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Context-free Grammars and Stable Multivariate Polynomials over Stirling Permutations

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Abstract

Haglund and Visontai established the stability of the multivariate Eulerian polynomials as the generating polynomials of Stirling permutations, which serves as a unification of the results of Bóna, Brenti, Janson, Kuba, and Panholzer concerning Stirling permutations. Let $B_n(x)$ be the generating polynomials of the descent statistic over Legendre-Stirling permutations, and let $T_n(x) = 2^n C_n(x/2)$, where $C_n(x)$ are the second-order Eulerian polynomials. Haglund and Visontai proposed the problems of finding stable multivariate refinements of the polynomials $B_n(x)$ and $T_n(x)$. We provide solutions to these two problems by using context-free grammars. Moreover, the grammars enable us to obtain combinatorial interpretations of the multivariate polynomials in terms of Legendre-Stirling permutations and marked Stirling permutations.

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Keywords: Legendre-Stirling permutation, marked Stirling permutation, stable multivariate polynomial, context-free grammar, descent, plateau, ascent

1 Introduction

This paper presents an approach to the construction of stable combinatorial polynomials from the perspective of context-free grammars. The framework of using context-free grammars to generate combinatorial polynomials was proposed in [9]. We find context-free grammars leading to stable multivariate polynomials over Legendre-Stirling permutations and marked Stirling permutations. These stable multivariate polynomials provide solutions to two problems raised by Haglund and Visontai [17] in their study of stable multivariate refinements of the second-order Eulerian polynomials.

Let us first give an overview of the second-order Eulerian polynomials. These polynomials were defined by Gessel and Stanley [13] as the generating functions of the descent statistic over Stirling permutations. Let $[n]_2$ denote the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. A permutation $\pi = \pi_1\pi_2 \cdots \pi_{2n-1}\pi_{2n}$ of $[n]_2$ is called a Stirling permutation if π satisfies the following condition: if $\pi_i = \pi_j$ then $\pi_k > \pi_i$

whenever $i < k < j$. For $1 \le i \le 2n$, we say that i is a descent of π if $i = 2n$ or $1 \le i < 2n$ and $\pi_i > \pi_{i+1}$. Analogously, i is called an ascent of π if $i = 1$ or $1 < i \le 2n$ and $\pi_{i-1} < \pi_i$. For the sake of consistency, we set $\pi_0 = \pi_{2n+1} = 0$. Let Q_n denote the set of Stirling permutations on $[n]_2$. Let $C(n,k)$ be the number of Stirling permutations of $[n]_2$ with k descents, and let

$$C_n(x) = \sum_{k=1}^{n} C(n,k)x^k.$$

Gessel and Stanley [13] showed that

$$\sum_{n=0}^{\infty} S(n+k,k)x^n = \frac{C_n(x)}{(1-x)^{2k+1}},$$

where $S(n,k)$, as usual, denotes the Stirling number of the second kind. The numbers $C(n,k)$ are called the second-order Eulerian numbers by Graham, Knuth and Patashnik [14], and the polynomials $C_n(x)$ are called the second-order Eulerian polynomials by Haglund and Visontai [17]. Besides the connections with enumeration of Stirling permutations, the second-order Eulerian number $C(n,k)$ has other combinatorial interpretations, such as the number of Riordan trapezoidal words of length n with k distinct letters [23], the number of rooted plane trees on $n+1$ nodes with k leaves [18] and the number of matchings of the complete graphs on $2n$ vertices with $n-k$ left-nestings [20].

The Stirling permutations were further studied by Bóna [1], Brenti [8], Janson [18] and Janson, Kuba and Panholzer [19]. Bóna [1] introduced the notion of a plateau of a stirling permutation and studied the plateau statistic. Given a Stirling permutation $\pi = \pi_1\pi_2 \ldots \pi_{2n} \in Q_n$, the index $1 < i \le 2n$ is called a plateau of π if $\pi_{i-1} = \pi_i$. Bóna showed that the number of ascents, the number of descents and the number of plateaux have the same distribution over Q_n. Analogous to real-rootedness of the classical Eulerian polynomials, Bóna [1] proved the real-rootedness of the second-order Eulerian polynomials $C_n(x)$.

Theorem 1.1 *For any positive integer n, the roots of the polynomial $C_n(x)$ are all real, distinct, and non-positive.*

It should be noted that the real-rootedness of $C_n(x)$ is essentially the real rootedness of the generating function of generalized Stirling permutations obtained by Brenti [8]. A permutation π of the multiset $\{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}$ is called a generalized Stirling permutation of rank n if π satisfies the same betweenness condition for a Stirling permutation. Let Q_n^* denote the set of generalized Stirling permutations of rank n. In particular, if $r_1 = r_2 = \cdots = r_n = r$ for some r, then π is called an r-Stirling permutation of order n. Let $Q_n(r)$ denote the set of r-Stirling permutations of order n. It is clear that 1-Stirling permutations are ordinary permutations and 2-Stirling permutations are Stirling permutations. Brenti [8] showed that the descent generating polynomials over Q_n^* have only real roots.

Janson [18] defined the following trivariate generating function

$$C_n(x,y,z) = \sum_{\pi \in Q_n} x^{\mathrm{des}(\pi)} y^{\mathrm{asc}(\pi)} z^{\mathrm{plat}(\pi)},$$

where $\mathrm{des}(\pi)$, $\mathrm{asc}(\pi)$, and $\mathrm{plat}(\pi)$ denote the number of descents, the number of ascents, and the number of plateaux of π, respectively, and proved that

$C_n(x, y, z)$ is symmetric in x, y, z. This implies the equidistribution of these three statistics derived by Bóna.

The symmetric property of $C_n(x, y, z)$ was further extended to r-Stirling permutations by Janson, Kuba and Panholzer [19]. For an r-Stirling permutation, they introduced the notion of a j-plateau. For an r-Stirling permutation $\pi = \pi_1\pi_2 \ldots \pi_{nr}$ and an integer $1 \le j \le r-1$, a number $1 \le i < nr$ is called a j-plateau of π if $\pi_i = \pi_{i+1}$ and there are $j-1$ indices $l < i$ such that $\pi_l = \pi_i$, i.e., the number π_i appears j times up to the i-th position of π. Let j-plat(π) denote the number of j-plateaux of π. Meanwhile, define a descent and an ascent of π similar as ordinary permutations, and let des(π) and asc(π) denote the number of descents and ascents of π. Janson, Kuba and Panholzer [19] showed that the distribution of $(\text{des}, 1-\text{plat}, 2-\text{plat}, \ldots, (r-1)-\text{plat}, \text{asc})$ is symmetric over the set of r-Stirling permutations.

Based on the theory of stable multivariate polynomials recently developed by Borcea and Brändén [3–5], Haglund and Visontai [17] presented a unified approach to the stability of the generating functions of Stirling permutations and r-Stirling permutations. A polynomial $f(z_1, z_2, \ldots, z_n) \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ is said to be stable, if whenever the imaginary part $\text{Im}(z_i) > 0$ for all i then $f(z_1, z_2, \ldots, z_n) \neq 0$. Clearly, a univariate polynomial $f(z) \in \mathbb{R}[z]$ has only real roots if and only if it is stable.

For the case of univariate real polynomials, Pólya and Schur [22] characterized all diagonal operators preserving stability or real-rootedness. Recently, Borcea and Brändén [3–5] characterized all linear operators preserving stability of multivariate polynomials, see also the survey of Wagner [26]. This implies a characterization of linear operators preserving stability of univariate polynomials.

A multivariate polynomial is called multiaffine if the degree of each variable is at most 1. Borcea and Brändén showed that each of the operators preserving stability for multiaffine polynomials has a simple form. Using this property, Haglund and Visontai [17] obtained a stable multiaffine refinement of the second-order Eulerian polynomial $C_n(x)$. The similar methods are employed for other related combinatorial structures, see [2, 6, 7, 24, 25] for a few of other instances.

Given a Stirling permutation $\pi = \pi_1\pi_2 \cdots \pi_{2n} \in Q_n$, let

$$\begin{aligned} A(\pi) &= \{i|\pi_{i-1} < \pi_i, 1 \le i \le 2n\}, \\ D(\pi) &= \{i|\pi_i > \pi_{i+1}, 1 \le i \le 2n\}, \\ P(\pi) &= \{i|\pi_{i-1} = \pi_i, 1 \le i \le 2n\} \end{aligned}$$

denote the set of ascents, the set of descents and the set of plateaux of π, respectively. Let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$ and $Z = (z_1, z_2, \ldots, z_n)$. Define

$$C_n(X, Y, Z) = \sum_{\pi \in Q_n} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

Haglund and Visontai [17] proved the stability of $C_n(X, Y, Z)$.

