Congruences for the Number of Cubic Partitions Derived from Modular Forms

William Y.C. Chen[1] and Bernard L.S. Lin[2]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

[1]chen@nankai.edu.cn, [2]linlishuang@cfc.nankai.edu.cn

Abstract

We obtain congruences for the number $a(n)$ of cubic partitions using modular forms. The notion of cubic partitions is introduced by Kim in connection with Ramanujan's cubic continued fractions. Chan has shown that $a(n)$ has several analogous properties to the number $p(n)$ of partitions, including the generating function, the continued fraction, and congruence relations. To be more specific, we show that $a(25n+22) \equiv 0 \pmod 5$, $a(49n+15) \equiv a(49n+29) \equiv a(49n+36) \equiv a(49n+43) \equiv 0 \pmod 7$. Furthermore, we prove that $a(n)$ takes infinitely many even values and infinitely odd values.

Keywords: cubic partition, congruence, modular form, Ramanujan's cubic continued fraction, parity.

AMS Classification: 11F33, 11P83

1 Introduction

The main objective of this paper is to study congruence relations for the number of cubic partitions by constructing suitable modular forms. The number of cubic partitions, denoted by $a(n)$, originated from the work of Chan [6] in connection with Ramanujan's cubic continued fraction which is often denoted by

$$G(q) := \frac{q^{1/3}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+} \, \frac{q^3+q^6}{1} \, {}_{+\cdots}, \quad |q| < 1.$$

On page 366 of his Lost Notebook, Ramanujan claimed that there are many properties of $G(q)$ which are analogous to Rogers-Ramanujan continued fraction $R(q)$ [21]

$$R(q) := \frac{q^{1/5}}{1} \, {}_{+} \frac{q}{1} {}_{+} \frac{q^2}{1} {}_{+} \frac{q^3}{1} {}_{+\cdots}, \quad |q| < 1.$$

Motivated by Ramanujan's observation, many new results on $G(q)$ analogous to those for $R(q)$ have been found, see, e.g., Chan [9]. To give an overview of recent results on $a(n)$, it is informative to recall relevant background on the generating function of $p(n)$ and the Rogers-Ramanujan continued fraction $R(q)$.

Ramanujan obtained many theorems on $R(q)$, see Andrews and Berndt [1]. In particular, he discovered the following beautiful identities on $R(q)$ and $1/R(q)$.

$$\frac{1}{R(q)} - 1 - R(q) = \frac{(q^{1/5};q^{1/5})_\infty}{q^{1/5}(q^5;q^5)_\infty} \tag{1.1}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}. \tag{1.2}$$

Here $(q;q)_\infty$ is the usual notation for $\prod_{n=1}^{\infty}(1-q^n)$.

Berndt [5, p.165] gave a beautiful proof of the following classical identity of Ramanujan by using the continued fraction $R(q)$:

$$\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5}\sum_{n=0}^{\infty} p(5n+4)q^n = 5. \tag{1.3}$$

Dividing (1.2) by (1.1), we get

$$\frac{(q;q)_\infty^6}{q^{4/5}(q^{1/5};q^{1/5})_\infty(q^5;q^5)_\infty^5}$$
$$= R^4(q) - R^3(q) + 2R^2(q) - 3R(q) + 5 + \tfrac{3}{R(q)} + \tfrac{2}{R^2(q)} + \tfrac{1}{R^3(q)} + \tfrac{1}{R^4(q)}. \tag{1.4}$$

Now, (1.3) can be easily deduced from (1.4) by extracting the integer powers of $q^n, n \geq 0$ from both sides of above identity since $R(q)$ has only terms in the form of $q^{n+1/5}$. Ramanujan's congruence on $p(n)$ modulo 5 can be derived directly from (1.3)

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.5}$$

Recently, using two identities of Ramanujan [21] on $G(q)$, see also Berndt [4, p.345, Entry 1], Chan [6] has found the following identities on $G(q)$ and $1/G(q)$ analogous to the above identities (1.1) and (1.2):

$$\frac{1}{G(q)} - 1 - 2G(q) = \frac{(q^{1/3};q^{1/3})_\infty(q^{2/3};q^{2/3})_\infty}{q^{1/3}(q^3;q^3)_\infty(q^6;q^6)_\infty}, \tag{1.6}$$

$$\frac{1}{G^3(q)} - 7 - 8G^3(q) = \frac{(q;q)_\infty^4(q^2;q^2)_\infty^4}{q(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}. \tag{1.7}$$

