The Log-Behavior of $\sqrt[n]{p(n)}$ and $\sqrt[n]{p(n)/n}$

William Y.C. Chen[1] and Ken Y. Zheng[2]

Center for Combinatorics, LPMC-TJKLC
Nankai Univercity
Tianjin 300071, P. R. China

Email:[1]chenyc@tju.edu.cn, [2]kenzheng@aliyun.com

Abstract

Let $p(n)$ denote the partition function. Desalvo and Pak proved the log-concavity of $p(n)$ for $n > 25$ and the inequality $\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}$ for $n > 1$. Let $r(n) = \sqrt[n]{p(n)/n}$ and Δ be the difference operator respect to n. Desalvo and Pak pointed out that their approach to proving the log-concavity of $p(n)$ may be employed to prove a conjecture of Sun on the log-convexity of $\{r(n)\}_{n\geq 61}$, as long as one finds an appropriate estimate of $\Delta^2 \log r(n-1)$. In this paper, we obtain a lower bound for $\Delta^2 \log r(n-1)$, leading to a proof of this conjecture. From the log-convexity of $\{r(n)\}_{n\geq 61}$ and $\{\sqrt[n]{n}\}_{n\geq 4}$, we are led to a proof of another conjecture of Sun on the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 27}$. Furthermore, we show that $\lim_{n\to+\infty} n^{\frac{5}{2}}\Delta^2 \log \sqrt[n]{p(n)} = 3\pi/\sqrt{24}$. Finally, by finding an upper bound of $\Delta^2 \log \sqrt[n-1]{p(n-1)}$, we prove an inequality on the ratio $\frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}$ analogous to the above inequality on the ratio $\frac{p(n-1)}{p(n)}$.

Keywords: partition function, log-convex sequence, Hardy-Ramanujan-Rademacher formula, Lehmer's error bound

AMS Subject Classifications: 05A20

1 Introduction

In this paper, we study the log-behavior of the sequences $\sqrt[n]{p(n)}$ and $\sqrt[n]{p(n)/n}$, where $p(n)$ is the number of partitions of n. Recently, by using the Hardy-Ramanujan-Rademacher formula of $p(n)$ (see [6, 7, 10]) and Lehmer's error bound (see [8, 9]), Desalvo and Pak [5] gave an estimate for $-\Delta^2 \log p(n-1)$, and then found an upper and lower bound for this estimate, finally proved that $p(n)$ is log-concave for $n > 25$. They also proved the following inequality conjectured by Chen [2].

Theorem 1.1 *For $n > 1$,*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}. \tag{1.1}$$

Desalvo and Pak [5] showed that

$$\lim_{n\to+\infty} -n^{\frac{3}{2}}\Delta^2 \log p(n-1) = \pi/\sqrt{24}, \tag{1.2}$$

and proposed the following conjecture.

Conjecture 1.1 *For $n \geq 45$,*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{1.3}$$

In view of (1.2), the coefficient $\frac{\pi}{\sqrt{24}}$ in (1.3) is the best possible. Chen, Wang and Xie [4] proved the above conjecture by showing that for $n \geq 5000$,

$$-\Delta^2 \log p(n-1) < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2. \tag{1.4}$$

The proof of (1.4) requires Desalvo and Pak's upper bound of $-\Delta^2 \log p(n-1)$ for $n \geq 50$,

$$\begin{aligned}-\Delta^2 \log p(n-1) <& \frac{24\pi}{(24(n-1)-1)^{3/2}} + \frac{288\pi(-3+\pi\sqrt{24(n-1)-1})}{(24(n-1)-1)^{3/2}(-6+\pi\sqrt{24(n-1)-1})^2} \\ &- \frac{864}{(24(n+1)-1)^2} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}.\end{aligned} \tag{1.5}$$

For $n \geq 50$, the upper bound in (1.5) can be relaxed to

$$\frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2 - \frac{1}{n^2} + \frac{3}{n^{5/2}} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}.$$

By using the Lambert W function, it can be shown that $-\frac{1}{n^2} + \frac{3}{n^{5/2}} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}} < 0$ when $n \geq 5000$, and therefore we arrive at the upper bound in the form of (1.4). Let $r(n) = \sqrt[n]{p(n)/n}$. Desalvo and Pak also considered the log-behavior of $r(n)$. A positive sequence $\{a_n\}$ is log-convex if it satisfies that for $n \geq 1$,

$$a_n^2 - a_{n-1}a_{n+1} \leq 0.$$

Conversely, a positive sequence $\{a_n\}$ is log-concave if it satisfies that for $n \geq 1$,

$$a_n^2 - a_{n-1}a_{n+1} \geq 0.$$

Desalvo and Pak noticed that the log-convexity of $\{r(n)\}_{n\geq 61}$ conjectured by Sun [11] can be derived from an estimate for $\Delta^2 \log r(n-1)$, see [5, Final Remark 7.7]. They also remarked that their approach to bounding $-\Delta^2 \log p(n-1)$ does not seem to apply to $\Delta^2 \log r(n-1)$. In this paper, we obtain a lower bound for $\Delta^2 \log r(n-1)$, leading to a proof of the log-convexity of $\{r(n)\}_{n\geq 61}$.

Theorem 1.2 *The sequence* $\{r(n)\}_{n\geq 61}$ *is log-convex.*

The log-convexity of $\{r(n)\}_{n\geq 61}$ implies the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 27}$, because the sequence $\{\sqrt[n]{n}\}_{n\geq 4}$ is log-convex [11]. It is known that $\lim_{n\to+\infty}\sqrt[n]{p(n)}=1$. For a combinatorial proof of this fact, see Andrews [1]. The log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 27}$ was conjectured by Sun [11]. He also proposed the conjecture that $\{\sqrt[n]{p(n)}\}_{n\geq 6}$ is strictly decreasing, which has been proved by Wang and Zhu [12]. It is easy to see that the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 27}$ implies the decreasing property.