Theorem 1.2 *The polynomial $C_n(X, Y, Z)$ is stable.*

It is worth mentioning that, as observed by Haglund and Visontai, the recurrence relation between $C_{n-1}(X, Y, Z)$ and $C_n(X, Y, Z)$ can be used to derive the symmetry of $C_n(X,Y,Z)$, which implies the symmetry of $C_n(x,y,z)$ obtained by Janson, Kuba and Panholzer [19].

Moreover, Haglund and Visontai extended the stability of $C_n(X,Y,Z)$ to generating polynomials of r-Stirling permutations by taking the j-plateau statistic into consideration. Let $P_j(\pi)$ denote the set of j-plateaux of π. For $i = 1,2,\ldots,r-1$, let $Z_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,n})$. Haglund and Visontai [17] obtained the following stable multivariate polynomial over r-Stirling permutations

$$E_n(X,Y,Z_1,\ldots,Z_{r-1}) = \sum_{\pi\in Q_n(r)} \left(\prod_{i\in D(\pi)} x_{\pi_i}\right)\left(\prod_{i\in A(\pi)} y_{\pi_i}\right)\prod_{j=1}^{r-1}\left(\prod_{i\in P_j(\pi)} z_{j,\pi_i}\right).$$

They also obtained a similar stable multivariate polynomial for generalized Stirling permutations.

Motivated by the real-rootedness of $C_n(x)$ and its stable multivariate refinement $C_n(X,Y,Z)$, Haglund and Visontai further considered the problem of finding stable multivariate polynomials as refinements of the generating polynomials of the descent statistic over Legendre-Stirling permutations. The Legendre-Stirling permutations were introduced by Egge [12] as a generalization of Stirling permutations in the study of Legendre-Stirling numbers of the second kind. For any $n \geq 1$, let M_n be the multiset $\{1,1,\bar{1},2,2,\bar{2},\ldots,n,n,\bar{n}\}$. A permutation $\pi = \pi_1\pi_2\ldots\pi_{3n}$ on M_n is called a Legendre-Stirling permutation if whenever $i<j<k$ and $\pi_i = \pi_k$ are both unbarred, then $\pi_j > \pi_i$. For a Legendre-Stirling permutation π on M_n, we say that i is a descent if either $i = 3n$ or $\pi_i > \pi_{i+1}$. Let $B_{n,k}$ denote the number of Legendre-Stirling permutations of M_n with k descents. Define

$$B_n(x) = \sum_{k=1}^{2n-1} B_{n,k}x^k.$$

Egge proved the real-rootedness of $B_n(x)$.

Theorem 1.3 *For $n > 0$, $B_n(x)$ has distinct, real, non-positive roots.*

In order to derive a stable multivariate refinement of $B_n(x)$, we introduce an approach of generating stable polynomials by a sequence of grammars. Based on the Stirling grammar given by Chen and Fu [10], we find a sequence $G_1, G_2, \ldots$ of context-free grammars to generate Legendre-Stirling permutations. Let D_n denote the differential operator associated with the grammar G_n, which leads to a stable multivariate refinement $B_n(X,Y,Z,U,V)$ of $B_n(x)$, that is,

$$B_n(X,Y,Z,U,V) = D_{2n}D_{2n-1}\ldots D_2D_1(x_0),$$

where $U = (u_1,u_2,\ldots,u_n)$ and $V = (v_1,v_2,\ldots,v_n)$, respectively. Then by applying Borcea and Brändén's characterization of linear operators and the grammatical interpretation of $B_n(X,Y,Z,U,V)$, we prove the stability of $B_n(X,Y,Z,U,V)$. On the other hand, according to the grammars, we obtain the following combinatorial interpretation

$$B_n(X,Y,Z,U,V) = \sum_{\pi} \prod_{i\in X(\pi)} x_{\pi_i} \prod_{i\in Y(\pi)} y_{\pi_i} \prod_{i\in Z(\pi)} z_{\pi_i} \prod_{i\in U(\pi)} u_{\pi_i} \prod_{i\in V(\pi)} v_{\pi_i},$$

where π runs over all Legendre-Stirling permutations on M_n. Here $X(\pi)$, $Y(\pi)$, $Z(\pi)$, $U(\pi)$ and $V(\pi)$ are defined as follows: For a Legendre-Stirling permutation

π on M_n, define

$$\begin{aligned}
X(\pi) &= \{i \,|\, \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears for the first time}\}, \\
Y(\pi) &= \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is unbarred}\}, \\
Z(\pi) &= \{i \,|\, \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears for the second time}\}, \\
U(\pi) &= \{i \,|\, \pi_{i-1} \leq \pi_i \text{ and } \pi_i \text{ is barred}\}, \\
V(\pi) &= \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is barred}\}.
\end{aligned}$$

Here we set $\pi_0 = \pi_{3n+1} = 0$. Then the real-rootedness of $B_n(x)$ is a consequence of the stability of $B_n(X, Y, Z, U, V)$ by setting $v_i = y_i = y$ and $x_i = z_i = u_i = 1$ for $0 \leq i \leq n$.

Haglund and Visontai also raised the question of finding stable multivariate refinements of the polynomials $T_n(x)$, which are given by

$$T_n(x) = 2^n C_n\left(\frac{x}{2}\right) = \sum_k 2^{n-k} C(n,k) x^k, \tag{1.1}$$

where $C(n, k)$ and $C_n(x)$, as before, denote the second-order Eulerian numbers and the second-order Eulerian polynomials respectively. The polynomials $T_n(x)$ were introduced by Riordan [23].

In view of the relation (1.1) between $T_n(x)$ and $C_n(x)$, we mark the Stirling permutations by some rule. We consider the following multivariate polynomials

$$T_n(X, Y, Z) = \sum_\pi \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i},$$

where π ranges over marked Stirling permutations of $[n]_2$. We shall show that the polynomials $T_n(X, Y, Z)$ are stable. The polynomial $T_n(x)$ becomes the specialization of $T_n(X, Y, Z)$ by setting $x_i = z_i = 1$ and $y_i = x$ for $0 \leq i \leq n$. This implies that $T_n(x)$ is real-rooted.

This paper is organized as follows. In Section 2, we give an overview of differential operators associated with context-free grammars. We find context-free grammars to generate the polynomials $C_n(X, Y, Z)$. In Section 3, we give context-free grammars to generate the multivariate polynomials $T_n(X, Y, Z)$. In Section 4, we obtain context-free grammars that lead to the multivariate generating polynomials $B_n(X, Y, Z, U, V)$. In Section 5, based on Borcea and Brändén's characterization of linear operators preserving stability, we prove that the formal derivative with respect to the grammar who generates $T_n(X, Y, Z)$ preserving stability for multiaffine polynomials. This leads to stability of $T_n(X, Y, Z)$. In Section 6, we provide an approach to find a new stability preserving operator when a grammar is not suitable to prove the stability of polynomials. In particular, we prove the stability of multivariate polynomials $B_n(X, Y, Z, U, V)$.

2 Context-free grammars

In this section, we give an overview of the idea of using context-free grammars G to generate combinatorial polynomials and combinatorial structures as developed in [9]. A context-free grammar G over an alphabet A is defined to be a set of production rules. Roughly speaking, a production rule means to substitute a letter in the alphabet A by a polynomial in A over a field. Given a context-free grammar, one may define a formal derivative D as a linear operator on polynomials in A, where the action of D on a letter is defined by the substitution rule of the grammar, the action of D on a sum of two polynomials u and v is defined by linear extension:

$$D(u+v) = D(u) + D(v),$$

and the action of D on the product of u and v is defined as Leibnitz rule, that is,

$$D(uv) = D(u)v + uD(v).$$

Many combinatorial polynomials can be generated by context-free grammars. Meanwhile, context-free grammars can be used to generate combinatorial structures. More precisely, one may use a word on an alphabet to label a combinatorial structure such that the context-free grammar serves as the procedure to recursively generate the combinatorial structures. Such a labeling of a combinatorial structure is called a grammatical labeling in [10].

For example, the grammar

$$G = \{a \to ax,\ x \to x\}$$

is used in [9] to generate the set of partitions of $[n]$ and the Stirling polynomials,

$$S_n(x) = \sum_{i=0}^{n} S(n,k)x^k,$$

where $S(n,k)$ denotes the Stirling number of the second kind. For a partition P, we label a block of P by letter x and label the partition itself by letter a, and we define the weight of a partition by the product of its labels. So a partition P with k blocks has the weight $w(P) = ax^k$. For example, the partition $\{\{1,2\},\{3\}\}$ is labeled as follows

$$\begin{array}{ccc} \{1,2\} & \{3\} & \\ x & x & a \end{array}.$$

In the above notation, we write a partition $P = \{P_1, P_2, \ldots, P_k\}$ of $[n]$ in such a way that the blocks are ordered in the increasing order of their minimum elements. Moreover, we put the letter a at the end of the partition.