Motivated by the idea of Berndt, Chan derived the following identity by dividing both sides of (1.7) by (1.6) and then setting $q \to q^3$:

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = q^2\frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}\left(4G^2(q^3) - 2G(q^3) + 3 + \frac{1}{G(q^3)} + \frac{1}{G^2(q^3)}\right). \tag{1.8}$$

Observing that the powers of q in $G(q^3)$ are in the form of $3n+1$, we find

$$\sum_{n=0}^{\infty}\left[q^{3n}\right]\left(4G^2(q^3) - 2G(q^3) + 3 + \frac{1}{G(q^3)} + \frac{1}{G^2(q^3)}\right)q^{3n} = 3.$$

It is now natural to define a function $a(n)$ by the left hand side of (1.8)

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}, \tag{1.9}$$

and it is natural to expect $a(n)$ to have analogous properties to $p(n)$.

Extracting those terms whose powers of q are in the form of $3n+2$ on both sides of (1.8), and then simplifying and setting $q^3 \to q$, Chan established the following elegant identity analogous to (1.3)

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}. \tag{1.10}$$

The above identity immediately leads to the following congruence

$$a(3n+2) \equiv 0 \pmod{3}, \tag{1.11}$$

which is analogous to Ramanujan's congruence (1.5) for $p(n)$.

From the point of view of partitions, it is obvious from the generating function (1.9) that $a(n)$ is the number of partition pairs (λ, μ) where $|\lambda| + |\mu| = n$ and μ only has even parts. Chan has called $a(n)$ a certain partition function. Kim [11] called such partitions counted by $a(n)$ cubic partitions owing to the fact that $a(n)$ is close related to Ramanujan's cubic continued fraction.

Based on the cubic partition interpretation of $a(n)$, Chan [8] asked whether there exist a function analogous to Dyson's rank that leads to a combinatorial interpretation of the congruence (1.5). Kim [11] discovered a crank function $N_V^a(m,n)$ for cubic partitions. Let $M'(m,N,n)$ be the number of cubic partitions of n with crank $\equiv m \pmod{N}$, Kim proved that

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod{3},$$

which implies (1.11).

Our main results are concerned with congruences for $a(n)$ modulo 5 and 7 which are in the spirit of Ramanujan's classical congruences modulo 7 and 11. Recall that Ramanujan obtained more general congruences modulo 5^k:

$$p(5^k n + r_k) \equiv 0 \pmod{5^k}, \tag{1.12}$$

where $k \geq 1$ and $5^k r_k \equiv 1 \pmod{24}$. In analogy with Ramanujan's congruences, Chan considered the general congruences for $a(n)$ modulo powers of 3. Employing the method of Hirschhorn and Hunt [10] to prove (1.12), Chan [7] derived the following congruence as a consequence of (1.10).

Theorem 1.1. *For $k \geq 1$,*

$$a(3^k n + c_k) \equiv 0 \pmod{3^{k+\delta(k)}}, \tag{1.13}$$

where c_k is the reciprocal modulo 3^k of 8, and $\delta(k) = 1$ if k is even and $\delta(k) = 0$ otherwise.

In the general case, Ramanujan conjectured that there are only three choices for a prime l such that the congruence $p(ln+c) \equiv 0 \pmod{l}$ holds, namely, $l = 5, 7, 11$. This conjecture has been confirmed by Ahlgren and Boylan [3] based on the work of Kiming and Olsson [12]. Chan [7] raised the problem of finding simple congruences for $a(n)$ besides $a(3n+2) \equiv 0$ mod 3. Recently, Sinick [22] has shown that there does not exist other primes l such that $a(ln+c) \equiv 0 \pmod{l}$ except that $l = 3$. In analogy with the results for $p(n)$ due to Ono [17] and Ahlgren [2], Chan [8] obtained the following theorem concerning congruences for $a(n)$ modulo powers of a prime.

Theorem 1.2. *Let $m \geq 5$ be prime and j a positive integer. Then a positive proportion of the primes $Q \equiv -1 \pmod{128m^j}$ have the property that*

$$a\left(\frac{mQn+1}{8}\right) \equiv 0 \pmod{m^j},$$

for every n coprime to Q.