It should be noted that there is another approach to proving the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 27}$. Chen, Guo and Wang [3] introduced the notion of a ratio log-convex sequence and showed that ratio log-convexity implies log-convexity under an initial condition. A sequence $\{a_n\}_{n\geq k}$ is called ratio log-convex if $\{a_{n+1}/a_n\}_{n\geq k}$ is log-convex, or, equivalently, for $n\geq k$,

$$\log a_{n+2}-3\log a_{n+1}+3\log a_n-\log a_{n-1}\geq 0.$$

Chen, Wang and Xie [4] showed that that for any $r\geq 1$, one can determine a number $n(r)$ such that for $n>n(r)$, $(-1)^{r-1}\Delta^r\log p(n)$ is positive. For $r=3$, it can be shown that for $n\geq 116$,

$$\Delta^3\log p(n-1)>0.$$

Since

$$\Delta^3\log p(n-1)=\log p(n+2)-3\log p(n+1)+3\log p(n)-\log p(n-1),$$

it is evident that $\{p(n)\}_{n\geq 116}$ is ratio log-convex. So we are led to the following assertion.

Theorem 1.3 *The sequence* $\{\sqrt[n]{p(n)}\}_{n\geq 27}$ *is log-convex.*

In the spirit of the inequality (1.3) on $\frac{p(n-1)}{p(n)}$, we obtain the following inequality on $\frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}$.

Theorem 1.4 *For* $n\geq 2$, *we have*

$$\frac{\sqrt[n]{p(n)}}{\sqrt[n+1]{p(n+1)}}\left(1+\frac{3\pi}{\sqrt{24}n^{5/2}}\right)>\frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}. \tag{1.6}$$

Desalvo and Pak [5] have shown that the limit of $-n^{\frac{3}{2}}\Delta^2\log p(n)$ is $\pi/\sqrt{24}$, see (1.2). By bounding $\Delta^2\log\sqrt[n]{p(n)}$, we derive the following limit of $n^{\frac{5}{2}}\Delta^2\log\sqrt[n]{p(n)}$, which is analogous to (1.2),

$$\lim_{n\to+\infty}n^{\frac{5}{2}}\Delta^2\log\sqrt[n]{p(n)}=3\pi/\sqrt{24}. \tag{1.7}$$

From the above relation (1.7), it can be seen that the coefficent $\frac{3\pi}{\sqrt{24}}$ in (1.6) is the best possible.

This paper is organized as follows. In Section 2, we show that $\{r(n)\}_{n\geq 61}$ is log-convex. In Section 3, we find the limit of $n^{\frac{5}{2}}\Delta^2 \log \sqrt[n]{p(n)}$ and give the inequality (1.6).

2 The Log-convexity of $r(n)$

In this section, we obtain a lower bound of $\Delta^2 \log r(n-1)$ and prove the log-convexity of $\{r(n)\}_{n\geq 61}$. First, we follow the approach of Desalvo and Pak to give an expression of $\Delta^2 \log r(n-1)$ as a sum of $\Delta^2 \widetilde{B}(n-1)$ and $\Delta^2 \widetilde{E}(n-1)$, where $\Delta^2 \widetilde{B}(n-1)$ makes a major contribution to $\Delta^2 \log r(n-1)$ with $\Delta^2 \widetilde{E}(n-1)$ being the error term, that is, $\Delta^2 \widetilde{B}(n-1)$ converges to $\Delta^2 \log r(n-1)$. The expressions for $B(n)$ and $E(n)$ will be given later. In this setting, we derive a lower bound of $\Delta^2 \widetilde{B}(n-1)$. By Lehmer's error bound, we give an upper bound for $|\Delta^2 \widetilde{E}(n-1)|$. Combining the lower bound for $\Delta^2 \widetilde{B}(n-1)$ and the upper bound for $\Delta^2 \widetilde{E}(n-1)$, we are led to a lower bound for $\Delta^2 \log r(n-1)$. By proving the positivity of this lower bound for $\Delta^2 \log r(n-1)$, we reach the log-convexity of $\{r(n)\}_{n\geq 61}$.

The strict log-convexity of $\{r(n)\}_{n\geq 61}$ can be restated as the following relation for $n \geq 61$,

$$\log r(n+1) + \log r(n-1) - 2\log r(n) > 0,$$

that is, for $n \geq 61$,

$$\Delta^2 \log r(n-1) > 0.$$

For $n \geq 1$ and any positive integer N, the Hardy-Ramanujan-Rademacher formula reads

$$p(n) = \frac{d}{\mu^2} \sum_{k=1}^{N} A_k^{\star}(n) \left[\left(1 - \frac{k}{\mu}\right) e^{\frac{\mu}{k}} + \left(1 + \frac{k}{\mu}\right) e^{-\frac{\mu}{k}} \right] + R_2(n, N), \tag{2.1}$$

where $d = \frac{\pi^2}{6\sqrt{3}}$, $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$, $A_k^{\star}(n) = k^{-\frac{1}{2}} A_k(n)$, $A_k(n)$ is a sum of 24th roots of unity with initial values $A_1(n) = 1$ and $A_2(n) = (-1)^n$, $R_2(n, N)$ is the remainder. Lehmer's error bound for $R_2(n, N)$ is given by

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu}\right)^3 \sinh \frac{\mu}{N} + \frac{1}{6} - \left(\frac{N}{\mu}\right)^2 \right]. \tag{2.2}$$

Let us give an outline of Desalvo and Pak's approach to proving the log-concavity of $\{p(n)\}_{n>25}$. Setting $N = 2$ in (2.1), they expressed $p(n)$ as

$$p(n) = T(n) + R(n), \tag{2.3}$$

where

$$T(n) = \frac{d}{\mu(n)^2}\left[\left(1-\frac{1}{\mu(n)}\right)e^{\mu(n)} + \frac{(-1)^n}{\sqrt{2}}e^{\frac{\mu(n)}{2}}\right], \tag{2.4}$$

$$R(n) = \frac{d}{\mu(n)^2}\left[\left(1+\frac{1}{\mu(n)}\right)e^{-\mu(n)} - \frac{(-1)^n}{\sqrt{2}}\frac{2}{\mu(n)} + \frac{(-1)^n}{\sqrt{2}}\left(1+\frac{2}{\mu(n)}\right)e^{-\frac{\mu(n)}{2}}\right] + R_2(n,2). \tag{2.5}$$

