An Iterated Map for the Lebesgue Identity

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Abstract. We present a simple iteration for the Lebesgue identity on partitions, which leads to a refinement involving the alternating sums of partitions.

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We find a simple iterated map for the classical Lebesgue identity on partitions. As an application of this iterated map, we give a refinement of the partition interpretation of this identity involving alternating sums of partitions. Recall that the q-shifted factorials are defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in \mathbb{Z},$$

where $|q|<1$. The Lebesgue identity reads

$$\sum_{k=0}^{\infty} \frac{(-aq;q)_k}{(q;q)_k} q^{\binom{k+1}{2}} = (-aq^2;q^2)_\infty(-q;q)_\infty, \tag{1}$$

see, for example, Andrews [2]. There are several combinatorial proofs of the Lebesgue identity. Ramamani and Venkatachaliengar [8] found a bijection for the following generalization of (1),

$$\sum_{m=0}^{\infty} q^{m(m+1)/2} \frac{(z;q)_m}{(q;q)_m} \alpha^m = (z;q)_\infty(-\alpha q;q)_\infty \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n(-\alpha q;q)_n}.$$

Bessenrodt [3] gave a combinatorial interpretation in terms of 2-modular diagrams. Alladi and Gordon [1] provided another bijection which implies the Lebesgue identity. Pak modified the construction of Alladi and Gordon to give a direct correspondence by using standard MacMahon diagrams [7]. Fu [5] discovered a bijective proof of the following extension of (1) by applying the insertion algorithm of Zeilberger:

$$\sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(q;q)_n} b^n q^{\binom{n+1}{2}} = (-bq;q)_\infty \sum_{k=0}^{\infty} \frac{(ab)^k q^{k(k+1)}}{(q;q)_k(-bq;q)_k}.$$

Rowell [9] presented a combinatorial proof which leads to the following finite form of (1):

$$\sum_{n=0}^{L} \begin{bmatrix} L \\ n \end{bmatrix}_q (-aq;q)_n q^{n(n+1)/2} = \sum_{k=0}^{L} \begin{bmatrix} L \\ k \end{bmatrix}_{q^2} (-q;q)_{L-k} a^k q^{k(k+1)}.$$

Recently, Little and Sellers [6] have established the relation (1) by using weighted Pell tilings.

To describe our bijection, we follow the terminology in [2]. A partition is meant to be a non-increasing finite sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The entries λ_i are called the parts of λ. The number of parts of λ is denoted by $\ell(\lambda)$, and sum of parts is denoted by $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$. The conjugate partition of λ is denoted by λ'. The partition with no parts is denoted by $\varnothing$.

Denote the left hand side of the Lebesgue identity (1) by $f(a,q)$. It is easily seen that

$$f(a,q) = \sum_{(\alpha,\beta)\in P} a^{\ell(\beta)} q^{|\alpha|+|\beta|},$$

where P denotes the set of pairs (α, β) of partitions with distinct parts such that $\ell(\alpha)$ is not less than the largest part of β. The corresponding diagram is illustrated by Figure 1.

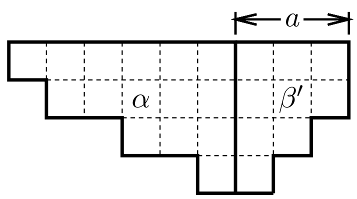

Figure 1: A pair $(\alpha, \beta) \in P$

Clearly, the right hand side of (1) has the following combinatorial interpretation

$$\sum_{(\mu,\nu)\in Q} a^{\ell(\nu)} q^{|\mu|+|\nu|},$$

where Q is the set of pairs (μ, ν) of partitions with distinct parts such that ν has only even parts.