Using the above grammatical labeling of a partition, we deduce that

$$D^n(a) = \sum_{P} w(P) = \sum_{k=1}^{n} S(n,k)ax^k = aS_n(x). \tag{2.1}$$

Here P runs over the partitions on $[n]$. Many properties of the Stirling polynomials follow from the above expression in terms of the differential operator D with respect to the grammar G.

Let us explain how the grammar works for the generation of partitions. For $n = 1$, there is one partition of $[1]$, that is, $\{\{1\}\}$, whose label is ax. Assume that we have generated all the partitions of $[n-1]$ by applying the operator D^{n-2} to $\{\{1\}\}$ with the initial grammatical labeling.

Let us give an example to demonstrate the action of the differential operator D with respect to the grammar G to a partition of $[n]$ with the aforementioned grammatical labeling. Consider the following partition of $\{1,2,3,4,5,6\}$

$$\begin{array}{cccc} \{1,3,6\} & \{2,5\} & \{4\} & \\ x & x & x & a \end{array}.$$

If we apply the substitution rule to the letter a, then we get ax which we rewrite as xa, where a still serves as the label of the new partition, and x stands for a new block $\{7\}$. In this case, we get a partition

$$\begin{array}{ccccc} \{1,3,6\} & \{2,5\} & \{4\} & \{7\} & \\ x & x & x & x & a \end{array}.$$

If we apply the substitution rule to the second letter x, then we get x. In this case, we insert the element 7 in the second block, and we are led to the following partition with a consistent grammatical labeling

$$\begin{array}{cccc} \{1,3,6\} & \{2,5,7\} & \{4\} & \\ x & x & x & a \end{array},$$

where by a consistent labeling we mean a relabeling that is consistent with the original definition of the grammatical labeling.

Starting with the empty partition with label a, we get

$$\begin{aligned}
D(a) &= \overset{\{1\}}{x}\,a,\\
D^2(a) &= \overset{\{1\}}{x}\,\overset{\{2\}}{x}\,a + \overset{\{1,2\}}{x}\,a,\\
D^3(a) &= \overset{\{1\}}{x}\,\overset{\{2\}}{x}\,\overset{\{3\}}{x}\,a + \overset{\{1\}}{x}\,\overset{\{2,3\}}{x}\,a + \overset{\{1,3\}}{x}\,\overset{\{2\}}{x}\,a + \overset{\{1,2\}}{x}\,\overset{\{3\}}{x}\,a + \overset{\{1,2,3\}}{x}\,a\,.
\end{aligned}$$

As the second example, we consider the Eulerian grammar

$$G = \{x \to xy,\ y \to xy\}$$

introduced by Dumont [11].

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n]$, let

$$\begin{aligned}
A(\pi) &= \{i \,|\, \pi_{i-1} < \pi_i\},\\
D(\pi) &= \{i \,|\, \pi_i > \pi_{i+1}\}
\end{aligned}$$

denote the set of ascents and the set of descents of π, respectively. Here, as usual, we set $\pi_0 = \pi_{n+1} = 0$. Let $A(n,k)$ denote the Eulerian number, that is, the number of permutations on $[n]$ with k descents.

In order to show how to use the Eulerian grammar to generate permutations, Chen and Fu [10] introduced a grammatical labeling of a permutation π on $[n]$: If i is an ascent of π, then π_{i-1} is labeled by x; If i is a descent, π_i is labeled by y. The weight of π is defined as the product of labels of elements in π, that is,

$$w(\pi) = x^{|A(\pi)|}y^{|D(\pi)|}.$$

For example, the grammatical labeling of the permutation $\pi = 325641$ is as follows:

$$\begin{array}{ccccccc} & 3 & 2 & 5 & 6 & 4 & 1 \\ x & y & x & x & y & y & y \end{array}.$$

Thus the weight of π equals $w(\pi) = x^3y^4$. This grammatical labeling leads to the following expression of the Eulerian polynomials. Dumont obtained an equivalent form in terms of cyclic permutations and gave an inductive proof.

Theorem 2.1 (Dumont) *Let D denote the formal derivative with respect to the Eulerian grammar. For $n \geq 1$, we have*

$$D^n(x) = \sum_{m=1}^{n} A(n,m) y^m x^{n+1-m}.$$

Let us now consider the grammar to generate Stirling permutations. Chen and Fu [10] introduced the grammar

$$G = \{x \to x^2 y, y \to x^2 y\}.$$

They defined a grammatical labeling of Stirling permutation π in Q_n as follows: Let $1 \leq i \leq 2n$. If $i \in A(\pi)$ or $i \in P(\pi)$, the element π_{i-1} is labeled by x; If $i \in D(\pi)$, the element π_i is labeled by y. Meanwhile, the weight of π, denoted by $w(\pi)$, is defined as the product of labels of elements in π. For example, the Stirling permutation $\pi = 233211$ has the following grammatical labeling

$$\begin{array}{ccccccc} & 2 & 3 & 3 & 2 & 1 & 1 \\ x & x & x & y & y & x & y \end{array}.$$

Then the weight of π is $w(\pi) = x^4 y^3$.

By using this grammatical labeling of Sitrling permutations, they proved that the second-order Eulerian polynomials can be obtained by the Stirling grammar G.

Theorem 2.2 (Chen and Fu) *Let D denote the formal derivative with respect to the Stirling grammar. For $n \geq 1$, we have*

$$D^n(x) = \sum_{m=1}^{n} C(n,m) x^{2n+1-m} y^m.$$

We shall give two sequences of grammars based on the Eulerian grammar and the Stirling grammar to solve the problems of Haglund and Visontai [17]. On one hand, we use these grammars to construct multivariate polynomials over Legendre-Stirling permutations and marked Stirling permutations. On the other hand, we use the grammars to construct stability preserving operators leading to the stability of the multivariate polynomials.

3 Marked Stirling permutations

In this section, we obtain a stable multivariate refinement of the polynomial $T_n(x)$, denoted by $T_n(X,Y,Z)$, which is defined as the generating functions of marked Stirling permutations on $[n]_2$. This provides a solution to the problem of Haglund and Visontai.

In order to prove the stability of $T_n(X,Y,Z)$, we find grammars $G_1, G_2, \ldots$ that can be used to generate $T_n(X,Y,Z)$. More precisely, define

$$G_n = \{x_i, z_i \to x_n y_n z_n, y_i \to 2x_n y_n z_n \mid 0 \leq i \leq n-1\}.$$

Let D_n denote the formal derivative with respect to G_n. Using a grammatical labeling of marked Stirling permutations, we shall show that the polynomial

$T_n(X,Y,Z)$ can be generated by $D_1, D_2, \ldots, D_n$. The stability of $T_n(X,Y,Z)$ can be established in Section 6 by using the operators $D_1, D_2, \ldots, D_n$.

A marked Stirling permutation is defined as follows. Given a Stirling permutation $\pi = \pi_1\pi_2\cdots\pi_{2n}$, if π_i is an element of π such that π_i occurs the second time in π and $\pi_i < \pi_{i+1}$, then we may mark the element π_i. We denote a marked element i by $\bar{i}$. A marked Stirling permutation is a Stirling permutation with some elements marked according to the above rule. Let $\bar{Q}_n$ denote the set of marked Stirling permutations on $[n]_2$. For example, there is only one marked Stirling permutation on $[1]_2$: 11, whereas there are four marked Stirling permutations on $[2]_2$:

$$2211, 1221, 1122, 1\bar{1}22.$$

Let $T(n,k)$ be the number of marked Stirling permutations on $[n]_2$ with k descents. Clearly,

$$T(n,k) = 2^{n-k} \cdot C(n,k),$$

where $C(n,k)$ denotes the second-order Eulerian number. Recall that $T_n(x)$ is defined by

$$T_n(x) = 2^n \cdot C_n\left(\frac{x}{2}\right) = \sum_{k=0}^{n} 2^{n-k} C(n,k) x^k.$$

Hence $T_n(x)$ is the generating function of marked Stirling permutations on $[n]_2$, that is,

$$T_n(x) = \sum_{k=0}^{n} T(n,k)x^k = \sum_{\pi \in \bar{Q}_n} x^{|D(\pi)|}.$$

In fact, Riordan [23] introduced the polynomials $T_n(x)$ and proved that $T_n(1)$ equals the Schröder number, namely, the number of series-reduced rooted trees with $n+1$ labeled leaves.