They have shown that

$$\left|\Delta^2 \log p(n-1) - \Delta^2 \log T(n-1)\right| = \left|\Delta^2 \log\left(1+\frac{R(n-1)}{T(n-1)}\right)\right| < e^{-\frac{\pi\sqrt{2n}}{10\sqrt{3}}}. \tag{2.6}$$

and

$$\left|\Delta^2 \log T(n-1) - \Delta^2 \log \frac{d}{\mu(n-1)^2}\left(1-\frac{1}{\mu(n-1)}\right)e^{\mu(n-1)}\right| < e^{-\frac{\pi\sqrt{2n}}{10\sqrt{3}}}. \tag{2.7}$$

It follows that $\Delta^2 \log \frac{d}{\mu(n-1)^2}\left(1-\frac{1}{\mu(n-1)}\right)e^{\mu(n-1)}$ converges to $\Delta^2 \log p(n-1)$. Finally, they use $-\Delta^2 \log \frac{d}{\mu(n-1)^2}\left(1-\frac{1}{\mu(n-1)}\right)e^{\mu(n-1)}$ to estimate $-\Delta^2 \log p(n-1)$, leading to the log-concavity of $\{p(n)\}_{n>25}$.

In this paper, we use an alternative decomposition of $p(n)$. Setting $N = 2$ in (2.1), we can express $p(n)$ as

$$p(n) = \widetilde{T}(n) + \widetilde{R}(n), \tag{2.8}$$

where

$$\widetilde{T}(n) = \frac{d}{\mu(n)^2}\left(1-\frac{1}{\mu(n)}\right)e^{\mu(n)}, \tag{2.9}$$

$$\widetilde{R}(n) = \frac{d}{\mu(n)^2}\Bigg[\left(1+\frac{1}{\mu(n)}\right)e^{-\mu(n)} + \frac{(-1)^n}{\sqrt{2}}\left(1-\frac{2}{\mu(n)}\right)e^{\frac{\mu(n)}{2}} + \frac{(-1)^n}{\sqrt{2}}\left(1+\frac{2}{\mu(n)}\right)e^{-\frac{\mu(n)}{2}}\Bigg] + R_2(n,2). \tag{2.10}$$

Based on the decomposition (2.8) for $p(n)$, one can express $\Delta^2 \log r(n-1)$ as follows:

$$\Delta^2 \log r(n-1) = \Delta^2 \widetilde{B}(n-1) + \Delta^2 \widetilde{E}(n-1), \tag{2.11}$$

where

$$\widetilde{B}(n) = \frac{1}{n}\log \widetilde{T}(n) - \frac{1}{n}\log n, \tag{2.12}$$

$$\widetilde{y}_n = \widetilde{R}(n)/\widetilde{T}(n), \tag{2.13}$$

$$\widetilde{E}(n) = \frac{1}{n}\log(1+\widetilde{y}_n). \tag{2.14}$$

The following lemma will be used to give a lower bound and an upper bound of $\Delta^2 \widetilde{B}(n-1)$.

Lemma 2.1 *Suppose $f(x)$ has a continuous second derivative for $x \in [n-1, n+1]$. Then there exists $c \in (n-1, n+1)$ such that*

$$\Delta^2 f(n-1) = f(n+1) + f(n-1) - 2f(n) = f^{''}(c). \tag{2.15}$$

If $f(x)$ has an increasing second derivative, then

$$f''(n-1) < \Delta^2 f(n-1) < f''(n+1). \tag{2.16}$$

Conversely, if $f(x)$ has a decreasing second derivative, then

$$f''(n+1) < \Delta^2 f(n-1) < f''(n-1). \tag{2.17}$$

Proof. Set $\varphi(x) = f(x+1) - f(x)$. By the mean value theorem, there exists a number $\xi \in (n-1, n)$ such that

$$f(n+1) + f(n-1) - 2f(n) = \varphi(n) - \varphi(n-1) = \varphi^{'}(\xi).$$

Again, applying the mean value theorem to $\varphi^{'}(\xi)$, there exists a number $\theta \in (0,1)$ such that

$$\varphi^{'}(\xi) = f'(\xi+1) - f'(\xi) = f^{''}(\xi+\theta).$$

Let $c = \xi + \theta$. Then we get (2.15), which yields (2.16) and (2.17). ∎

In order to give a lower bound for $\Delta^2 \log r(n-1)$ and obtain the limit of $n^{\frac{5}{2}} \Delta^2 \log \sqrt[n]{p(n)}$, we need the following lower and upper bounds for $\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1)$.

Lemma 2.2 *Let*

$$B_1(n) = \frac{72\pi}{(n+1)(24n+23)^{3/2}} - \frac{4\log(\mu(n-1))}{(n-1)^3}, \tag{2.18}$$

$$B_2(n) = \frac{72\pi}{(n-1)(24n-25)^{3/2}} - \frac{4\log(\mu(n+1))}{(n+1)^3} + \frac{5}{(n-1)^3}. \tag{2.19}$$

For $n \geq 40$, we have

$$B_1(n) < \Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1) < B_2(n). \tag{2.20}$$

Proof. By the definition (2.9), we may write

$$\frac{\log \widetilde{T}(n)}{n} = \sum_{i=1}^{4} f_i,$$

where

$$
\begin{aligned}
f_1(n) &= \frac{\mu(n)}{n}, \\
f_2(n) &= -\frac{3\log\mu(n)}{n}, \\
f_3(n) &= \frac{\log(\mu(n)-1)}{n}, \\
f_4(n) &= \frac{\log d}{n}.
\end{aligned}
$$

Thus

$$
\Delta^2 \frac{1}{n-1}\log \widetilde{T}(n-1) = \sum_{i=1}^{4} \Delta^2 f_i(n-1). \tag{2.21}
$$