For a triple of partitions (α, β, γ) where $(\alpha, \beta) \in P$ and γ is a partition with even parts such that $\ell(\gamma) \geq \ell(\beta)$ or $\gamma = \phi$, we define a map ϕ: $(\alpha, \beta, \gamma) \to (\mu, \lambda, \nu)$ as follows:

Case 1: The smallest part of β equals 1. Decrease each part of α by 1 to form a partition μ. Change the 1-part of β to an $(\ell(\alpha)+1)$-part and decrease each part of the resulting partition by 2 to generate a partition λ. Then add two $\ell(\beta)$-parts to the conjugate partition γ' to produce a conjugate partition ν'. This operation can be visualized as moving the rightmost square of β' to the bottom of α, then shifting the diagram below the x-axis to the right by one column, and finally moving up the diagram on the right side of the y-axis by two rows. See Figure 2 for an illustration, where $\alpha = (6,5,3,1)$, $\beta = (4,3,1)$, $\gamma = (2,2,2,2)$, $\mu = (5,4,2)$, $\lambda = (3,2,1)$, and $\nu = (4,4,4,2)$.

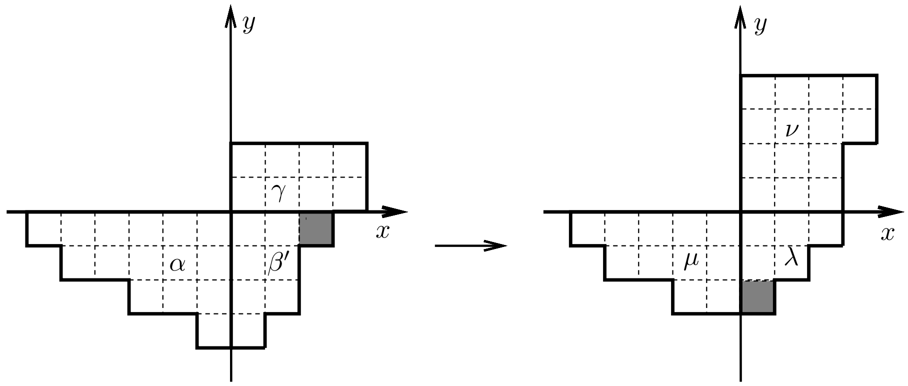

Figure 2: An example

Case 2: The smallest part of β is larger than 1. Set $\mu = \alpha$ and move up the diagram of β' by two rows to generate the two conjugate partitions λ' and ν'.

To recover (α, β, γ) from (μ, λ, ν), we first move down the diagram on the right side of the y-axis by two rows to obtain a triple $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$. If $\bar{\beta}_1 \leq \ell(\bar{\alpha})$, we then have $(\alpha, \beta, \gamma) = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})$. Otherwise, we further shift the diagram below the x-axis to the left by one column and move the bottom square to the right of $\bar{\beta}'$. Thus, ϕ is invertible.

Starting from $(\alpha, \beta, \varnothing)$, we can iterate the above map until λ becomes empty. This gives a pair (μ, ν) of partitions that belongs to Q. This completes the combinatorial proof of the Lebesgue identity.

The above map leads to a refinement of the Lebesgue identity (1). Define the alternating sum of a partition λ by

$$|\lambda|_a = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots .$$

This statistic has occurred in the study of refinements of Euler's partition theorem, see [4]. Notice that when the parts of α are distinct, the alternating sum of α equals to the number of odd parts of its conjugate partition. Denote by $n_o(\lambda)$ the number of odd parts of a partition λ. It is straightforward to check that the map ϕ preserves the difference $n_o(\alpha') - n_o(\beta)$. Therefore, our bijection leads to the following refinement of the combinatorial interpretation of the Lebesgue identity.

Theorem 1 *Let P denote the set of pairs (α, β) of partitions with distinct parts such that $\ell(\alpha)$ is not less than the largest part of β, and let Q denote the set of pairs (μ, ν) of partitions with distinct parts such that ν has only even parts. Then for each nonnegative integer k, the number of pairs $(\alpha, \beta) \in P$ with $|\alpha|_a - n_o(\beta) = k$ is equal to the number of pairs $(\mu, \nu) \in Q$ with $|\mu|_a = k$.*

We notice that Bessenrodt's bijection [3] also keeps the difference $|\alpha|_a - n_o(\beta)$. We also note that our map can be viewed as a direct correspondence in the sense that it does not require Sylvester's bijection for Euler's identity, see the remark in [7].

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