We shall prove that the polynomials $T_n(x)$ can be generated by the grammar

$$G = \{x \to x^2 y, y \to 2x^2 y\}.$$

The proof relies on the following grammatical labeling of a marked Stirling permutation. Let π be a marked Stirling permutation on $[n]_2$. If $i \in D(\pi)$, we label π_i by y. If $i \in A(\pi)$ or $i \in P(\pi)$, we label π_{i-1} by x. The weight of a marked Stirling permutation π on $[n]_2$ with m descents is given by

$$w(\pi) = x^{2n+1-m} y^m.$$

Theorem 3.1 *Let G be the grammar $G = \{x \to x^2 y, y \to 2x^2 y\}$ and D be the formal derivative associated with G. For $n \geq 1$,*

$$D^n(x) = \sum_{k=1}^{n} T(n,k) x^{2n-k+1} y^k.$$

Setting $x = 1$, we have

$$D^n(x)|_{x=1} = T_n(y).$$

Proof. We aim to show that $D^n(x)$ equals the sum of the weights of marked Stirling permutations of $[n]_2$ by induction on n, that is,

$$D^n(x) = \sum_{\pi \in \bar{Q}_n} w(\pi). \tag{3.1}$$

For $n=1$, (3.1) follows from the fact that the weight of 11, the only marked Stirling permutation on $[1]_2$, is x^2y. Assume that (3.1) holds for $n-1$, that is,

$$D^{n-1}(x)=\sum_{\pi\in\bar{Q}_{n-1}} w(\pi).$$

We now use an example to demonstrate the action of D on a marked Stirling permutation of $[n-1]_2$. Let $\pi=12\bar{2}331$ with the following grammatical labeling

$$\begin{array}{ccccccccccccc} & 1 & & 2 & & \bar{2} & & 3 & & 3 & & 1 & \\ x & & x & & x & & x & & x & & y & & y \end{array}.$$

If we apply the substitution rule $x\rightarrow x^2y$ to the fourth letter x, then we insert the two elements 4 4 after $\bar{2}$. We keep all the old labels and assign the labels x and y to the two new letters 4 4 from left to right. It is not difficult to see that the generated marked Stirling permutation has a consistent grammatical labeling

$$\begin{array}{ccccccccccccccccc} & 1 & & 2 & & \bar{2} & & 4 & & 4 & & 3 & & 3 & & 1 & \\ x & & x & & x & & x & & x & & y & & x & & y & & y \end{array}.$$

If we apply the substitution rule $y\rightarrow 2x^2y$ to the first letter y, then we insert 44 after the second element 3. We change the label of the second element 3 from y into x and assign x and y to the two new elements 44 from left to right. According to the marking rule, the second element 3 may be marked or unmarked. These two choices correspond the coefficient 2 in the substitution rule $y\rightarrow 2x^2y$. So we are led to the following two marked Stirling permutations with consistent grammatical labelings,

$$\begin{array}{ccccccccccccccccc} & 1 & & 2 & & \bar{2} & & 3 & & 3 & & 4 & & 4 & & 1 & \\ x & & x & & x & & x & & x & & x & & x & & y & & y \end{array},$$

and

$$\begin{array}{ccccccccccccccccc} & 1 & & 2 & & \bar{2} & & 3 & & \bar{3} & & 4 & & 4 & & 1 & \\ x & & x & & x & & x & & x & & x & & x & & y & & y \end{array}.$$

In general, it can be verified that the action of D on weights of marked Stirling permutations in $\bar{Q}_{n-1}$ generates the weights of marked Stirling permutations in $\bar{Q}_n$. So we deduce that (3.1) holds for n, that is,

$$D^n(x)=D(D^{n-1}(x))=D\left(\sum_{\pi\in\bar{Q}_{n-1}} w(\pi)\right)=\sum_{\sigma\in\bar{Q}_n} w(\sigma).$$

Hence the proof is complete by induction. ∎

As a multivariate refinement of $T_n(x)$, we define the following generating function of marked Stirling permutations on $[n]_2$,

$$T_n(X,Y,Z)=\sum_{\pi\in\bar{Q}_n}\prod_{i\in A(\pi)} x_{\pi_i}\prod_{i\in D(\pi)} y_{\pi_i}\prod_{i\in P(\pi)} z_{\pi_i}.$$

Let

$$G_n=\{x_i,z_i\rightarrow x_ny_nz_n, y_i\rightarrow 2x_ny_nz_n \mid 0\le i\le n-1\}.$$

We give a grammatical labeling of a marked Stirling permutation. For a marked Stirling permutation π on $[n]_2$, if $i\in A(\pi)$, we label π_{i-1} by x_{π_i}; if $i\in D(\pi)$, we label π_i by y_{π_i}; and if $i \in P(\pi)$, we label π_{i-1} by z_{π_i}. Then the weight of π equals

$$w(\pi) = \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

The following theorem shows that the polynomials $T_n(X,Y,Z)$ can be generated by the grammars $G_1, G_2, \ldots, G_n$.

Theorem 3.2 *Let D_n denote the formal derivative associated with the grammar G_n. For $n \geq 1$,*

$$T_n(X,Y,Z) = D_n D_{n-1} \cdots D_1(z_0).$$

The proof of the above theorem is analogous to that of Theorem 3.1. Hence the details are omitted. Here we use an example to illustrate the action of D_4 on the above marked Stirling permutation $\pi = 12\bar{2}331$ with the grammatical labeling

$$\begin{array}{ccccccc} & 1 & 2 & \bar{2} & 3 & 3 & 1 \\ x_1 & x_2 & z_2 & x_3 & z_3 & y_3 & y_1 \end{array}.$$

Applying the substitution rule $x_3 \to x_4 y_4 z_4$ to π, we get a marked Stirling permutation by inserting the two elements 44 after $\bar{2}$ and the consistent grammatical labeling is given below:

$$\begin{array}{ccccccccc} & 1 & 2 & \bar{2} & 4 & 4 & 3 & 3 & 1 \\ x_1 & x_2 & z_2 & x_4 & z_4 & y_4 & z_3 & y_3 & y_1 \end{array}.$$

Similarly, applying the substitution rule $y_3 \to 2x_4 y_4 z_4$ leads to two marked Stirling permutations by inserting 44 after the second element 3, since the second element 3 can be marked. The consistent grammatical labelings are

$$\begin{array}{ccccccccc} & 1 & 2 & \bar{2} & 3 & 3 & 4 & 4 & 1 \\ x_1 & x_2 & z_2 & x_3 & z_3 & x_4 & z_4 & y_4 & y_1 \end{array},$$

and

$$\begin{array}{ccccccccc} & 1 & 2 & \bar{2} & 3 & \bar{3} & 4 & 4 & 1 \\ x_1 & x_2 & z_2 & x_3 & z_3 & x_4 & z_4 & y_4 & y_1 \end{array}.$$

For $n = 0$, the empty permutation is labeled by z_0. We have $T_0(X,Y,Z) = z_0$. For $n = 1, 2$, we have

$$\begin{aligned} T_1(X,Y,Z) = D_1(z_0) &= x_1 \overset{1}{z_1} \overset{1}{y_1}, \\ T_2(X,Y,Z) = D_2 D_1(z_0) &= D_2(x_1 \overset{1}{z_1} \overset{1}{y_1}) \\ &= x_2 \overset{2}{z_2} \overset{2}{y_2} \overset{1}{z_1} \overset{1}{y_1} + x_1 \overset{1}{x_2} \overset{2}{z_2} \overset{2}{y_2} \overset{1}{y_1} + x_1 \overset{1}{z_1} \overset{1}{x_2} \overset{2}{z_2} \overset{2}{y_2} \\ &\quad + x_1 \overset{1}{z_1} \overset{\bar{1}}{x_2} \overset{2}{z_2} \overset{2}{y_2} \\ &= y_1 z_1 x_2 y_2 z_2 + x_1 y_1 x_2 y_2 z_2 + 2x_1 z_1 x_2 y_2 z_2. \end{aligned}$$

4 Legendre-Stirling permutations

In this section, we give refinements of the Stirling grammar and the Eulerian grammar, and we show that these refined grammars can be used to generate stable multivariate polynomials. Define a sequence $G_1, G_2, \ldots$ of grammars as follows. For $n \geq 1$, let

$$G_{2n-1} = \{x_i, y_i, z_i, u_i, v_i \to u_n v_n \mid 0 \leq i < n\},$$

and let

$$G_{2n} = \{u_n \to x_n z_n u_n, v_n \to x_n y_n z_n,$$
$$x_i, y_i, z_i, u_i, v_i \to x_n y_n z_n \mid 0 \leq i < n\}.$$

Clearly, G_{2n-1} is a refinement of the Eulerian grammar, and G_{2n} is a refinement of the Stirling grammar.

Let D_n denote the formal derivative with respect to the grammar G_n. We give a grammatical labeling of Legendre-Stirling permutations, which leads to a combinatorial interpretation of the multivariate polynomial $D_{2n}D_{2n-1}\cdots D_1(x_0)$. In doing so, we introduce several statistics of a Legendre-Stirling permutation. In terms of these statistics, we obtain a multivariate polynomial $B_n(X, Y, Z, U, V)$ as a refinement of $B_n(x)$, which can be generated by the operators $D_1, D_2, \ldots, D_{2n}$.