Since

$$
f_1^{'''}(n) = \frac{\pi}{n(24n-1)^{3/2}}\left(-\frac{216}{n} + \frac{864}{24n-1} + \frac{36}{n^2} - \frac{1}{n^3}\right),
$$

we see that for $n \geq 1$, $f_1^{'''}(n) < 0$. Similarly, it can be checked that for $n \geq 4$, $f_2^{'''}(n) > 0$, $f_3^{'''}(n) < 0$, and $f_4^{'''}(n) > 0$. Consequently, for $n \geq 4$, $f_1^{''}(n)$ and $f_3^{''}(n)$ are decreasing, whereas $f_2^{''}(n)$ and $f_4^{''}(n)$ are increasing. Using Lemma 2.1, for each i, we can get a lower bound and an upper bound for $\Delta^2 f_i(n-1)$ in terms of $f_i^{''}(n-1)$ and $f_i^{''}(n+1)$. For example,

$$
f_1^{''}(n+1) < \Delta^2 f_1(n-1) < f_1^{''}(n-1).
$$

So, by (2.21) we find that

$$
\Delta^2 \frac{1}{n-1}\log \widetilde{T}(n-1) > f_1^{''}(n+1) + f_2^{''}(n-1) + f_3^{''}(n+1) + f_4^{''}(n-1), \tag{2.22}
$$

and

$$
\Delta^2 \frac{1}{n-1}\log \widetilde{T}(n-1) < f_1^{''}(n-1) + f_2^{''}(n+1) + f_3^{''}(n-1) + f_4^{''}(n+1), \tag{2.23}
$$

where

$$
f_1^{''}(n) = \frac{72\pi}{n(24n-1)^{3/2}} - \frac{12\pi}{n^2(24n-1)^{3/2}} + \frac{\pi}{3n^3(24n-1)^{3/2}}, \tag{2.24}
$$

$$
f_2^{''}(n) = -\frac{6\log\mu(n)}{n^3} + \frac{72}{(24n-1)n^2} + \frac{864}{n(24n-1)^2}, \tag{2.25}
$$

$$
\begin{aligned}
f_3^{''}(n) = &-\frac{4\pi^2}{(\mu(n)-1)^2(24n-1)n} + \frac{2\log(\mu(n)-1)}{n^3} \\
&- \frac{4\pi}{(\mu(n)-1)\sqrt{24n-1}n^2} - \frac{24\pi}{(\mu(n)-1)(24n-1)^{3/2}n},
\end{aligned} \tag{2.26}
$$

$$f_4''(n) = \frac{2\log d}{n^3}. \tag{2.27}$$

According to (2.24), one can check that for $n \geq 2$,

$$f_1''(n+1) > \frac{72\pi}{(n+1)(24n+23)^{3/2}} - \frac{12\pi}{(n+1)^2(24n+23)^{3/2}}. \tag{2.28}$$

An easy computation shows that for $n \geq 3$,

$$\mu(n) - 1 > \frac{2}{3}\mu(n-2). \tag{2.29}$$

Substituting (2.29) into (2.26) yields that

$$f_3''(n+1) > \frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{540}{(24n-25)^2(n-1)} - \frac{36}{(24n-25)(n-1)^2}. \tag{2.30}$$

Using (2.25) and (2.30), we find that

$$\begin{aligned} &f_2''(n-1) + f_3''(n+1) \\ &\quad > \frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{6\log(\mu(n-1))}{(n-1)^3} \\ &\qquad + \frac{324}{(n-1)(24n-25)^2} + \frac{36}{(n-1)^2(24n-25)} \end{aligned} \tag{2.31}$$

Apparently, for $n \geq 2$,

$$\frac{2}{(n+1)^3} - \frac{2}{(n-1)^3} > -\frac{12}{(n-1)^4},$$

so that

$$\begin{aligned} &\frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{6\log(\mu(n-1))}{(n-1)^3} \\ &\quad > \frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{2\log(\mu(n+1)-1)}{(n-1)^3} - \frac{4\log(\mu(n-1))}{(n-1)^3} \\ &\quad > -\frac{12\log(\mu(n+1)-1)}{(n-1)^4} - \frac{4\log(\mu(n-1))}{(n-1)^3}. \end{aligned} \tag{2.32}$$

Since, for $n \geq 2$,

$$\frac{324}{(n-1)(24n-25)^2} + \frac{36}{(n-1)^2(24n-25)} > \frac{2}{(n-1)^3}, \tag{2.33}$$

utilizing (2.31) and (2.32) yields that for $n \geq 3$,

$$f_2''(n-1) + f_3''(n+1) > -\frac{4\log(\mu(n-1))}{(n-1)^3} + \frac{2}{(n-1)^3} - \frac{12\log(\mu(n+1)-1)}{(n-1)^4}. \tag{2.34}$$

Using (2.27), (2.28) and (2.34), we deduce that

$$\begin{aligned}&f_1''(n+1)+f_2''(n-1)+f_3''(n+1)+f_4''(n-1)-B_1(n)\\&\qquad>\frac{2(1+\log d)}{(n-1)^3}-\frac{12\pi}{(n+1)^2(24n+23)^{3/2}}-\frac{12\log(\mu(n+1)-1)}{(n-1)^4}.\end{aligned}\tag{2.35}$$

Let $C(n)$ be the right hand side of (2.35). To prove (2.22), it is enough to show that $C(n)>0$ when $n\geq 40$. Since $\log x<x$ for $x>0$, and for $n\geq 3$

$$\mu(n+1)-1<\frac{\pi}{4}\sqrt{24n-24},\tag{2.36}$$

we get

$$-\frac{12\log(\mu(n+1)-1)}{(n-1)^4}>-\frac{12(\mu(n+1)-1)}{(n-1)^4}>-\frac{3\sqrt{24}\pi}{(n-1)^{7/2}}.\tag{2.37}$$

Note that for $n\geq 2$,

$$-\frac{12\pi}{(n+1)^2(24n+23)^{3/2}}>-\frac{\sqrt{24}\pi}{48(n-1)^{7/2}}.\tag{2.38}$$

Combining (2.37) and (2.38) gives for $n\geq 2$,

$$C(n)>\frac{2(1+\log d)}{(n-1)^3}-\frac{(3+1/48)\sqrt{24}\pi}{(n-1)^{7/2}}.\tag{2.39}$$