The above theorem implies that for every integer n there exists infinitely many non-nested arithmetic progressions $An+B$ for prime $m \geq 5$ and positive integer j such that

$$a(An+B) \equiv 0 \pmod{m^j}.$$

It should be noted that although the proof of Theorem 1.2 leads to some Ramanujan-type congruences modulo m^j, it does not cover all the congruences in form of $a(An+B) \equiv 0 \pmod{m^j}$. Chan [7] studied the case for $m = 3$, which is not in the scope of Theorem 1.2. This paper is devoted to finding concrete congruences for the cases $m = 5, 7$ and $j = 1$, which are also out of the range of Theorem 1.2 since the Q is larger than 1278 and 2686 for $m = 5$ and 7, respectively. To be precise, we derive the following congruences by constructing suitable modular forms.

Theorem 1.3. *For every nonnegative integer n, we have*

$$a(25n+22) \equiv 0 \pmod{5}.$$

It would be interesting to give a combinatorial interpretation of the above congruence by finding a suitable crank function. In the following theorem, we present some congruences modulo 7.

Theorem 1.4. *For every nonnegative integer n, we have*

$$a(49n+15) \equiv a(49n+29) \equiv a(49n+36) \equiv a(49n+43) \equiv 0 \pmod{7}.$$

The last section of this paper is focused on the parity of $a(n)$. Recall that Kolberg [13] has shown that $p(n)$ takes both even and odd values infinitely often. From numerical evidence, we conjecture that when n tends to infinity, half of the values of $a(1), a(2), \ldots a(n)$ are even and half of them are odd. While we have not been able to prove this conjecture, we shall show that there are infinitely many even values of $a(n)$ and there are infinitely many odd values of $a(n)$.

2 Preliminaries

To make this paper self-contained, we give an overview of the background relevant to the proofs of the congruences for $a(n)$ by using modular forms. For more details on the theory of modular forms, see for example, Koblitz [14] and Ono [18].

For a rational integer $N \geq 1$, the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| c \equiv 0 \pmod{N} \right\}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ act on the complex upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$$

by the linear fractional transformation

$$\gamma z := \frac{az+b}{cz+d}.$$

Suppose that k is a positive integer and χ is a Dirichlet character modulo N.

Definition 2.1. *Let $f(z)$ be a holomorphic function on $\mathbb{H}$ and satisfy the following relation for all $\gamma \in \Gamma_0(N)$ and all $z \in \mathbb{H}$,*

$$f(\gamma z) = \chi(d)(cz+d)^k f(z).$$

In addition, if $f(z)$ is also holomorphic at the cusps of $\Gamma_0(N)$, we call such a function $f(z)$ a modular form of weight k on $\Gamma_0(N)$.

The modular forms of weight k on $\Gamma_0(N)$ with Dirichlet character χ form a finite-dimensional complex vector space denoted by $M_k(\Gamma_0(N), \chi)$. For convenience, we write $M_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi)$ when χ is the trivial Dirichlet character.

Dedekind's eta function is defined by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$. It is well-known that $\eta(z)$ is holomorphic and does not vanish on $\mathbb{H}$.

A function $f(z)$ is called eta-quotient if it can be written in the form of

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z),$$

where $N \geq 1$ and each r_δ is an integer. The following two facts is useful to verify whether an eta-quotient is a modular form, see Ono [18, p.18].

Proposition 2.1. *If $f(z)=\prod_{\delta|N}\eta^{r_\delta}(\delta z)$ is an eta-quotient with*

$$k=\frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z},$$

satisfies the following conditions:

$$\sum_{\delta|N}\delta r_\delta \equiv 0 \pmod{24} \tag{2.1}$$

and

$$\sum_{\delta|N}\frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \tag{2.2}$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right)=\chi(d)(cz+d)^k f(z) \tag{2.3}$$

for each $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\in\Gamma_0(N)$. Here the character χ is defined by $\chi(d):=\left(\frac{(-1)^k s}{d}\right)$, where

$$s:=\prod_{\delta|N}\delta^{r_\delta}$$

and $\left(\frac{m}{n}\right)$ is Kronecker symbol.

Based on this proposition, for a given eta-quotient $f(z)$, by checking the conditions (2.1) and (2.2), one can show that $f(z)$ satisfies (2.3). Moreover, if k is a positive integer and $f(z)$ is holomorphic at the cusps of $\Gamma_0(N)$, then $f(z)\in M_k(\Gamma_0(N),\chi)$ because $\eta(z)$ is holomorphic and does not vanish on $\mathbb{H}$. Combined with the following proposition which gives the analytic orders of an eta-quotient at the cusps of $\Gamma_0(N)$, we can deduce that $f(z)$ is a modular form.

