Ratio Monotonicity of Polynomials Derived from Nondecreasing Sequences

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Abstract. The ratio monotonicity of a polynomial is a stronger property than log-concavity. Let P(x) be a polynomial with nonnegative and nondecreasing coefficients. We prove the ratio monotone property of P(x + 1), which leads to the log-concavity of P(x+c) for any $c \ge 1$ due to Llamas and Martínez-Bernal. As a consequence, we obtain the ratio monotonicity of the Boros-Moll polynomials obtained by Chen and Xia without resorting to the recurrence relations of the coefficients.

Keywords: log-concavity, ratio monotonicity, Boros-Moll polynomials.

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1 Introduction

This paper is concerned with the ratio monotone property of polynomials derived from nonnegative and nondecreasing sequences. A sequence $\{a_k\}_{0 \le k \le m}$ of positive real numbers is said to be unimodal if there exists an integer $r \ge 0$ such that

 $a_0 \leq \cdots \leq a_{r-1} \leq a_r \geq a_{r+1} \geq \cdots \geq a_m,$

and it is said to be spiral if

$$a_m \le a_0 \le a_{m-1} \le a_1 \le \dots \le a_{[\frac{m}{2}]},$$
 (1.1)

where $\left[\frac{m}{2}\right]$ stands for the largest integer less than $\frac{m}{2}$. We say that a sequence $\{a_k\}_{0 \le k \le m}$ is log-concave if for any $1 \le k \le m - 1$,

$$a_k^2 - a_{k+1}a_{k-1} \ge 0,$$

or equivalently,

$$\frac{a_0}{a_1} \le \frac{a_1}{a_2} \le \dots \le \frac{a_{m-1}}{a_m}.$$

It is easy to see that either log-concavity or the spiral property implies unimodality, while a log-concave sequence is not necessarily spiral, and vice versa.

A stronger property, which implies both log-concavity and the spiral property, was introduced by Chen and Xia [6] and is called the ratio monotonicity. A sequence of positive real numbers $\{a_k\}_{0 \le k \le m}$ is said to be ratio monotone if

$$\frac{a_m}{a_0} \le \frac{a_{m-1}}{a_1} \le \dots \le \frac{a_{m-i}}{a_i} \le \dots \le \frac{a_{m-[\frac{m-1}{2}]}}{a_{[\frac{m-1}{2}]}} \le 1$$
(1.2)

and

$$\frac{a_0}{a_{m-1}} \le \frac{a_1}{a_{m-2}} \le \dots \le \frac{a_{i-1}}{a_{m-i}} \le \dots \le \frac{a_{[\frac{m}{2}]-1}}{a_{m-[\frac{m}{2}]}} \le 1.$$
(1.3)

Given a polynomial $P(x) = a_0 + a_1 x + \cdots + a_m x^m$ with positive coefficients, we say that P(x) is log-concave (or ratio monotone) if $\{a_k\}_{0 \le k \le m}$ is log-concave (resp., ratio monotone).

Assume that P(x) is a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [3] proved the unimodality of P(x + 1) which implies the unimodality of the Boros-Moll polynomials. They posed the conjecture that the Boros-Moll polynomials are log-concave, which was confirmed by Kauers and Paule [8]. Alvarez et al. [1] showed that P(x+n) is also unimodal for any positive integer n. Wang and Yeh [12] obtained a stronger result that P(x + c) is unimodal for c > 0. Llamas and Martínez-Bernal [9] proved that P(x + c) is log-concave for $c \ge 1$.

In this paper, we prove that if P(x) is a polynomial with nonnegative and nondecreasing coefficients, then P(x + 1) is ratio monotone. This property implies the log-concavity of P(x + 1). Note that by a criterion for logconcavity due to Brenti [5], the log-concavity of P(x + 1) leads to the logconcavity of P(x+c) for $c \ge 1$, as established by Llamas and Martínez-Bernal [9]. The ratio monotonicity of P(x + 1) serves as a simple proof of the ratio monotonicity of the Boros-Moll polynomials obtained by Chen and Xia [7] without resorting to the recurrence relations of the coefficients.

2 The ratio monotone property

The main result of this paper is given below.

Theorem 2.1 If P(x) is a polynomial with nonnegative and nondecreasing coefficients, then P(x + 1) is ratio monotone.

To prove Theorem 2.1, we need three lemmas. The first lemma is a special case of [6, Lemma 2.1].