It is straightforward to show that the right hand side of (2.39) is positive if $n\geq 490$. For $40\leq n\leq 489$, it is routine to check that $C(n)>0$, and so $C(n)>0$ for $n\geq 40$. It follows from (2.35) that for $n\geq 40$,

$$\Delta^2\frac{1}{n-1}\log\widetilde{T}(n-1)>B_1(n).$$

To derive the upper bound for $\Delta^2\frac{1}{n-1}\log\widetilde{T}(n-1)$, we obtain the following upper bounds which can be verified directly. The proofs are omitted. For $n\geq 2$,

$$f_1''(n-1)<\frac{72\pi}{(n-1)[24n-25]^{3/2}},$$

$$f_2''(n+1)<-\frac{6\log\mu(n+1)}{(n+1)^3}+\frac{9}{2(n-1)^3},$$

$$\begin{aligned}f_3''(n-1)<&-\frac{4\pi^2}{(\mu(n-1))^2(24n-25)(n-1)}+\frac{2\log(\mu(n-1))}{(n-1)^3}\\&-\frac{4\pi}{\mu(n-1)\sqrt{24n-25}(n-1)^2}-\frac{24\pi}{\mu(n-1)(24n-25)^{3/2}(n-1)},\end{aligned}$$

$$f_2''(n+1)+f_3''(n-1)<\frac{3}{(n-1)^3}+\frac{12\log(\mu(n+1))}{(n-1)^4}-\frac{4\log(\mu(n+1))}{(n+1)^3},$$

$$f_4''(n+1) < 0.$$

Combining the above upper bounds, we conclude that for $n \geq 40$,

$$f_1''(n-1) + f_2''(n+1) + f_3''(n-1) + f_4''(n+1) < B_2(n).$$

This completes the proof. ∎

The following lemma gives an upper bound for $|\Delta^2 \widetilde{E}(n-1)|$.

Lemma 2.3 *For $n \geq 40$,*

$$|\Delta^2 \widetilde{E}(n-1)| < \frac{5}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}}. \tag{2.40}$$

Proof. By (2.14), we find that for $n \geq 2$,

$$\Delta^2 \widetilde{E}(n-1) = \frac{1}{n-1}\log(1+\widetilde{y}_{n-1}) + \frac{1}{n+1}\log(1+\widetilde{y}_{n+1}) - \frac{2}{n}\log(1+\widetilde{y}_n), \tag{2.41}$$

where

$$\widetilde{y}_n = \widetilde{R}(n)/\widetilde{T}(n).$$

To bound $|\Delta^2 \widetilde{E}(n-1)|$, it is necessary to bound $\widetilde{y}_n$. For this purpose, we first consider $\widetilde{R}(n)$, as defined by (2.10). Since $d < 1$ and $\mu(n) > 2$, for $n \geq 1$ we have

$$\begin{aligned} &\frac{d}{\mu(n)^2}\left[\left(1+\frac{1}{\mu(n)}\right)e^{-\mu(n)} + \frac{(-1)^n}{\sqrt{2}}\left(1-\frac{2}{\mu(n)}\right)e^{\frac{\mu(n)}{2}} + \frac{(-1)^n}{\sqrt{2}}\left(1+\frac{2}{\mu(n)}\right)e^{-\frac{\mu(n)}{2}}\right] \\ &\quad < \frac{1}{\mu(n)^2}\left(1+e^{\frac{\mu(n)}{2}}+1\right). \end{aligned}$$

For $N = 2$ and $n \geq 1$, Lehmer's bound (2.2) reduces to

$$|R_2(n,2)| < 4\left(1+\frac{4}{\mu(n)^3}e^{\frac{\mu(n)}{2}}\right).$$

By the definition of $\widetilde{R}(n)$,

$$|\widetilde{R}(n)| < \frac{1}{\mu(n)^2}\left(1+e^{\frac{\mu(n)}{2}}+1\right) + 4\left(1+\frac{4}{\mu(n)^3}e^{\frac{\mu(n)}{2}}\right) < 5 + \frac{9}{\mu(n)^2}e^{\frac{\mu(n)}{2}}. \tag{2.42}$$

Recalling the definition (2.9) of $\widetilde{T}(n)$, it follows from (2.42) that for $n \geq 1$,

$$|\widetilde{y}_n| < \frac{\mu(n)}{d(\mu(n)-1)}\left(5\mu(n)^2 e^{-\frac{2\mu(n)}{3}} + 9e^{-\frac{\mu(n)}{6}}\right)e^{-\frac{\mu(n)}{3}}. \tag{2.43}$$

Observe that for $n \geq 2$,

$$\left(5\mu(n)^2 e^{-\frac{2\mu(n)}{3}} + 9e^{-\frac{\mu(n)}{6}}\right)' < 0, \tag{2.44}$$

and

$$\left(\frac{d(\mu(n)-1)}{\mu(n)}\right)' > 0. \tag{2.45}$$

Since

$$5\mu^2(40)e^{-\frac{2\mu(40)}{3}} + 9e^{-\frac{\mu(40)}{6}} < \frac{d(\mu(40)-1)}{\mu(40)},$$

using (2.44) and (2.45), we deduce that for $n \geq 40$,

$$5\mu^2(n)e^{-\frac{2\mu(n)}{3}} + 9e^{-\frac{\mu(n)}{6}} < \frac{d(\mu(n)-1)}{\mu(n)}. \tag{2.46}$$

Now, it is clear from (2.43) and (2.46) that for $n \geq 40$,

$$|\widetilde{y}_n| < e^{-\frac{\mu(n)}{3}}. \tag{2.47}$$

In view of (2.47), for $n \geq 40$,

$$|\widetilde{y}_n| < e^{-\frac{\mu(40)}{3}} < \frac{1}{5}. \tag{2.48}$$

It is known that $\log(1+x) < x$ for $0 < x < 1$ and $-\log(1+x) < -x/(1+x)$ for $-1 < x < 0$. Thus, for $|x| < 1$,

$$|\log(1+x)| \leq \frac{|x|}{1-|x|}, \tag{2.49}$$

see also [5], and so it follows from (2.48) and (2.49) that for $n \geq 40$,

$$|\log(1+\widetilde{y}_n)| \leq \frac{|\widetilde{y}_n|}{1-|\widetilde{y}_n|} \leq \frac{5}{4}|\widetilde{y}_n|. \tag{2.50}$$

Because of (2.41), we see that for $n \geq 2$,

$$\left|\Delta^2 \widetilde{E}(n-1)\right| \leq \frac{1}{n-1}\left|\log(1+\widetilde{y}_{n-1})\right| + \frac{1}{n+1}\left|\log(1+\widetilde{y}_{n+1})\right| + \frac{2}{n}\left|\log(1+\widetilde{y}_n)\right|. \tag{2.51}$$