Proposition 2.2. *Let c,d and N be positive integers with $d|N$ and $(c,d)=1$. If $f(z)$ is an eta-quotient satisfying the conditions in Proposition 2.1 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta|N}\frac{(d,\delta)^2 r_\delta}{(d,\frac{N}{d})d\delta}.$$

In the other words, to prove that the above function $f(z)$ is holomorphic at the cusp $\frac{c}{d}$, it suffices to check that

$$\sum_{\delta|N}\frac{(d,\delta)^2 r_\delta}{\delta}\geq 0.$$

Let M be a positive integer and

$$f(z)=\sum_{n=0}^{\infty}a(n)q^n$$

be a function with rational integer coefficients. Define $ord_M(f(z))$ to be the smallest n such that $a(n)\not\equiv 0 \pmod{M}$. Sturm [23] provided the following powerful criterion to determine whether two modular forms are congruent modulo a prime by the verification of a finite number of cases.

Proposition 2.3. *Let p be a prime and $f(z), g(z) \in M_k(\Gamma_0(N))$ with rational integer coefficients. If*

$$ord_p(f(z) - g(z)) > \frac{kN}{12} \prod_d (1 + \frac{1}{d}),$$

where the product is over the prime divisors d of N. Then $f(z) \equiv g(z) \pmod p$, i.e., $ord_p(f(z) - g(z)) = \infty$.

We also need the following result due to Lovejoy [15].

Proposition 2.4. *Let*

$$f = \sum_{n=0}^{\infty} u(n) q^n$$

and

$$g = 1 + \sum_{n=1}^{\infty} v(mn) q^{mn}.$$

Define $w(n)$ by

$$fg = \sum_{n=0}^{\infty} w(n) q^n.$$

Let d be a residue class modulo m. Then,

(1) If $u(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for $0 \leq n \leq N$, then $w(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for $0 \leq n \leq N$.

(2) If $w(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for all n, then $u(mn+d) \equiv 0 \pmod{\mathrm{M}}$ for all n.

The following two propositions will also be used to construct modular forms, see Koblitz [14].

Proposition 2.5. *Suppose $f(z) \in M_k(\Gamma_0(N))$ with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} u(n) q^n.$$

Then for any positive integer $m|N$,

$$f(z)|U(m) := \sum_{n=0}^{\infty} u(mn) q^n$$

is the Fourier expansion of a modular form in $M_k(\Gamma_0(N))$.

Proposition 2.6. *Let χ_1 be a Dirichlet character modulo M, and let χ_2 be a primitive Dirichlet character modulo N. Let*

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(M, \chi_1)$$

and

$$g(z) = \sum_{n=0}^{\infty} a(n) \chi_2(n) q^n.$$

Then $g(z) \in M_k(MN^2, \chi_1 \chi_2^2)$. In particular, if $f(z) \in M_k(\Gamma_0(M))$ and χ_2 is quadratic, then $g(z) \in M_k(\Gamma_0(MN^2))$.

3 Congruences for the Number of Cubic Partitions

In this section, we give the proofs of Theorem 1.3 and Theorem 1.4 using the technique of modular forms due to Ono [16]. The following congruence relation is well-known, see, for example, Ono [16]. We include a proof for the sake of completeness.

Lemma 3.1. *If $p \geq 3$ is a prime, then*

$$\frac{(q;q)_\infty^p}{(q^p;q^p)_\infty} \equiv 1 \pmod{p}. \tag{3.1}$$

Proof. Using the binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

it is easily seen that

$$\frac{(1-X)^p}{1-X^p} = \frac{\sum_{k=0}^{p} \binom{p}{k} (-1)^k X^k}{1-X^p} \equiv \frac{1-X^p}{1-X^p} \equiv 1 \pmod{p},$$

since $p \,|\, \binom{p}{k}$ for $0 < k < p$. It follows that

$$\frac{(q;q)_\infty^p}{(q^p;q^p)_\infty} = \prod_{k=1}^{\infty} \frac{(1-q^k)^p}{(1-q^{kp})} \equiv 1 \pmod{p},$$

as desired. ∎

We first consider Theorem 1.3, that is, for $n \geq 0$,

$$a(25n+22) \equiv 0 \pmod{5}.$$

Proof. To establish the claimed congruence relation, we shall construct an eta-quotient with the following expansion in $q = e^{2\pi i z}$,

$$g(z) = \sum_{n \geq 0} b(n) q^n.$$

We assume that $g(z)$ satisfies the following conditions

(1) $g(z)$ is a modular form;

(2) If for all $n \geq 0$, $b(25n+25) \equiv 0 \pmod{5}$ then $a(25n+22) \equiv 0 \pmod{5}$;

(3) The function

$$g(z)|U(25) = \sum_{n \geq 0} b(25n) q^n \equiv 0 \pmod{5}$$

is also a modular form.