Lemma 2.2 Suppose that a, b, c, d, e, f are positive real numbers satisfying

$$\frac{a}{b} \le \frac{c}{d} \le \frac{e}{f}.$$

Then

$$\frac{a+c}{b+d} \le \frac{e+c}{f+d}.$$

Lemma 2.3 If B(x) is a ratio monotone polynomial, then so is (x+1)B(x).

Proof. Let

$$B(x) = \sum_{k=0}^{m} a_k x^k$$
 and $(x+1)B(x) = \sum_{k=0}^{m+1} b_k x^k$.

For each k we have $b_k = a_{k-1} + a_k$, where a_{-1} and a_{m+1} are set to 0.

When m = 2n, the ratio monotonicity of B(x) states that

$$\frac{a_{2n}}{a_0} \le \frac{a_{2n-1}}{a_1} \le \dots \le \frac{a_{2n-i}}{a_i} \le \dots \le \frac{a_{n+1}}{a_{n-1}} \le 1$$
(2.1)

and

$$\frac{a_0}{a_{2n-1}} \le \frac{a_1}{a_{2n-2}} \le \dots \le \frac{a_{i-1}}{a_{2n-i}} \le \dots \le \frac{a_{n-1}}{a_n} \le 1.$$
(2.2)

In order to show that (x + 1)B(x) is ratio monotone, we need to justify that

$$\frac{b_{2n+1}}{b_0} \le \frac{b_{2n}}{b_1} \le \dots \le \frac{b_{2n+1-i}}{b_i} \le \dots \le \frac{b_{n+1}}{b_n} \le 1$$
(2.3)

and

$$\frac{b_0}{b_{2n}} \le \frac{b_1}{b_{2n-1}} \le \dots \le \frac{b_i}{b_{2n-i}} \le \dots \le \frac{b_{n-1}}{b_{n+1}} \le 1.$$
(2.4)

We first consider (2.3). Since

$$\frac{a_{2n}}{a_0} \le \frac{a_{2n-1}}{a_1},$$

we see that

$$\frac{a_{2n}}{a_0} \le \frac{a_{2n-1} + a_{2n}}{a_1 + a_0},$$

that is,

3

For $1 \le i \le n-1$, from (2.1) we deduce that

$$\frac{a_{2n+1-i}}{a_{i-1}} \le \frac{a_{2n-i}}{a_i} \le \frac{a_{2n-i-1}}{a_{i+1}}.$$

By Lemma 2.2, we obtain

$$\frac{a_{2n+1-i} + a_{2n-i}}{a_i + a_{i-1}} \le \frac{a_{2n-i} + a_{2n-i-1}}{a_{i+1} + a_i}$$

or equivalently,

$$\frac{b_{2n+1-i}}{b_i} \le \frac{b_{2n-i}}{b_{i+1}}$$

In light of (2.1), we see that $a_{n+1} \leq a_{n-1}$, and thus we have

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1} + a_n}{a_n + a_{n-1}} \le 1.$$

Next, we prove (2.4). From $\frac{a_0}{a_{2n-1}} \leq \frac{a_1}{a_{2n-2}}$ it follows that

$$\frac{a_0}{a_{2n-1} + a_{2n}} \le \frac{a_1 + a_0}{a_{2n-2} + a_{2n-1}}$$

that is,

$$\frac{b_0}{b_{2n}} \le \frac{b_1}{b_{2n-1}}$$

For $2 \leq i \leq n-1$, in view of (2.2) we find that

$$\frac{a_{i-2}}{a_{2n-i+1}} \le \frac{a_{i-1}}{a_{2n-i}} \le \frac{a_i}{a_{2n-i-1}}.$$

By Lemma 2.2, we have

$$\frac{a_{i-1} + a_{i-2}}{a_{2n-i+1} + a_{2n-i}} \le \frac{a_i + a_{i-1}}{a_{2n-i} + a_{2n-i-1}},$$

which can be expressed as

$$\frac{b_{i-1}}{b_{2n-i+1}} \le \frac{b_i}{b_{2n-i}}$$

From (2.2) it is clear that $a_{n-2} \leq a_{n+1}$ and $a_{n-1} \leq a_n$, and hence

$$\frac{b_{n-1}}{b_{n+1}} = \frac{a_{n-1} + a_{n-2}}{a_{n+1} + a_n} \le 1.$$

The case m = 2n + 1 can be dealt with in the same manner. This completes the proof.