Applying (2.50) to (2.51), we obtain that for $n \geq 40$,

$$\left|\Delta^2 \widetilde{E}(n-1)\right| \leq \frac{5}{4}\left[\frac{|\widetilde{y}_{n-1}|}{n-1} + \frac{|\widetilde{y}_{n+1}|}{n+1} + \frac{2|\widetilde{y}_n|}{n}\right]. \tag{2.52}$$

Plugging (2.47) into (2.52), we infer that for $n \geq 40$,

$$\left|\Delta^2 \widetilde{E}(n-1)\right| < \frac{5}{4}\left[\frac{e^{-\frac{\mu(n-1)}{3}}}{n-1} + \frac{e^{-\frac{\mu(n+1)}{3}}}{n+1} + \frac{2e^{-\frac{\mu(n)}{3}}}{n}\right]. \tag{2.53}$$

But $\frac{1}{n}e^{-\frac{\mu(n)}{3}}$ is decreasing for $n \geq 1$, it follows from (2.53) that for $n \geq 40$,

$$\left|\Delta^2 \widetilde{E}(n-1)\right| < \frac{5}{n-1}e^{-\frac{\mu(n-1)}{3}}.$$

This proves (2.40). ∎

With the aid of Lemmas Lemma 2.2 and 2.3, we are ready to prove the log-convexity of $\{r(n)\}_{n\geq 61}$.

Proof of Theorem 1.2. To prove the strict log-convexity of $\{r(n)\}_{n\geq 61}$, we proceed to show that for $n \geq 61$,

$$\Delta^2 \log r(n-1) > 0.$$

Evidently, for $n \geq 40$,

$$\left(-\frac{\log n}{n}\right)''' > 0.$$

By Lemma 2.1,

$$-\Delta^2 \frac{\log(n-1)}{n-1} > \left(-\frac{\log(n-1)}{n-1}\right)'',$$

that is,

$$-\Delta^2 \frac{\log(n-1)}{n-1} > -\frac{2\log(n-1)}{(n-1)^3} + \frac{1}{(n-1)^3}. \tag{2.54}$$

It follows from (2.12) that

$$\Delta^2 \widetilde{B}(n-1) = \Delta^2 \frac{1}{n-1}\log \widetilde{T}(n-1) - \Delta^2 \frac{\log(n-1)}{n-1}.$$

Applying Lemma 2.2 and (2.54) to the above relation, we deduce that for $n \geq 40$,

$$\Delta^2 \widetilde{B}(n-1) > \widetilde{B}_1(n) - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3},$$

that is,

$$\Delta^2 \widetilde{B}(n-1) > \frac{72\pi}{(n+1)[24n+23]^{3/2}} - \frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3}. \tag{2.55}$$

By (2.11) and Lemma 2.3, we find that for $n \geq 40$,

$$\Delta^2 \log r(n-1) > \Delta^2 \widetilde{B}(n-1) - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}. \tag{2.56}$$

It follows from (2.55) and (2.56) that for $n \geq 40$,

$$\Delta^2 \log r(n-1)$$

$$> \frac{72\pi}{(n+1)[24n+23]^{3/2}} - \frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}.$$

Let $D(n)$ denote the right hand side of the above relation. Clearly, for $n \geq 5505$,

$$\frac{72\pi}{(n+1)[24n+23]^{3/2}} > \frac{3\pi}{\sqrt{24}(n+1)^{5/2}} > \frac{1}{(n-1)^{5/2}}. \tag{2.57}$$

To prove that $D(n) > 0$ for $n \geq 5505$, we wish to show that for $n \geq 5505$,

$$-\frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}} > -\frac{1}{(n-1)^{5/2}}. \tag{2.58}$$

Using the fact that for $x > 5504$, $\log x < x^{1/4}$, we deduce that for $n \geq 5505$,

$$\frac{4\log[\mu(n-1)]}{(n-1)^3} < \frac{4\sqrt[4]{\mu(n-1)}}{(n-1)^3} < \frac{4\sqrt[4]{\frac{\pi}{4}\sqrt{24n-24}}}{(n-1)^3} < \frac{6}{(n-1)^{23/8}}, \tag{2.59}$$

and

$$\frac{2\log(n-1)}{(n-1)^3} < \frac{2(n-1)^{1/4}}{(n-1)^3} < \frac{2}{(n-1)^{11/4}}. \tag{2.60}$$

Since $e^x > x^6/720$ for $x > 0$, we see that for $n \geq 2$,

$$\frac{1}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}} < \frac{1}{n-1}e^{-\frac{\pi\sqrt{23n}}{18}} < \frac{2094}{n^3(n-1)} < \frac{2094}{(n-1)^4}. \tag{2.61}$$

Combining (2.59), (2.60) and (2.61), we find that for $n \geq 5505$,

$$\begin{aligned} &-\frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}} \\ &> -\frac{6}{(n-1)^{23/8}} - \frac{2}{(n-1)^{11/4}} + \frac{3}{(n-1)^3} - \frac{10470}{(n-1)^4} \\ &> -\frac{6}{(n-1)^{23/8}} - \frac{2}{(n-1)^{11/4}} \\ &> -\frac{1}{(n-1)^{5/2}}. \end{aligned}$$

This proves the inequality (2.58). By (2.58) and (2.57), we obtain that $D(n) > 0$ for $n \geq 5505$. Verifying that $\Delta^2 \log r(n-1)$ for $61 \leq n \leq 5504$ completes the proof. ∎

3 An inequality on the ratio $\frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}$

In this section, we employ Lemma 2.2 and Lemma 2.3 to find the limit of $n^{\frac{5}{2}}\Delta^2 \log \sqrt[n]{p(n)}$. Then we give an upper bound for $\Delta^2 \log \sqrt[n-1]{p(n-1)}$. This leads to an inequality analogous to the inequality (1.3).