In order to satisfy the second condition, we consider the function $h(q)$ of the following form

$$h(q) = \prod_i (q^{25r_i}; q^{25r_i})_\infty^{s_i} \prod_j \Big(\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}\Big)^{t_j},$$

where r_i, s_i, t_j are integers. By the above Lemma 3.1, it is easily seen that for any integers r_i, s_i and t_j the expansion of $h(q)$ has the following form modulo 5,

$$h(q) \equiv 1 + \sum_{m\geq 1} c(m) q^{25m} \pmod 5,$$

where $c(m)$ are integers. Now, we set

$$g(z) = h(q) \sum_{n=0}^{\infty} a(n) q^{n+3} = h(q) \sum_{n\geq 3}^{\infty} a(n-3) q^n. \tag{3.2}$$

Since $h(q)$ is a series in q^{25} modulo 5 with constant term 1, we can make use of Proposition 2.4 (2) to deduce that if for all n, $b(25n+25) \equiv 0 \pmod 5$, we have $a(25n+22) \equiv 0 \pmod 5$.

We now proceed to determine the parameters in $g(z)$ to make it a modular form. Consider the case $r_1 = 1, s_1 = 1, r_2 = 2, s_2 = 1$ and $t_1 = 2$, namely,

$$g(z) = (q^{25}; q^{25})_\infty (q^{50}; q^{50})_\infty \Big(\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}\Big)^2 \sum_{n=0}^{\infty} a(n) q^{n+3}. \tag{3.3}$$

We are going to show that $g(z)$ satisfies the conditions (2.1) and (2.2) in Proposition 2.1. Recalling the definition of $\eta(z)$, we can rewrite $g(z)$ as an eta-quotient

$$\begin{aligned} g(z) &= \frac{\eta(25z)\eta(50z)}{\eta(z)\eta(2z)} \left(\frac{\eta^5(z)}{\eta(5z)}\right)^2 \\ &= \frac{\eta^9(z)\eta(25z)\eta(50z)}{\eta(2z)\eta^2(5z)}. \end{aligned}$$

The two conditions (2.1) and (2.2) can be expressed as follows,

$$\begin{aligned} \sum_{\delta|50} \delta r_\delta &= 9 - 2 - 5\times 2 + 25 + 50 \equiv 0 \pmod{24}, \\ \sum_{\delta|50} \frac{50}{\delta} r_\delta &= 50\times 9 - 25 - 10\times 2 + 2 + 1 \equiv 0 \pmod{24}. \end{aligned}$$

To make $g(z)$ a modular form, it remains to compute the order of $g(z)$ at cusps. By Proposition 2.2, it is easily verified that the order of $g(z)$ at the cusps of $\Gamma_0(50)$ are nonnegative, that is, for any $d|50$,

$$\sum_{\delta|50} \frac{(d,\delta)^2 r_\delta}{\delta} \geq 0.$$

So we have $g(z) \in M_4(\Gamma_0(50), \chi_1)$, where

$$\chi_1(d) = \left(\frac{25}{d}\right) = \left(\frac{5}{d}\right)^2 = 1$$

for $(d, 50) = 1$. This implies that $g(z) \in M_4(\Gamma_0(50))$.

Since we have proved that $g(z)$ satisfies the second condition, the following congruence is valid

$$a(25n + 22) \equiv 0 \pmod{5} \tag{3.4}$$

provided that for all $n \geq 0$,

$$b(25n + 25) \equiv 0 \pmod{5}. \tag{3.5}$$

Let us rewrite (3.5) as

$$\sum_{n\geq 1} b(25n)q^n \equiv 0 \pmod{5}. \tag{3.6}$$

By Proposition 2.5, we see that the summation on the left hand side of (3.6) is a modular form, that is,

$$g(z)|U(25) = \sum_{n\geq 1} b(25n)q^n \in M_4(\Gamma_0(50)).$$

Hence, by Proposition 2.3, we find that (3.6) is valid if (3.5) holds for

$$0 \leq n \leq \frac{4 \times 50}{12}(1 + \frac{1}{2})(1 + \frac{1}{5}) + 1 = 31.$$