The third lemma is concerned with an inequality of an increasing positive sequence. This inequality will be needed in the proof of Theorem 2.1.

Lemma 2.4 For any nondecreasing positive sequence $\{a_k\}_{0 \le k \le m}$, we have

$$\frac{m(m+1)}{2}a_m^2 + a_m a_{m-1} \ge \left(\sum_{k=0}^{m-2} \left(m-1-k\right)a_k\right)a_{m-1} + \left(\sum_{k=0}^m a_k\right)a_{m-2}.$$

Proof. Since $0 < a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m$, we have

$$\frac{m(m+1)}{2}a_m^2 + a_m a_{m-1} - \left(\sum_{k=0}^{m-2} (m-1-k)a_k\right)a_{m-1} - \left(\sum_{k=0}^m a_k\right)a_{m-2}$$

$$\geq \frac{m(m+1)}{2}a_m^2 + a_m a_{m-1} - \sum_{k=0}^{m-2} (m-1-k)a_m^2 - \sum_{k=1}^m a_m^2 - a_m a_{m-1},$$

which simplifies to zero, as desired.

Proof of Theorem 2.1. Use induction on the degree m of P(x). Let

$$P(x) = \sum_{k=0}^{m} a_k x^k,$$

where $0 < a_0 \le a_1 \le \dots \le a_{m-1} \le a_m$.

When m = 2, we have

$$P(x+1) = a_2 x^2 + (a_1 + 2a_2)x + a_0 + a_1 + a_2.$$

Note that $a_2 \leq a_0 + a_1 + a_2$, $a_0 + a_1 + a_2 \leq a_1 + 2a_2$. Therefore, the theorem holds for m = 2.

Now assume that the theorem holds for polynomials of degree m-1. We need to show that it is also true for polynomials P(x) of degree m. Suppose that

$$P(x+1) = \sum_{k=0}^{m} a_k (z+1)^k = \sum_{k=0}^{m} b_k z^k.$$
 (2.5)

We wish to prove that

$$\frac{b_m}{b_0} \le \frac{b_{m-1}}{b_1} \le \dots \le \frac{b_{m-i}}{b_i} \le \dots \le \frac{b_{m-[\frac{m-1}{2}]}}{b_{[\frac{m-1}{2}]}} \le 1$$
(2.6)

and

$$\frac{b_0}{b_{m-1}} \le \frac{b_1}{b_{m-2}} \le \dots \le \frac{b_{i-1}}{b_{m-i}} \le \dots \le \frac{b_{[\frac{m}{2}]-1}}{b_{m-[\frac{m}{2}]}} \le 1.$$
(2.7)

Let

$$Q(x) = \sum_{k=0}^{m-1} a_{k+1} x^k.$$

Then

$$P(x+1) = a_0 + (x+1)Q(x+1)$$

By the inductive hypothesis and Lemma 2.3, we deduce that the polynomial

$$(x+1)Q(x+1) = b_0 - a_0 + \sum_{k=1}^m b_k x^k$$

is ratio monotone. So we have

$$\frac{b_m}{b_0 - a_0} \le \frac{b_{m-1}}{b_1} \le \dots \le \frac{b_{m-i}}{b_i} \le \dots \le \frac{b_{m-[\frac{m-1}{2}]}}{b_{[\frac{m-1}{2}]}} \le 1$$
(2.8)

and

$$\frac{b_0 - a_0}{b_{m-1}} \le \frac{b_1}{b_{m-2}} \le \dots \le \frac{b_{i-1}}{b_{m-i}} \le \dots \le \frac{b_{\lfloor \frac{m}{2} \rfloor - 1}}{b_{m-\lfloor \frac{m}{2} \rfloor}} \le 1.$$
(2.9)

Clearly, (2.6) follows from (2.8). To prove (2.7), it remains to show that

$$\frac{b_0}{b_{m-1}} \le \frac{b_1}{b_{m-2}}.$$

From (2.5), we see that

$$b_0 = \sum_{k=0}^m a_k, \qquad b_{m-1} = a_{m-1} + ma_m,$$

and

$$b_1 = \sum_{k=0}^{m} k a_k, \quad b_{m-2} = a_{m-2} + (m-1)a_{m-1} + \binom{m}{2}a_m.$$