On the other hand, in Section 6, we shall use the operators $D_1, D_2, \ldots, D_{2n}$ to prove the stability of $B_n(X, Y, Z, U, V)$. This leads to a solution to the problem of Haglund and Visontai.

We shall also show that the grammars $G_1, G_3, \ldots, G_{2n-1}$ can be used to generate the stable multivariate polynomial $C_n(X, Y, Z)$, introduced by Haglund and Vistonai [17] as a stable multivariate refinement of second-order Eulerian polynomial $C_n(x)$.

Recall that M_n denote the multiset $\{1, 1, \bar{1}, 2, 2, \bar{2}, \ldots, n, n, \bar{n}\}$. Let L_n denote the set of Legendre-Stirling permutations on M_n. For a Legendre-Stirling permutation $\pi = \pi_1\pi_2\ldots\pi_{3n} \in L_n$, define

$$\begin{aligned}
X(\pi) &= \{i \,|\, \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears the first time}\}, \\
Y(\pi) &= \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is unbarred}\}, \\
Z(\pi) &= \{i \,|\, \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears the second time}\}, \\
U(\pi) &= \{i \,|\, \pi_{i-1} \leq \pi_i \text{ and } \pi_i \text{ is barred}\}, \\
V(\pi) &= \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is barred}\}.
\end{aligned}$$

Here we set $\pi_0 = \pi_{3n+1} = 0$ as usual.

For example, let $\pi = \bar{1}1\bar{2}2332\bar{3}1$. Then we have $X(\pi) = \{2, 4, 5\}$, $Y(\pi) = \{6, 9\}$, $Z(\pi) = \{6\}$, $U(\pi) = \{1, 3, 8\}$ and $V(\pi) = \{8\}$.

Based on the above statistics, we define

$$B_n(X, Y, Z, U, V) = \sum_{\pi \in L_n} \prod_{i \in X(\pi)} x_{\pi_i} \prod_{i \in Y(\pi)} y_{\pi_i} \prod_{i \in Z(\pi)} z_{\pi_i} \prod_{i \in U(\pi)} u_{\pi_i} \prod_{i \in V(\pi)} v_{\pi_i}. \tag{4.1}$$

For example, we have only 2 Legendre-Stirling permutations on M_1, $11\bar{1}$ and $\bar{1}11$. The corresponding sum terms are $x_1z_1u_1v_1$ and $x_1y_1z_1u_1$ respectively. This implies

$$B_1(X, Y, Z, U, V) = x_1y_1z_1u_1 + x_1z_1u_1v_1.$$

Similarly, we have 40 Legendre-Stirling permutations on M_2. Here we omit the sum terms corresponding to permutations, just give the following expression:

$$\begin{aligned}
B_2(X,Y,Z,U,V) = {} & 2x_2y_2z_2u_2x_1z_1u_1 + x_2y_2z_2u_2x_1y_1z_1 + x_2y_2z_2u_2x_1y_1u_1 \\
& + x_2y_2z_2u_2y_1z_1u_1 + x_2y_2z_2u_2x_1u_1v_1 + x_2y_2z_2u_2z_1u_1v_1 \\
& + x_2y_2z_2u_2x_1z_1v_1 + 2x_2z_2u_2v_2x_1z_1u_1 + x_2z_2u_2v_2x_1y_1z_1 \\
& + x_2z_2u_2v_2x_1y_1u_1 + x_2z_2u_2v_2y_1z_1u_1 + x_2z_2u_2v_2x_1z_1v_1 \\
& + x_2z_2u_2v_2x_1u_1v_1 + x_2z_2u_2v_2z_1u_1v_1 + 4x_2y_2z_2u_2v_2x_1z_1 \\
& + 4x_2y_2z_2u_2v_2x_1u_1 + 4x_2y_2z_2u_2v_2u_1z_1 + 2x_2y_2z_2u_2v_2x_1y_1 \\
& + 2x_2y_2z_2u_2v_2y_1z_1 + 2x_2y_2z_2u_2v_2y_1u_1 + 2x_2y_2z_2u_2v_2x_1v_1 \\
& + 2x_2y_2z_2u_2v_2u_1v_1 + 2x_2y_2z_2u_2v_2z_1v_1.
\end{aligned}$$

We now give a grammatical labeling of a Legendre-Stirling permutation. Let π be a Legendre-Stirling permutation in L_n. For the sake of convenience, we set $\pi_0 = \pi_{3n+1} = 0$. For $i \in X(\pi)$, $i \in Z(\pi)$ or $i \in U(\pi)$, we label π_{i-1} by x_{π_i}, z_{π_i} or u_{π_i}, respectively; For $i \in Y(\pi)$ or $i \in V(\pi)$, we label π_i by y_{π_i} or v_{π_i}, respectively. And the weight of π is defined as the product of these letters labeled on entries of π and denoted by $w(\pi)$. For example, the grammatical labeling of the aforementioned Legendre-Stirling permutation $\pi = 1\bar{2}\bar{1}2332\bar{3}1$ is given below:

$$\begin{array}{cccccccccc}
 & 1 & \bar{2} & \bar{1} & 2 & 3 & 3 & 2 & \bar{3} & 1 \\
x_1 & u_2 & v_2 & x_2 & x_3 & z_3 & y_3 & u_3 & v_3 & y_1
\end{array}.$$

The following theorem shows that the polynomials $B_n(X,Y,Z,U,V)$ can be generated by the grammars $G_1, G_2, \ldots, G_{2n}$.

Theorem 4.1 *For $n \geq 1$, let D_n denote the differential operator with respect to the grammar G_n, then we have*

$$D_{2n}D_{2n-1}\cdots D_1(x_0) = B_n(X,Y,Z,U,V). \tag{4.2}$$

Proof. Since $B_n(X,Y,Z,U,V)$ is defined to be the sum of weights of Legendre-Stirling permutations in L_n, we proceed by induction to show that

$$D_{2n}D_{2n-1}\cdots D_1(x_0) = \sum_{\pi \in L_n} w(\pi). \tag{4.3}$$

It is clear that (4.3) holds for $n = 0$, since the empty permutation is labeled by x_0. For $n \geq 1$, we assume that (4.3) holds for $n-1$, that is,

$$D_{2n-2}D_{2n-3}\cdots D_1(x_0) = \sum_{\pi \in L_{n-1}} w(\pi).$$

Note that any Legendre-Stirling permutation on M_n can be obtained from a Legendre-Stirling permutation on M_{n-1} by inserting nn and $\bar{n}$. We use examples to illustrate that the application of the operator $D_{2n}D_{2n-1}$ reflects the insertions of nn and $\bar{n}$.

Consider the Legendre-Stirling permutation $\pi = \bar{1}1\bar{2}2332\bar{3}1$ with the following grammatical labeling:

$$\begin{array}{cccccccccc}
 & \bar{1} & 1 & \bar{2} & 2 & 3 & 3 & 2 & \bar{3} & 1 \\
u_1 & x_1 & u_2 & x_2 & x_3 & z_3 & y_3 & u_3 & v_3 & y_1
\end{array}.$$

Let w be the weight of the above grammatical labeling, that is,

$$w = u_1x_1u_2x_2x_3z_3y_3u_3v_3y_1.$$

Let us consider the action of D_7 on w. Recall that

$$G_7 = \{x_i, y_i, z_i, u_i, v_i \to u_4v_4 \mid i = 1, 2, 3\}.$$

Consider a substitution rule that replaces a letter s by u_4v_4. Assume that π_k is labeled by s, where $0 \le k \le 9$. This rule corresponds to an insertion of $\bar{4}$ after the entry π_k in π. Then the element π_k is relabeled by u_4, and the element $\bar{4}$ is labeled by v_4.

For example, the substitution rule $z_3 \to u_4v_4$ corresponds to the insertion of $\bar{4}$ after the first element 3 in π. After the insertion, we obtain a Legendre-Stirling permutation with a consistent grammatical labeling:

$$\begin{array}{ccccccccccc} \bar{1} & 1 & \bar{2} & 2 & 3 & \bar{4} & 3 & 2 & \bar{3} & 1 \\ u_1 & x_1 & u_2 & x_2 & x_3 & u_4 & v_4 & y_3 & u_3 & v_3 & y_1 \end{array}.$$

As for the action of D_8, consider the above permutation $\sigma = \bar{1}1\bar{2}23\bar{4}32\bar{3}1$. Let w' denote the weight of σ, that is,

$$w' = u_1x_1u_2x_2x_3u_4v_4y_3u_3v_3y_1.$$

The two substitution rules $u_4 \to x_4z_4u_4$ and $v_4 \to x_4y_4z_4$ of G_8 correspond to the insertions of the element 44 into σ before $\bar{4}$ or after $\bar{4}$, respectively, resulting in two Legendre-Stirling permutations: $\bar{1}1\bar{2}2344\bar{4}32\bar{3}1$ or $\bar{1}1\bar{2}23\bar{4}4432\bar{3}1$.