Applying Lemma 3.1 with $p = 5$ to (3.3), we have

$$\sum_{n\geq 3} b(n)q^n \equiv (q^{25}; q^{25})_\infty (q^{50}; q^{50})_\infty \sum_{n=0}^{\infty} a(n)q^{n+3} \pmod{5}. \tag{3.7}$$

Using the above relation and Proposition 2.4 (1), we see that the verification of (3.5) on $b(n)$ for $0 \leq n \leq 31$ can be reduced to the verification of (3.4) on $a(n)$ for a the same range $0 \leq n \leq 31$. It is readily checked that (3.4) holds for $0 \leq n \leq 31$. This completes the proof. ∎

Now, we turn to the proof of Theorem 1.4, namely,

$$a(49n + 15) \equiv a(49n + 29) \equiv a(49n + 36) \equiv a(49n + 43) \equiv 0 \pmod{7}. \tag{3.8}$$

Proof of Theorem 1.4. Following the above procedure in the proof of Theorem 1.3, by the generating function of $a(n)$, we construct an eta-quotient

$$\begin{aligned} h(z) &= \frac{\eta(49z)\eta(98z)}{\eta(z)\eta(2z)} \left(\frac{\eta^7(z)}{\eta(7z)}\right)^2 \\ &= \frac{\eta^{13}(z)\eta(49z)\eta(98z)}{\eta(2z)\eta^2(7z)} \\ &= \left(\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}\right)^2 (q^{49}; q^{49})_\infty (q^{98}; q^{98})_\infty \sum_{n=0}^{\infty} a(n)q^{n+6}. \end{aligned} \tag{3.9}$$

Setting $N = 98$, we see that $h(z)$ satisfies the conditions (2.1) and (2.2) in Proposition 2.1, namely,

$$\begin{aligned}\sum_{\delta|98} \delta r_\delta &= 13 - 2 - 2 \times 7 + 49 + 98 \equiv 0 \pmod{24},\\ \sum_{\delta|98} \frac{98}{\delta} r_\delta &= 98 \times 13 - 49 - 14 \times 2 + 2 + 1 \equiv 0 \pmod{24}.\end{aligned}$$

Moreover, it is not difficult to verify that the order of $h(z)$ at the cusps of $\Gamma_0(98)$ are non-negative by Proposition 2.2, that is, for any $d|98$,

$$\sum_{\delta|98} \frac{(d,\delta)^2 r_\delta}{\delta} \geq 0.$$

Hence we deduce that $h(z)$ is a modular form, i.e., $h(z) \in M_6(\Gamma_0(98), \chi_2)$. Moreover,

$$\chi_2(d) = \left(\frac{49}{d}\right) = \left(\frac{7}{d}\right)^2 = 1$$

for $(d, 98) = 1$. This implies that $h(z) \in M_6(\Gamma_0(98))$.

Write

$$h(z) = \sum_{n\geq 6} c(n) q^n.$$

Applying Lemma 3.1 with $p = 7$, (3.9) becomes

$$\sum_{n\geq 6} c(n) q^n \equiv (q^{49}; q^{49})_\infty (q^{98}; q^{98})_\infty \sum_{n\geq 6} a(n-6) q^n \pmod 7. \tag{3.10}$$

Since $(q^{49}; q^{49})_\infty (q^{98}; q^{98})_\infty$ can be expanded as series in q^{49} with constant term 1, we can make use of Proposition 2.4 (2) to deduce that the four congruences in (3.8) can be derived from the corresponding congruences for $c(n)$, i.e., for $n \geq 0$,

$$c(49n + 21) \equiv c(49n + 35) \equiv c(49n + 42) \equiv c(49n + 49) \equiv 0 \pmod 7. \tag{3.11}$$

Observing that the arithmetic progressions $49n + 21, 49n + 35, 49n + 42, 49n + 49$ in the above congruences are divided by 7, we construct another function based on $c(n)$ as follows

$$u(z) = \sum_{n\geq 1} d(n) q^n = \sum_{n\geq 1} c(7n) q^n.$$

By Proposition 2.5, we have $u(z) \in M_6(\Gamma_0(98))$. Obviously, (3.11) can be restated as

$$d(7n + 3) \equiv d(7n + 5) \equiv d(7n + 6) \equiv d(7n + 7) \equiv 0 \pmod 7. \tag{3.12}$$

As will be seen, one can combine the above four congruences into a single congruence relation. Define

$$v(z) = \sum_{n\geq 1} e(n) q^n = \sum_{n\geq 1} d(n) q^n - \sum_{n\geq 1} \left(\frac{n}{7}\right) d(n) q^n. \tag{3.13}$$