Consequently, it suffices to show that

$$\frac{\sum_{k=0}^{m} a_k}{a_{m-1} + ma_m} \le \frac{\sum_{k=0}^{m} ka_k}{a_{m-2} + (m-1)a_{m-1} + \binom{m}{2}a_m},$$

or equivalently,

$$\left(\sum_{k=0}^{m} ka_k\right)a_{m-1} + \left(\sum_{k=0}^{m} mka_k\right)a_m - \left(\sum_{k=0}^{m} a_k\right)a_{m-2} - \left(\sum_{k=0}^{m} (m-1)a_k\right)a_{m-1} - \left(\sum_{k=0}^{m} \binom{m}{2}a_k\right)a_m \ge 0.$$

The left hand side of the above inequality can be simplified to

$$\left(\sum_{k=0}^{m} \frac{2k-m+1}{2}a_k\right) m a_m + \left(\sum_{k=0}^{m} (k-m+1)a_k\right) a_{m-1} - \left(\sum_{k=0}^{m} a_k\right) a_{m-2},$$

which can be rewritten as a sum of

$$\left(\sum_{k=0}^{m-1} \frac{2k - m + 1}{2} a_k\right) m a_m \tag{2.10}$$

and

$$\frac{m(m+1)}{2}a_m^2 + a_m a_{m-1} - \left(\sum_{k=0}^{m-2} \left(m-1-k\right)a_k\right)a_{m-1} - \left(\sum_{k=0}^m a_k\right)a_{m-2}.$$
(2.11)

By Lemma 2.4, the sum in (2.11) is nonnegative. The sum in (2.10) is also nonnegative, since

$$\sum_{k=0}^{m-1} \frac{2k - m + 1}{2} a_k = \sum_{k=\left[\frac{m-1}{2}\right]+1}^{m-1} \frac{2k - m + 1}{2} a_k - \sum_{k=0}^{\left[\frac{m-1}{2}\right]} \frac{m - 1 - 2k}{2} a_k$$
$$= \sum_{k=0}^{m-2 - \left[\frac{m-1}{2}\right]} \frac{m - 1 - 2k}{2} a_{m-1-k} - \sum_{k=0}^{\left[\frac{m-1}{2}\right]} \frac{m - 1 - 2k}{2} a_k$$
$$= \sum_{k=0}^{\left[\frac{m-1}{2}\right]} \frac{m - 1 - 2k}{2} (a_{m-1-k} - a_k),$$

which is nonnegative, and thus the proof is complete.

Theorem 2.1 leads to the following result of Llamas and Martínez-Bernal [9], since the ratio monotonicity implies log-concavity of P(x + 1) and the log-concavity of P(x + 1) implies the log-concavity of P(x + c) for $c \ge 1$ by a criterion of Brenti [4, 5].

Corollary 2.5 If P(x) is a polynomial with nonnegative and nondecreasing coefficients, then for any $c \ge 1$ the polynomial P(x+c) is log-concave and has no internal zero coefficients.

Theorem 2.1 also serves as a simple proof of the ratio monotonicity of the Boros-Moll polynomials $P_m(x)$, which were introduced by Boros and Moll [2] in their study of the following quartic integral

$$\int_0^{+\infty} \frac{1}{(t^4 + 2xt^2 + 1)^{m+1}} dt = \frac{\pi}{2^{m+3/2}(x+1)^{m+1/2}} P_m(x).$$

Let

$$c_k(m) = 2^{-2m+k} \binom{2m-2k}{m-k} \binom{m+k}{k}$$

Boros and Moll showed that

$$P_m(x) = \sum_{k=0}^m c_k(m)(x+1)^k.$$
 (2.12)

They also observed that, for $0 \le k \le m - 1$,

$$\frac{c_k(m)}{c_{k+1}(m)} = \frac{(2m-2k-1)(k+1)}{(m-k)(m+k+1)} < 1.$$

Thus, $P_m(x-1)$ is a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [2] proved that $P_m(x)$ is unimodal for any $m \ge 0$, and Moll [10] conjectured that $P_m(x)$ is log-concave for any m. This conjecture was confirmed by Kauers and Paule [8]. The ratio monotonicity of $P_m(x)$ was established by Chen and Xia and the proof is quite technical and heavily depends on inequalities on the coefficients. The proof of Theorem 2.1 shows that the log-concavity and ratio monotonicity only depend on the nondecreasing property of the coefficients of $P_m(x-1)$.

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