Theorem 3.1 *Let $\alpha = 3\pi/\sqrt{24}$. We have*

$$\lim_{n\to+\infty} n^{\frac{5}{2}}\Delta^2 \log \sqrt[n]{p(n)} = \alpha. \tag{3.1}$$

Proof. Using (2.8), that is, the $N = 2$ case of the Hardy-Ramanujan-Rademacher formula for $p(n)$, we find that

$$\begin{aligned}\log \sqrt[n]{p(n)} &= \frac{1}{n}\log[\widetilde{T}(n) + \widetilde{R}(n)]\\ &= \frac{1}{n}\log \widetilde{T}(n) + \frac{1}{n}\log\left(1 + \frac{\widetilde{R}(n)}{\widetilde{T}(n)}\right)\\ &= \frac{1}{n}\log \widetilde{T}(n) + \frac{1}{n}\log(1 + \widetilde{y}_n),\end{aligned}$$

where $\widetilde{T}(n)$ and y_n are given by (2.9) and (2.13). By the definition (2.14) of $\widetilde{E}(n)$, we get

$$\Delta^2 \log \sqrt[n-1]{p(n-1)} = \Delta^2 \frac{1}{n-1}\log \widetilde{T}(n-1) + \Delta^2 \widetilde{E}(n-1). \tag{3.2}$$

Applying Lemma 2.2, we obtain that for $n \geq 40$,

$$B_1(n) < \Delta^2 \frac{1}{n-1}\log \widetilde{T}(n-1) < B_2(n), \tag{3.3}$$

where

$$\begin{aligned}B_1(n) =& \frac{72\pi}{(n+1)[24n+23]^{3/2}} - \frac{4\log[\mu(n-1)]}{(n-1)^3},\\ B_2(n) =& \frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{4\log[\mu(n+1)]}{(n+1)^3} + \frac{5}{(n-1)^3}.\end{aligned}$$

It is easily seen that

$$\lim_{n\to+\infty} \frac{72\pi(n-1)^{5/2}}{(n+1)[24n+23]^{3/2}} = \alpha, \tag{3.4}$$

$$\lim_{n\to+\infty} \frac{\log \mu}{(n-1)^{1/2}} = 0. \tag{3.5}$$

By (3.4) and (3.5), we see that

$$\lim_{n\to+\infty}(n-1)^{\frac{5}{2}}B_1(n)=\lim_{n\to+\infty}(n-1)^{\frac{5}{2}}B_2(n)=\alpha. \tag{3.6}$$

Combining (3.3) and (3.6) gives

$$\lim_{n\to+\infty}(n-1)^{\frac{5}{2}}\Delta^2\frac{1}{n-1}\log\widetilde{T}(n-1)=\alpha. \tag{3.7}$$

From Lemma 2.3, we know that for $n\geq 40$,

$$-\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}<\Delta^2\widetilde{E}(n-1)<\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}.$$

By the fact that

$$\lim_{n\to+\infty}(n-1)^{\frac{3}{2}}e^{-\frac{\pi\sqrt{24n}}{18}}=0,$$

we get

$$\lim_{n\to+\infty}(n-1)^{\frac{5}{2}}\Delta^2\widetilde{E}(n-1)=0. \tag{3.8}$$

Using (3.2), (3.7) and (3.8), we deduce that

$$\lim_{n\to+\infty}n^{\frac{5}{2}}\Delta^2\log\sqrt[n]{p(n)}=\alpha,$$

as required. ∎

To prove Theorem 1.4, we need the following upper bound for $\Delta^2\log\sqrt[n-1]{p(n-1)}$.

Theorem 3.2 *For $n\geq 2095$,*

$$\Delta^2\log\sqrt[n-1]{p(n-1)}<\frac{3\pi}{\sqrt{24}n^{5/2}+3\pi}. \tag{3.9}$$

Proof. By the upper bound of $\Delta^2\frac{1}{n-1}\log\widetilde{T}(n-1)$ given in Lemma 2.2, the upper bound of $\Delta^2\widetilde{E}(n-1)$ given in Lemma 2.3 and the relation (3.2), we get the following upper bound of $\Delta^2\log\sqrt[n-1]{p(n-1)}$ for $n\geq 40$,

$$\Delta^2\log\sqrt[n-1]{p(n-1)}<\frac{72\pi}{(n-1)[24n-25]^{3/2}}+\frac{5}{(n-1)^3}-\frac{4\log[\mu(n+1)]}{(n+1)^3}+\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}.$$

To prove (3.9), we claim that for $n\geq 2095$,

$$\frac{72\pi}{(n-1)[24n-25]^{3/2}}+\frac{5}{(n-1)^3}-\frac{4\log[\mu(n+1)]}{(n+1)^3}+\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}<\frac{3\pi}{\sqrt{24}n^{5/2}+3\pi}. \tag{3.10}$$

First, we show that for $n \geq 60$,

$$\frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} < \frac{1}{(n-1)^3}. \tag{3.11}$$

For $0 < x \leq \frac{1}{48}$, it can be checked that

$$\frac{1}{(1-x)^{3/2}} < 1 + \frac{3}{2}x + \frac{3}{8}x^{\frac{3}{2}}. \tag{3.12}$$

In the notation $\alpha = 3\pi/\sqrt{24}$, we have

$$\frac{72\pi}{(n-1)(24n-25)^{3/2}} = \frac{\alpha}{(n-1)n^{3/2}(1-\frac{25}{24n})^{3/2}}. \tag{3.13}$$

Setting $x = \frac{25}{24n}$, we have $x \leq \frac{1}{48}$ for $n \geq 60$. Applying (3.12) to the right hand side of (3.13), we find that for $n \geq 60$,

$$\frac{\alpha}{(n-1)n^{3/2}(1-\frac{25}{24n})^{3/2}} < \frac{\alpha}{(n-1)n^{3/2}}\left[1 + \frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}}\right], \tag{3.14}$$

so that for $n \geq 60$,

$$\begin{aligned} &\frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} \\ &< \frac{\alpha}{(n-1)n^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} + \frac{\alpha}{(n-1)n^{3/2}}\left[\frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}}\right]. \end{aligned} \tag{3.15}$$

To prove (3.11), we proceed to show that the right hand side of (3.15) is bounded by $\frac{1}{(n-1)^3}$. For $n \geq 2$, we obtain that

$$\begin{aligned} &\frac{\alpha}{(n-1)n^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} \\ &= \frac{\alpha}{(n-1)n^{3/2}} - \frac{\alpha}{n^{5/2}+\alpha} \\ &= \frac{\alpha n^{3/2}+\alpha^2}{(n^{5/2}+\alpha)(n-1)n^{3/2}} \\ &= \frac{\alpha}{(n^{5/2}+\alpha)(n-1)} + \frac{\alpha^2}{(n^{5/2}+\alpha)(n-1)n^{3/2}}. \end{aligned}$$