Since $\left(\frac{n}{7}\right) = 1$ for $n \equiv 1, 2, 4 \pmod 7$, $\left(\frac{n}{7}\right) = -1$ for $n \equiv 3, 5, 6 \pmod 7$ and $\left(\frac{n}{7}\right) = 0$ for $n \equiv 0 \pmod 7$, $v(z)$ can be expressed as

$$v(z) = \sum_{\left(\frac{n}{7}\right)=-1} 2d(n)q^n + \sum_{n\equiv 0 \pmod 7} d(n)q^n. \tag{3.14}$$

To prove (3.12), it suffices to show that $v(z) \equiv 0 \pmod 7$.

Denote the second summation in (3.13) by

$$w(z) = \sum_{n\geq 1} \left(\frac{n}{7}\right) d(n)q^n.$$

By Proposition 2.6, we see that $w(z)$ is a modular form. In other words, $w(z) \in M_6(\Gamma_0(4802))$. Since $u(z)$ is also a modular form, we obtain

$$v(z) = u(z) - w(z) \in M_6(\Gamma_0(4802).$$

By Proposition 2.3, we find that $v(z) \equiv 0 \pmod 7$ can be verified by a finite number of cases. To be precise, we need to check that

$$e(n) \equiv 0 \pmod 7$$

holds for $0 \leq n \leq 4117$. In view of (3.14), we only need to verify that (3.12) holds for $0 \leq n \leq \lceil\frac{4117-1}{7}\rceil - 1 = 587$. Since $d(n) = c(7n)$, it suffices to check (3.11) for $0 \leq n \leq 587$. Finally, using (3.10) and Proposition 2.4 (1), it is necessary to verify Theorem 1.4 holds only for $0 \leq n \leq 587$, which is an easy task. This completes the proof. ∎

4 The Parity of $a(n)$

In this section, we show that the function $a(n)$ takes infinitely many even values and infinitely many odd values. For the partition function $p(n)$, it has been conjectured by Parkin and Shanks [19] that the parities of $p(1), p(2), \ldots, p(N)$ are equidistributed when N tends to infinity. Using Euler's recurrence formula for $p(n)$,

$$p(n) + \sum_{0<\omega_j\leq n} (-1)^j p(n - \omega(j)) = 0, \tag{4.1}$$

where $\omega(j) = j(3j - 1)/2, -\infty < j < \infty$, Kolberg [13] proved that $p(n)$ takes both even and odd values infinitely often.

To prove the analogous property for $a(n)$, we need Jacobi's identity

$$\sum_{n=0}^{\infty} (-1)^n (2n + 1)q^{n(n+1)} = (q^2; q^2)_\infty^3 \tag{4.2}$$

and Gauss's identity

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}. \tag{4.3}$$

We now have the following recurrence relation modulo 2.

Theorem 4.1.

$$a(n)+\sum_{0<k+k^2\le n} a(n-k-k^2)\equiv \Delta(n) \pmod 2, \tag{4.4}$$

where $\Delta(n)=1$*, if* $n=s(s+1)/2$ *for some integer* s *and* $\Delta(n)=0$*, otherwise.*

Proof. Multiplying both sides of (1.9) by $(q^2;q^2)_\infty^3$, we get

$$(q^2;q^2)_\infty^3\sum_{n=0}^\infty a(n)q^n=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{4.5}$$

Substituting (4.2) and (4.3) into both sides of (4.5), we find that

$$\begin{aligned}\sum_{n=0}^\infty q^{n(n+1)/2} &= \sum_{n=0}^\infty(-1)^n(2n+1)q^{n(n+1)}\sum_{n=0}^\infty a(n)q^n\\ &\equiv \sum_{n=0}^\infty q^{n(n+1)}\sum_{n=0}^\infty a(n)q^n \pmod 2. \end{aligned} \tag{4.6}$$

Equating coefficients of q^n on both sides of (4.6) gives (4.4). This completes the proof. ∎

With the aid of the formula (4.4) and following the idea of Kolberg [13] for $p(n)$, we obtain the following theorem for $a(n)$.

