries equal to 0 or 2 where 0's are omitted.

Step 3. Put the $k-1$ A-triangles obtained in Step 2 together to form an A-triangle T.

The resulting A-triangle corresponds to an anti-lecture hall composition μ such that $\mu_1 = 2k-2$ and $\lfloor \lambda_i / i \rfloor$ are even for all i.

It is easily verified that the map θ is reversible. This completes the proof. ∎

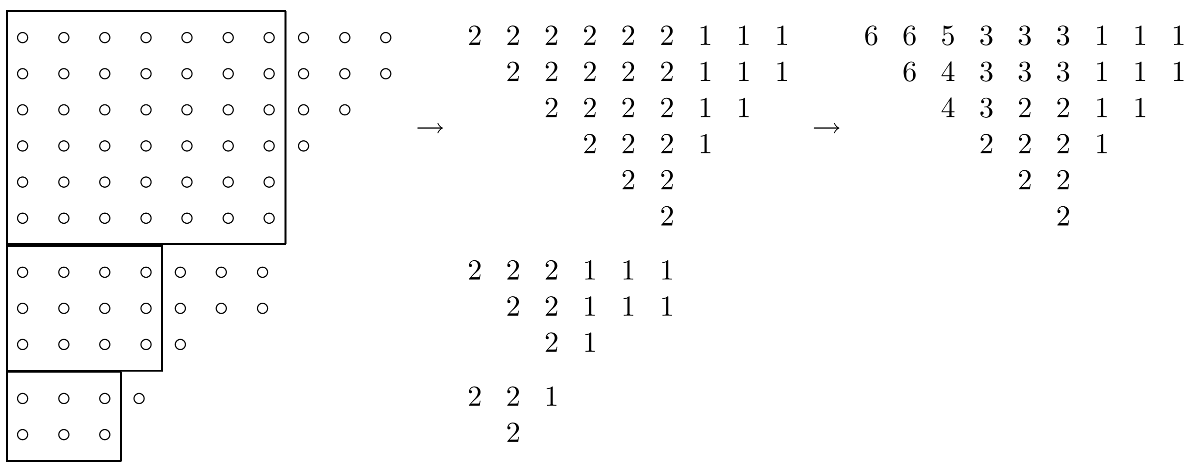
For example, let

$$\lambda = (10, 10, 9, 8, 7, 7, 7, 7, 5, 4, 3)$$

be a partition in $R_4(77)$. Then the corresponding anti-lecture hall composition in $Q_6(77)$ equals

$$\mu = (6, 12, 13, 11, 12, 14, 4, 3, 2).$$

The successive Durfee rectangles of λ are exhibited as follows.



Second Proof of Theorem 4.1. Examining Corteel and Savage's second bijection γ from A to $D \times E$, we see that it maps an anti-lecture hall composition of n in A with the first part not exceeding $2k-1$ to a pair (α, β) in $D \times E$ such that β is an anti-lecture hall composition in E with the first part β_1 not exceeding $2k-2$ and the sum of parts of α and β equals n. In other words, γ is a bijection between F_{2k-1} and $D \times Q_{2k-2}$. Together with Theorem 4.3, we are led to a bijection between F_{2k-1} and $D \times R_k$.

On the other hand, there is a combinatorial interpretation of the left hand side of (1.5) in terms of the Durfee dissection of a partition, given by Andrews [3]. We observe that technique of Andrews easily extends to Durfee rectangle dissection of a partition. In this way, we find that the generating function of partitions in $R_k(n)$ is given by

$$\sum_{n=0}^{\infty} |R_k(n)| q^n = \sum_{N_1 \geq N_2 \geq \ldots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \ldots + N_{k-1}^2 + N_1 + \ldots + N_{k-1}}}{(q)_{N_1 - N_2} \ldots (q)_{N_{k-2} - N_{k-1}} (q)_{N_{k-1}}}. \tag{4.27}$$

Setting $a = 1$ in the generalization of the Rogers-Ramanujan identity (1.5) gives

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_1 + \cdots + N_{k-1}}}{(q)_{N_1 - N_2} \ldots (q)_{N_{k-2} - N_{k-1}} (q)_{N_{k-1}}} = \frac{(q, q^{2k}, q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty}.$$

Hence the generating function of partitions in $R_k(n)$ can be expressed as follows

$$\sum_{n=0}^{\infty} |R_k(n)| q^n = \frac{(q, q^{2k}, q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty}. \tag{4.28}$$

By the bijection between $F_{2k-1}(n)$ and $D \times R_k(n)$ we conclude that

$$\sum_{n=0}^{\infty} |F_{2k-1}(n)| q^n = \frac{(-q;q)_\infty (q, q^{2k}, q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty}. \tag{4.29}$$

It is easy to see that the right hand side of the above identity is the generating function of overpartitions in $H_{2k+1}(n)$. This completes the proof. ∎

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] G.E. Andrews, An analytic generalization of the Rogers-Ramanujan identities for odd moduli, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 4082–4085.

[2] G.E. Andrews, Problems and prospects for basic hypergeometric series, in: R. Askey. Theory and Application of Special Functions, Academic Press, New York 1975, 191–224.

[3] G.E. Andrews, Partitions and Durfee dissection, Amer. J. Math., 101 (1979), 735–742.

[4] M. Bousquet-Mélou and K. Eriksson, A refinement of the lecture hall theorem, J. Combin. Theory Ser. A, 86 (1999), 63–84.

[5] S. Corteel and C.D. Savage, Anti-lecture hall compositions, Discrete Math., 263 (2003), 275–280.

[6] S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc., 356 (2004), 1623–1635.

[7] S. Corteel and O. Mallet, Overpartitions, lattice paths, and Rogers-Ramanujan identities, J. Combin. Theory Ser. A, 114 (2007), 1407–1437.

[8] B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, Amer. J. Math., 83 (1961), 393–399.