It remains to consider the substitution rules of G_8 that are of the form $s \to x_4y_4z_4$, where $s \in \{x_i, y_i, z_i, u_i, v_i \mid i = 1, 2, 3\}$. Suppose that σ_i is the element in σ that is labeled by s. The substitution rule $s \to x_4y_4z_4$ corresponds to the insertion of 44 into σ after σ_i. Let τ denote the resulting permutation obtained from σ after the insertion. Then one can obtain a consistent grammatical labeling of τ by relabeling σ_i by x_4 and assigning the two labels z_4 and y_4 to the inserted two element 44 from left to right. For example, by applying the substitution rule $u_2 \to x_4y_4z_4$, we obtain the Legendre-Stirling permutation by inserting 44 after the first element 1 with consistent grammatical labeling:

$$\begin{array}{ccccccccccccc} \bar{1} & 1 & 4 & 4 & \bar{2} & 2 & 3 & \bar{4} & 3 & 2 & \bar{3} & 1 \\ u_1 & x_1 & x_4 & z_4 & y_4 & x_2 & x_3 & u_4 & v_4 & y_3 & u_3 & v_3 & y_1 \end{array}.$$

In general, it can be verified that the action of $D_{2n}D_{2n-1}$ on weights of the Legendre-Stirling permutations in L_{n-1} generates the weights of Legendre-Stirling permutations in L_n. So we conclude that (4.3) holds for n, that is,

$$D_{2n}D_{2n-1}\cdots D_1(x_0) = \sum_{\pi \in L_n} w(\pi).$$

Thus (4.3) holds for all n. This completes the proof. ∎

For $n = 0$, the empty permutation is labeled by x_0, and $B_1(X, Y, Z, U, V)$ can be calculated as follows,

$$D_1(x_0) = \overset{\bar{1}}{u_1}\, v_1,$$

$$\begin{aligned} B_1(X, Y, Z, U, V) = D_2D_1(x_0) &= \overset{\bar{1}}{u_1}\, \overset{1}{x_1}\, \overset{1}{z_1}\, y_1 + \overset{1}{x_1}\, \overset{1}{z_1}\, \overset{\bar{1}}{u_1}\, v_1 \\ &= u_1x_1y_1z_1 + x_1z_1u_1v_1. \end{aligned}$$

We note that the grammars $G_2, G_4, \ldots$ are related to the polynomials $C_n(X,Y,Z)$ introduced by Haglund and Vistonai [17], as defined by

$$C_n(X,Y,Z) = \sum_{\pi \in Q_n} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

Clearly, $C_1(X,Y,Z) = x_1 y_1 z_1$. Based on the combinatorial interpretation of $C_n(X,Y,Z)$, Haglund and Visontai [17] established the following recurrence relation for $n \geq 1$:

$$C_{n+1}(X,Y,Z) = x_{n+1} y_{n+1} z_{n+1} \partial C_n(X,Y,Z), \tag{4.4}$$

where

$$\partial = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial}{\partial y_i} + \sum_{i=1}^{n} \frac{\partial}{\partial z_i}. \tag{4.5}$$

The following theorem shows that the grammar D_{2n} has the same effect as the operator $x_n y_n z_n \partial$ when acting on $C_{n-1}(X,Y,Z)$.

Theorem 4.2 *For $n \geq 0$,*

$$D_{2n+2}(C_n(X,Y,Z)) = x_{n+1} y_{n+1} z_{n+1} \partial C_n(X,Y,Z). \tag{4.6}$$

The relation (4.6) implies that

$$D_{2n+2} D_{2n} \cdots D_4 D_2(z_0) = C_{n+1}(X,Y,Z).$$

To prove (4.6), we observe the following property of the formal derivative D with respect to a grammar G. The verification is a straightforward computation.

Proposition 4.3 *Let X denote the set of variables of a grammar G. For a polynomial f in X, we have*

$$D(f) = \sum_{x \in X} \frac{\partial f}{\partial x} \cdot D(x).$$

Proof of Theorem 4.2. Recall that

$$G_{2n+2} = \{x_i, y_i, z_i \to x_{n+1} y_{n+1} z_{n+1} \mid 1 \leq i \leq n\}.$$

According to Proposition 4.3, we obtain that

$$\begin{aligned} D_{2n+2}(C_n(X,Y,Z)) &= \sum_{i=1}^{n} D(x_i) \frac{\partial C_n}{\partial x_i} + \sum_{i=1}^{n} D(y_i) \frac{\partial C_n}{\partial y_i} + \sum_{i=1}^{n} D(z_i) \frac{\partial C_n}{\partial z_i} \\ &= x_{n+1} y_{n+1} z_{n+1} \left(\sum_{i=1}^{n} \frac{\partial C_n}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial C_n}{\partial y_i} + \sum_{i=1}^{n} \frac{\partial C_n}{\partial z_i} \right). \end{aligned}$$

So we get (4.6). This completes the proof. ∎

5 The stability of $T_n(X,Y,Z)$

In this section, we prove the stability of the multivariate polynomials $T_n(X,Y,Z)$ by showing the related formal derivatives with respect to the generating grammars are stability preserving operators. The proof relies on the characterization of stability preserving linear operators on multiaffine polynomials due to Borcea and Brändén [4].

Recall that a multivariate polynomial $f(z_1,z_2,\ldots,z_n)$ is called multiaffine if the degree of any variable in f is at most 1. An operator T is called a stability preserver for multiaffine polynomials if $T(f)$ is either stable or identically 0 for any stable multiaffine polynomial $f\in\mathbb{C}[z_1,z_2,\ldots,z_n]$.

Theorem 5.1 (Borcea and Brändén) *Let T denote a linear operator acting on the polynomials in $\mathbb{C}[z_1,z_2,\ldots,z_n]$. If*

$$T\left(\prod_{i=1}^{n}(z_i+w_i)\right)\in\mathbb{C}[z_1,z_2,\ldots,z_n,w_1,\ldots,w_n]$$

is stable, then T is a stability preserver for multiaffine polynomials.

To prove the stability of $T_n(X,Y,Z)$, we use the grammatical expression

$$T_n(X,Y,Z)=D_nD_{n-1}\cdots D_1(z_0)$$

in Theorem 3.2, where D_n is the formal derivative with respect to the grammar

$$G_n=\{x_i,z_i\rightarrow x_ny_nz_n,\, y_i\rightarrow 2x_ny_nz_n \mid 0\le i<n\}.$$

We shall show that D_n is a stability preserver, this proves the stability of $T_n(X,Y,Z)$.

Theorem 5.2 *For $n\ge 1$, $T_n(X,Y,Z)$ is stable.*

Proof. Let

$$F=\prod_{i=0}^{n-1}(x_i+u_i)(y_i+v_i)(z_i+w_i), \tag{5.1}$$

and let

$$\xi=\sum_{i=0}^{n}\left(\frac{1}{x_i+u_i}+\frac{2}{y_i+v_i}+\frac{1}{z_i+w_i}\right). \tag{5.2}$$

We have

$$\begin{aligned}
D_n(F)&=\sum_{i=0}^{n}D(x_i)\frac{\partial F}{\partial x_i}+\sum_{j=0}^{n}D(y_j)\frac{\partial F}{\partial y_j}+\sum_{k=0}^{n}D(z_k)\frac{\partial F}{\partial z_k}\\
&=\sum_{i=0}^{n}x_ny_nz_n\frac{F}{x_i+u_i}+\sum_{j=0}^{n}2x_ny_nz_n\frac{F}{y_i+v_i}+\sum_{k=0}^{n}x_ny_nz_n\frac{F}{z_k+w_k}\\
&=x_ny_nz_n\xi F.
\end{aligned}$$

To prove that D_n preserves stability of multiaffine polynomials, we assume that x_i, y_i, z_i, u_i, v_i and w_i have positive imaginary parts for all $0\le i\le n$. We proceed to show $D_n(F)\neq 0$.

Under the above assumptions, for $0 \leq i \leq n$, $x_i + u_i$, $y_i + v_i$ and $z_i + w_i$ also have positive imaginary parts. It follows that $\frac{1}{x_i+u_i}$, $\frac{2}{y_i+v_i}$ and $\frac{1}{z_i+w_i}$ have negative imaginary parts. By the definition (5.1), we see that $F \neq 0$. By (5.2), we find that $\xi \neq 0$ has a negative imaginary part. Hence $D_n(F) \neq 0$. Thus D_n is a stability preserver. This completes the proof. ∎

6 The stability of $B_n(X, Y, Z, U, V)$

In this section, we prove the stability of the multivariate polynomials $B_n(X, Y, Z, U, V)$. Unlike the proof of $T_n(X, Y, Z)$, the formal derivatives with respect to the grammars do not preserve stability. Fortunately, as for the multiaffine polynomials that we are concerned with, the formal derivatives in our case are equivalent to linear operators which turn out to be stability preserving.