Since $n^{5/2}+\alpha > (n-1)^{5/2}$ and $n^{3/2} > (n-1)^{3/2}$ for $n \geq 2$, in this case we have

$$\frac{\alpha}{(n-1)n^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} < \frac{\alpha}{(n-1)^{7/2}} + \frac{\alpha}{(n-1)^5}. \tag{3.16}$$

Applying (3.16) to (3.15), we obtain that for $n \geq 60$,

$$\frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi}$$

$$< \frac{\alpha}{(n-1)^{7/2}} + \frac{\alpha^2}{(n-1)^5} + \frac{\alpha}{(n-1)n^{3/2}}\left[\frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}}\right]. \tag{3.17}$$

Since $\frac{75}{48n} < \frac{2}{n-1}$ and $\frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}} < \frac{1}{(n-1)^{3/2}}$ for $n \geq 2$, it follows from (3.17) that for $n \geq 60$,

$$\frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi}$$

$$< \frac{\alpha}{(n-1)^{7/2}} + \frac{\alpha^2}{(n-1)^5} + \frac{2\alpha}{(n-1)^{7/2}} + \frac{\alpha}{(n-1)^4}.$$

Using the fact that $\alpha < 2$, we see that

$$\frac{3\alpha}{(n-1)^{7/2}} + \frac{\alpha^2}{(n-1)^5} + \frac{\alpha}{(n-1)^4} < \frac{6}{(n-1)^{7/2}} + \frac{4}{(n-1)^5} + \frac{2}{(n-1)^4}. \tag{3.18}$$

For $n \geq 60$, it is easily checked that the right hand side of (3.18) is bounded by $\frac{1}{(n-1)^3}$. This confirms (3.11).

To prove the claim (3.10), it is enough to show that for $n \geq 2095$,

$$\frac{1}{(n-1)^3} < \frac{4\log[\mu(n+1)]}{(n+1)^3} - \frac{5}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}. \tag{3.19}$$

From (2.61) it can be seen that for $n \geq 2095$,

$$\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}} < \frac{5}{(n-1)^3}. \tag{3.20}$$

Since $4\log[\mu(n+1)] > 18$ for $n \geq 2095$, it follows from (3.20) that for $n \geq 2095$,

$$\frac{4\log[\mu(n+1)]}{(n+1)^3} - \frac{5}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}$$

$$> \frac{18}{(n+1)^3} - \frac{10}{(n-1)^3} > \frac{1}{(n-1)^3}.$$

So we obtain (3.19). This completes the proof. ∎

We are now in a position to finish the proof of Theorem 1.4.

Proof of Theorem 1.4. It is known that for $x > 0$,

$$\frac{x}{1+x} < \log(1+x),$$

so that for $n \geq 1$,

$$\frac{3\pi}{\sqrt{24}n^{5/2} + 3\pi} < \log\left(1 + \frac{3\pi}{\sqrt{24}n^{5/2}}\right).$$

In light of the above relation, Theorem 3.2 implies that for $n \geq 2095$,

$$\Delta^2 \log \sqrt[n-1]{p(n-1)} < \log\left(1 + \frac{3\pi}{\sqrt{24}n^{5/2}}\right),$$

that is,

$$\sqrt[n+1]{p(n+1)}\ \sqrt[n-1]{p(n-1)} < \left(1 + \frac{3\pi}{\sqrt{24}n^{5/2}}\right)(\sqrt[n]{p(n)})^2.$$

It can be checked that the above inequality holds for $2 \leq n \leq 2095$. This completes the proof of the theorem. ∎

We remark that $\alpha = 3\pi/\sqrt{24}$ is the smallest possible number for the inequality in Theorem 1.4. Suppose that $0 < \beta < \alpha$. By Theorem 3.1, there exists an integer N so as to for $n > N$,

$$n^{5/2}\Delta^2 \log \sqrt[n-1]{p(n-1)} > \beta.$$

It follows that

$$\Delta^2 \log \sqrt[n-1]{p(n-1)} > \frac{\beta}{n^{5/2}} > \log\left(1 + \frac{\beta}{n^{5/2}}\right),$$

which implies that for $n > N$,

$$\frac{\sqrt[n]{p(n)}}{\sqrt[n+1]{p(n+1)}}\left(1 + \frac{\beta}{n^{5/2}}\right) < \frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}.$$

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] G.E. Andrews, Combinatorial proof of a partition function limit, Amer. Math. Monthly., 78 (1971), 276-278.

[2] W.Y.C. Chen, Recent developments on log-concavity and q-log-concavity of combinatorial polynomials, DMTCS Proceedings of 22nd International Conference on Formal Power Series and Algebraic Combinatorics, 2010.

[3] W.Y.C. Chen, J.J.F. Guo and L.X.W. Wang, Infinitely-logarithmically monotonic combinatorial sequences. Adv. Appl. Math., 52 (2014), 99-120.

[4] W.Y.C. Chen, L.X.W. Wang and G.Y.B. Xie, Finite differences of the logarithm of the partition function. Math. Comp., to appear.

[5] S. Desalvo and I. Pak, Log-concavity of the partition function, Ramanujan J., to appear.

[6] G.H. Hardy, Twelve Lectures on Subjects Suggested by His Life and Work, Cambridge University Press, Cambridge, 1940.

[7] G.H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc., 17 (1918), 75-175.

[8] D.H. Lehmer, On the series for the partition function, Trans. Amer. Math. Soc., 43 (1938), 271-292.

[9] D.H. Lehmer, On the remainders and convergence of the series for the partition function, Trans. Amer. Math. Soc., 46 (1939), 362-373.

[10] H. Rademacher, A convergent series for the partition function $p(n)$, Proc. Nat. Acad. Sci, 23 (1937), 78-84.

[11] Z.-W. Sun, On a sequence involving sums of primes, Bull. Aust. Math. Soc. 88 (2013), 197-205.

[12] Y. Wang, B.-X. Zhu, Proofs of some conjectures on monotonicity of number-theoretic and combinatorial sequences, Sci. China Math. 57 (2014), 2429-2435.