Theorem 4.2. *There are infinitely many integers* n *such that* $a(n)$ *is even and there are infinitely many integers* n *such that* $a(n)$ *is odd.*

Proof. We prove by contradiction. Assume that there exists m such that $a(n)$ is odd for any $n\ge m$. Without loss of generality, we may assume that m is an even integer greater than 2. It is easy to show that there exists an integer $m^2+2m\le t\le m^2+3m+1$ such that $\Delta(t)=0$. Setting $t=m^2+2m+\delta$, where $0\le\delta\le m+1$. Substituting $n=t$ into (4.4) yields

$$a(m^2+2m+\delta)+a(m^2+2m+\delta-2)+\cdots+a(m+\delta)\equiv 0 \pmod 2. \tag{4.7}$$

But the left hand side of (4.7) is the sum of $m+1$ odd numbers, so it is also odd since m is even. This leads to a contradiction with parity of the right hand side of (4.7).

On the other hand, assume that $a(n)$ is even for any $n\ge m$, where $m\ge 5$. It is easy to verify the following inequalities for $k\ge 10$,

$$k(k-1)+1<\frac{\sqrt{2}k\big(\sqrt{2}k+1\big)}{2}<k(k+1).$$

Therefore, there are no integers $e(e+1)$ in the following interval with more than $k-\lceil\sqrt{2}k/2\rceil-2$ elements,

$$\left[k(k-1)+1,\frac{\lceil\sqrt{2}k\rceil\big(\lceil\sqrt{2}k\rceil+1\big)}{2}\right].$$

Choose k such that $k-\lceil\sqrt{2}k/2\rceil-2>m$ and set

$$t=\frac{\lceil\sqrt{2}k\rceil(\lceil\sqrt{2}k\rceil+1)}{2}.$$

Let r be the largest integer k such that $k^2 + k \leq t$. It follows that

$$t - r - r^2 > m.$$

Substituting $n = t$ into (4.4), we obtain that

$$a(t) + a(t-2) + \cdots + a(t - r - r^2) \equiv \Delta(t) \equiv 1 \pmod 2.$$

This is impossible since the left hand side of above congruence are a sum of even numbers. This completes the proof. ∎

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] G.E. Andrews and B.C. Berndt, Ramanujan's Lost Notebook, Part I, Springer, New York, 2005.

[2] S. Alhgren, The partition function modulo composite integers M, Math. Ann., 318 (2000), 795–803.

[3] S. Alhgren and M. Boylan, Arithmetic properties of the partition function, Invent. Math., 153 (2003), 487–502.

[4] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.

[5] B.C. Berndt, Number Theory in the Spirit of Ramanujan, Amer. Math. Soc., Providence, 2006.

[6] H.-C. Chan, Ramanujan's cubic continued fraction and a generalization of his "most beautiful identity", Int. J. Number Theory, to appear.

[7] H.-C. Chan, Ramanujan's cubic continued fraction and Ramanujan type congruences for a ceratin partition function, Int. J. Number Theory, to appear.

[8] H.-C. Chan, Distribution of a certain partition function modulo powers of primes, preprint, 2008.

[9] H.H. Chan, On Ramanujan's cubic continued fraction, Acta Arith., 73 (1995), 343–355.

[10] M.D. Hirschhorn and D.C. Hunt, A simple proof of the Ramanujan conjecture for powers of 5, J. Reine Angew. Math., 326 (1981), 1–17.

[11] B. Kim, A crank analog on a certain kind of partition function arising from the cubic continued fraction, preprint, 2008.

[12] I. Kiming and J. Olsson, Congruences like Ramanujan's for powers of the partition function, Arch. Math., 59 (1992), 348–360.

[13] O. Kolberg, Note on the parity of the partition function, Math. Scand., 7 (1959), 377–378.

[14] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag, 1984.

[15] J. Lovejoy, Ramanujan-type congruences for three colored Frobenius partitions, J. Number Theory, 86 (2000), 283–290.

[16] K. Ono, Congruences for Frobenius partitions, J. Number Theory, 57 (1996), 170–180.

[17] K. Ono, Distribution of the partition function modulo m, Ann. Math., 151 (2000), 293–307.

[18] K. Ono, The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series, CBMS Regional Conference Series in Mathematics, 102, Amer. Math. Soc., Province, RI, 2004.

[19] T.R. Parkin and D. Shanks, On the distribution of parity in the partition function, Math. Comp., 21 (1967), 466–480.

[20] S. Ramanujan, Congruence propertities of partitions, Proc. Lond. Math. Soc. (2), 19 (1919), 207–210.

[21] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delpi, 1988.

[22] J. Sinick, Ramanujan congruences for a class of eta quotients, arXiv:math.NT/0810.1931.

[23] J. Sturm, On the congruence of modular forms, Lecture Notes in Math. 1240 (1984), 275–280.