More specifically, the idea goes as follows. Let $G_1, G_2, \ldots$ be context-free grammars, and $D_1, D_2, \ldots$ be the formal derivatives with respect to $G_1, G_2, \ldots$. Suppose that we wish to prove the stability of the multivariate polynomials

$$f_n = D_n D_{n-1} \cdots D_1(x),$$

for $n \geq 1$, where $D_1, D_2, \ldots$ may not be stability preserving. We aim to construct stability preservers $T_1, T_2, \ldots$ such that

$$T_n T_{n-1} \cdots T_1(x) = D_n D_{n-1} \cdots D_1(x).$$

Once such stability preservers $T_1, T_2, \ldots$ are found, it can be asserted that the multivariate polynomials f_n are stable. The following lemma provides a way to find such operators T_n.

Lemma 6.1 *Let G be a context-free grammar over the alphabet $X \cup Y$, where*

$$X = \{x_1, x_2, \ldots, x_r\}$$

and

$$Y = \{y_1, y_2, \ldots, y_s\}.$$

Let D denote the formal derivative with respect to G. Assume that $D(x_i)$ contains a factor x_i for $i = 1, 2, \ldots, r$, namely, $x_i \to x_i h_i(X, Y)$ is a substitution rule in G. Let T denote the following operator

$$T = \sum_{i=1}^{r} h_i(X, Y) I + \sum_{j=1}^{s} D(y_j) \frac{\partial}{\partial y_j},$$

where I denotes the identify operator. Let $g(Y)$ be any polynomial in Y and let $f(X, Y) = x_1 x_2 \ldots x_r g(Y)$. Then we have

$$D(f(X, Y)) = T(f(X, Y)).$$

Proof. By Proposition 4.3, we find that

$$\begin{aligned}
D(f(X,Y)) &= \sum_{i=1}^{r} D(x_i)\frac{\partial f(X,Y)}{\partial x_i} + \sum_{j=1}^{s} D(y_j)\frac{\partial f(X,Y)}{\partial y_j} \\
&= \sum_{i=1}^{r} x_i h_i(X,Y)\cdot \frac{f(X,Y)}{x_i} + \sum_{j=1}^{s} D(y_j)\frac{\partial f(X,Y)}{\partial y_j} \\
&= \sum_{i=1}^{r} h_i(X,Y)f(X,Y) + \sum_{j=1}^{s} D(y_j)\frac{\partial f(X,Y)}{\partial y_j},
\end{aligned}$$

which equals $T(f(X,Y))$. This completes the proof. ∎

For example, recall that the grammar

$$G = \{a \to ax, x \to x\}$$

can be used to obtain the Stirling polynomials $S_n(x)$, that is,

$$D^n(a) = aS_n(x).$$

Let $X = \{a\}$ and $Y = \{x\}$. Then D satisfies the conditions in Lemma 6.1. Thus $D(af(x)) = T(af(x))$ for any polynomial $f(x)$, where the operator T is given by

$$T = x\left(I + \frac{\partial}{\partial x}\right).$$

In particular, we have

$$T(aS_n(x)) = D(aS_n(x)).$$

In fact, the above operator T corresponds to the following recurrence relation for $S_n(x)$:

$$S_n(x) = T(S_{n-1}(x)),$$

which is equivalent to the recurrence relation of $S(n,k)$:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{6.1}$$

where $n \geq k > 1$. Harper [15] proved that $S_n(x)$ has only real roots for $n \geq 1$. Liu and Wang [21] showed that T preserves the real-rootedness of polynomials in x.

As a generalization of the real-rootedness of $S_n(x)$, we consider the stability of the multivariate polynomials $S_n(a, x_1, x_2, \cdots, x_n)$, which can be viewed as a refinement of the Stirling polynomial $S_n(x)$. Let

$$G_n = \{a \to ax_n, x_i \to x_n, 1 \leq i < n\}.$$

and let D_n denote the formal derivative associated with G_n. It will be shown that for $n \geq 1$, $S_n(a, x_1, x_2, \ldots, x_n)$ can be generated by $G_1, G_2, \ldots, G_n$.

The polynomial $S_n(a, x_1, x_2, \ldots, x_n)$ is defined by using the following grammatical labeling of a partition $P = \{P_1, P_2, \ldots, P_k\}$ of $[n]$. The partition itself is labeled by the letter a and a block P_i is labeled by the letter x_{m_i}, where m_i is the maximal element in P_i. The weight of P is given by the product of all labelings in P, that is,

$$w(P) = a\prod_{i=1}^{k} x_{m_i}.$$

Denote by $S_n(a, x_1, x_2, \ldots, x_n)$ the sum of weights of partitions of $[n]$. Clearly, $S_n(a, x_1, x_2, \ldots, x_n)$ is the generating function of partitions of $[n]$ involving not only the number of blocks, but also the maximal elements of the blocks.

For example, for $n = 1, 2, 3$, we have

$$S_1(a, x_1) = ax_1,$$

$$S_2(a, x_1, x_2) = ax_1x_2 + ax_2,$$

$$S_3(a, x_1, x_2, x_3) = ax_1x_2x_3 + 2ax_2x_3 + ax_1x_3 + ax_3.$$

The following theorem gives a grammatical expression of $S_n(a, x_1, x_2, \ldots, x_n)$ in terms of $D_1, D_2, \ldots, D_n$.

Theorem 6.2 *For $n \geq 1$,*

$$S_n(a, x_1, x_2, \ldots, x_n) = D_n D_{n-1} \cdots D_1(a). \tag{6.2}$$

The proof of the above theorem is analogous to that of (2.1). Here we use the same example to demonstrate the action of D_7 on a partition of $\{1, 2, 3, 4, 5, 6\}$. Recall that

$$G_7 = \{a \to ax_7, x_i \to x_7, 1 \leq x \leq 6\}.$$

Consider the following partition along with its grammatical labeling:

$$\begin{array}{cccc} \{1,3,6\} & \{2,5\} & \{4\} & \\ x_6 & x_5 & x_4 & a \end{array}.$$

Applying the substitution rule $a \to ax_7$ to the above partition leads to a partition with a consistent grammatical labeling:

$$\begin{array}{ccccc} \{1,3,6\} & \{2,5\} & \{4\} & \{7\} & \\ x_6 & x_5 & x_4 & x_7 & a \end{array}.$$

Similarly, applying the substitution rule $x_5 \to x_7$ to the partition, we get the following partition with a consistent grammatical labeling

$$\begin{array}{cccc} \{1,3,6\} & \{2,5,7\} & \{4\} & \\ x_6 & x_7 & x_4 & a \end{array}.$$

In fact, the above arguments are sufficient to justify the expression (6.2).

It should be noticed that the relation (6.2) cannot be directly used to prove the stability of $S_n(a, x_1, x_2, \ldots, x_n)$, since the operator D_n does not preserve stability in general. Take D_2 as an example. Consider the polynomial $(a + 1)(x_1 + 1)$, which is clearly stable. But

$$D_2((a + 1)(x_1 + 1)) = x_2(ax_1 + 2a + 1),$$

is not stable since it vanishes for $a = i, x_1 = i - 2$. That says that D_2 is not stability preserving.

Fortunately, we can find a stability preserving operator T_n in place of D_n for the purpose of justifying the stability of $S_n(a, x_1, x_2, \ldots, x_n)$. It is easy to see that $S_n(a, x_1, x_2, \ldots, x_n)$ can be written as $ah(X)$, where $h(X)$ is a multivariate polynomial in $x_1, x_2, \ldots, x_n$ that is independent of the variable a. Let

$$T_n = x_n I + x_n \sum_{i=1}^{n} \frac{\partial}{\partial x_i}. \tag{6.3}$$

According to Lemma 6.1, for each $n \geq 1$, we have

$$T_n(S_n(a, x_1, x_2, \ldots, x_n)) = D_n(S_n(a, x_1, x_2, \ldots, x_n)).$$

The following theorem shows that $S_n(a, x_1, x_2, \cdots, x_n)$ can be obtained by $T_1, T_2, \ldots, T_n$.

Theorem 6.3 *For $n \geq 1$, we have*

$$S_n(a, x_1, x_2, \ldots, x_n) = T_n T_{n-1} \ldots T_1(a). \tag{6.4}$$

It should be noticed that one can give a direct interpretation of (6.4) based on the expression (6.3). Using the operators $T_1, T_2, T_3, \ldots$, we prove the stability of $S_n(a, x_1, x_2, \ldots, x_n)$.

Theorem 6.4 *For $n \geq 1$, the multivariate polynomial $S_n(a, x_1, x_2, \ldots, x_n)$ is stable.*

Proof. It suffices to show that the linear operator T_n preserves stability for multiaffine polynomials. By Theorem 5.1, it is enough to prove that $T_n(F)$ is stable, where

$$F = (a + u) \prod_{i=1}^{n-1} (x_i + v_i).$$

Let

$$\xi = 1 + \sum_{i=1}^{n-1} \frac{1}{x_i + v_i}.$$

Then we have

$$\begin{aligned} T_n(F) &= x_n F + x_n \sum_{i=1}^{n-1} \frac{\partial F}{\partial x_i} \\ &= x_n F + x_n F \sum_{i=1}^{n-1} \frac{1}{x_i + v_i} \\ &= x_n \xi F. \end{aligned}$$

To prove that $T_n(F)$ is stable, we assume that $a, u, x_1, x_2, \ldots, x_n$ and $v_1, v_2, \ldots, v_n$ have positive imaginary parts. It remains to show that $T_n(F) \neq 0$.

Under the above assumptions, for $1 \leq i \leq n$, $x_i + v_i$ has a positive imaginary part. It follows that $\frac{1}{x_i + v_i}$ has a negative imaginary part. Furthermore, the imaginary part of ξ is also negative. Thus we have $F \neq 0$ and $\xi \neq 0$. Consequently, $T_n(F) \neq 0$. This completes the proof. ∎

Next we prove the stability of $B_n(X, Y, Z, U, V)$, where $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, $Z = (z_1, z_2, \ldots, z_n)$, $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$. Recall that

$$B_n(X, Y, Z, U, V) = D_{2n} D_{2n-1} \ldots D_1(x_0),$$

where

$$G_{2n-1} = \{x_i, y_i, z_i, u_i, v_i \to u_n v_n \mid 1 \leq i < n\},$$

and

$$G_{2n} = \{u_n \to x_n z_n u_n, v_n \to x_n y_n z_n,$$
$$x_i, y_i, z_i, u_i, v_i \to x_n y_n z_n \mid 1 \le i < n\}.$$

For $1 \le k \le 2n$, let D_k denote the formal derivative with respect to G_k. We shall show that for $n \ge 1$, D_{2n-1} is stability preserving. However, for $n \ge 1$, the operator D_{2n} does not preserve stability. For example, the polynomial $(u_n + 1)(v_n + 1)$ is clearly stable, but

$$D_{2n}((u_n + 1)(v_n + 1)) = x_n z_n (u_n(v_n + z_n + 1) + z_n)$$

is not stable since it vanishes for $z_n = i + 2, u_n = 1, v_n = i - 4$. As a remedy of this problem, we find a stability preserving operator T_n that is equivalent to D_{2n} while acting on polynomials of the form $u_n \cdot g$, where g is a polynomial in X, Y, Z, U and V that is independent of u_n.

Theorem 6.5 *For $n \ge 1$, the multivariate polynomial $B_n(X, Y, Z, U, V)$ is stable.*

Proof. For $1 \le k \le 2n$, let

$$f_k = D_k D_{k-1} \cdots D_1(z_0),$$

which is a polynomial in

$$A_k = \{x_i, y_i, z_i, u_i, v_i \mid 1 \le i \le \lfloor (k+1)/2 \rfloor\}.$$

So $f_{2n} = B_n(X, Y, Z, U, V)$. For $1 \le k \le 2n$, it can be seen that f_k is multiaffine. We proceed to prove the stability of f_{2n} by induction on n. The stability of z_0 is evident. For $n \ge 1$, assume that f_{2n-2} is stable. Let us consider the actions of D_{2n-1} and D_{2n}.

First, we show that D_{2n-1} preserves stability for multiaffine polynomials. Let

$$A'_k = \{x'_i, y'_i, z'_i, u'_i, v'_i \mid 1 \le i \le \lfloor (k+1)/2 \rfloor\}.$$

According to Theorem 5.1, it suffices to show that the polynomial $D_{2n-1}(F)$ is stable, where

$$F = \prod_{i=1}^{n}(x_i + x'_i) \prod_{i=1}^{n}(y_i + y'_i) \prod_{i=1}^{n}(z_i + z'_i) \prod_{i=1}^{n}(u_i + u'_i) \prod_{i=1}^{n}(v_i + v'_i).$$

Let

$$\xi = \sum_{i=1}^{n-1} \left(\frac{1}{x_i + x'_i} + \frac{1}{y_i + y'_i} + \frac{1}{z_i + z'_i} + \frac{1}{u_i + u'_i} + \frac{1}{v_i + v'_i} \right).$$

By Proposition 4.3,

$$\begin{aligned}
D_{2n-1}(F) &= \sum_{i=1}^{n-1} D_{2n-1}(x_i)\frac{\partial F}{\partial x_i} + \sum_{i=1}^{n-1} D_{2n-1}(y_i)\frac{\partial F}{\partial y_i} + \sum_{i=1}^{n-1} D_{2n-1}(z_i)\frac{\partial F}{\partial z_i} \\
&\quad + \sum_{i=1}^{n-1} D_{2n-1}(u_i)\frac{\partial F}{\partial u_i} + \sum_{i=1}^{n-1} D_{2n-1}(v_i)\frac{\partial F}{\partial v_i} \\
&= u_n v_n \sum_{i=1}^{n-1}\left(\frac{F}{x_i+x_i'} + \frac{F}{y_i+y_i'} + \frac{F}{z_i+z_i'} + \frac{F}{u_i+u_i'} + \frac{F}{v_i+v_i'}\right), \\
&= u_n v_n \xi F.
\end{aligned}$$

Assume that all the variables in A_{2n} and A'_{2n} have positive imaginary parts. We need to show that $D_{2n-1}(F) \neq 0$. It is easily seen that $\xi \neq 0$ and $F \neq 0$. Hence $D_{2n-1}(F) \neq 0$. This proves that D_{2n-1} is stability preserving. It follows from the induction hypothesis that f_{2n-1} is stable.

Next we turn to the operator D_{2n}. Define

$$T_n = x_n z_n I + x_n y_n z_n \sum_{i=1}^{n-1}\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} + \frac{\partial}{\partial z_i} + \frac{\partial}{\partial u_i}\right) + x_n y_n z_n \sum_{i=1}^{n} \frac{\partial}{\partial v_i}.$$

Since f_{2n-1} can be written in the form $u_n g$, where g is a polynomial in B that is independent of u_n, using Lemma 6.1, we find that

$$f_{2n} = D_{2n}(f_{2n-1}) = T_n(f_{2n-1}).$$

To prove that T_n preserves stability for multiaffine polynomials, let

$$F = \prod_{i=1}^{n}(x_i+x_i')\prod_{i=1}^{n}(y_i+y_i')\prod_{i=1}^{n}(z_i+z_i')\prod_{i=1}^{n}(u_i+u_i')\prod_{i=1}^{n}(v_i+v_i').$$

Then we have

$$\begin{aligned}
T_n(F) &= x_n y_n z_n F \sum_{i=1}^{n-1}\left(\frac{1}{x_i+x_i'} + \frac{1}{y_i+y_i'} + \frac{1}{z_i+z_i'} + \frac{1}{u_i+u_i'}\right) \\
&\quad + x_n y_n z_n F \sum_{i=1}^{n} \frac{1}{v_i+v_i'} + x_n z_n F \\
&= x_n y_n z_n \xi F,
\end{aligned}$$

where

$$\xi = \frac{1}{y_n} + \sum_{i=1}^{n-1}\left(\frac{1}{x_i+x_i'} + \frac{1}{y_i+y_i'} + \frac{1}{z_i+z_i'} + \frac{1}{u_i+u_i'}\right) + \sum_{i=1}^{n}\frac{1}{v_i+v_i'}.$$

Assume that all the variables in A_{2n} and A'_{2n} have positive imaginary parts. By Lemma 5.1, it suffices to verify that $T_n(F) \neq 0$. For $1 \leq i \leq n$, since $x_i+x_i', y_i+y_i', z_i+z_i', u_i+u_i'$, and v_i+v_i' have positive imaginary parts, we see that

$$\frac{1}{x_i+x_i'}, \frac{1}{y_i+y_i'}, \frac{1}{z_i+z_i'}, \frac{1}{u_i+u_i'}, \quad \text{and} \quad \frac{1}{v_i+v_i'}$$

have negative imaginary parts. Similarly, $\frac{1}{y_n}$ has a negative imaginary part. Thus we deduce that $\xi \neq 0$ and $F \neq 0$. Consequently, $T_n(F) \neq 0$. This leads to the stability of $T_n(F)$. Finally, in light of Lemma 5.1, we conclude that f_{2n} is stable. This completes the proof. ∎

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