Combinatorial Telescoping for an Identity of Andrews on Parity in Partitions

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Abstract

Following the method of combinatorial telescoping for alternating sums given by Chen, Hou and Mu, we present a combinatorial telescoping approach to partition identities on sums of positive terms. By giving a classification of the combinatorial objects corresponding to a sum of positive terms, we establish bijections that lead a telescoping relation. We illustrate this idea by giving a combinatorial telescoping relation for a classical identity of MacMahon. Recently, Andrews posed a problem of finding a combinatorial proof of an identity on the q-little Jacobi polynomials which was derived based on a recurrence relation. We find a combinatorial classification of certain triples of partitions and a sequence of bijections. By the method of cancelation, we see that there exists an involution for a recurrence relation that implies the identity of Andrews.

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1 Introduction

In his study of parities in partition identities, Andrews [2] obtained the following identity on the little q-Jacobi polynomials [5, p. 27]:

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1} \\ -q \end{matrix}; q, -q\right) = (-1)^n q^{\binom{n+1}{2}} \sum_{j=-n}^{n} (-1)^j q^{-j^2}. \tag{1.1}$$

Let $G_n(q)$ denote the sum on the left hand side of (1.1). Andrews [2] established the following recurrence relation for $n \geq 1$,

$$G_n(q) + q^n G_{n-1}(q) = 2q^{-\binom{n}{2}}, \tag{1.2}$$

from which (1.1) can be easily deduced. As one of the fifteen open problems, Andrews asked for a combinatorial proof of the above identity (1.1).

In this paper, we give a combinatorial treatment of a homogeneous recurrence relation for the sum

$$F_n(q) = q^{\binom{n}{2}}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1} \\ -q \end{matrix}; q, -q\right),$$

which is a consequence of recurrence relation (1.2). More precisely, for $n \geq 2$ we have

$$F_n(q) + (q^{2n-1} - 1)F_{n-1}(q) - q^{2n-3}F_{n-2}(q) = 0. \tag{1.3}$$

It is readily seen that identity (1.1) is an immediate consequence of (1.3).

The main objective of this paper is to present a combinatorial treatment of the recurrence relation (1.3). To this end, we present the method of combinatorial telescoping for sums of positive terms, which is a variant of the method of combinatorial telescoping for alternating sums. In this framework, we find a classification of certain triples of partitions and a sequence of bijections, leading to a combinatorial proof of the above recurrence relation (1.3).

Recall that Chen, Hou and Mu [3] presented the method of combinatorial telescoping for alternating sums. Consider the alternating sum

$$\sum_{k=0}^{\infty}(-1)^k f(k). \tag{1.4}$$

A combinatorial telescoping for the above alternating sum means a classification of certain combinatorial objects along with a sequence of bijections. This method can be used to show that the above alternating sum satisfies a recurrence relation, and it applies to many q-series identities on alternating sums such as Watson's identity [10]

$$\sum_{k=0}^{\infty}(-1)^k \frac{1-aq^{2k}}{(q;q)_k(aq^k;q)_\infty} a^{2k} q^{k(5k-1)/2} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n}, \tag{1.5}$$

and Sylvester's identity [11]

$$\sum_{k=0}^{\infty}(-1)^k q^{k(3k+1)/2} x^k \frac{1-xq^{2k+1}}{(q;q)_k(xq^{k+1};q)_\infty} = 1. \tag{1.6}$$

In this paper, we consider a summation of the following form

$$\sum_{k=0}^{\infty} f(k). \tag{1.7}$$

Suppose that $f(k)$ is a weighted count of a set A_k, that is,

$$f(k) = \sum_{\alpha\in A_k} w(\alpha).$$

We wish to find sets B_k and H_k ($k \geq 0$) with a weight assignment w such that there exists a weight preserving bijection

$$\phi_k \colon A_k \cup H_k \to B_k \cup H_{k+1}, \tag{1.8}$$

where $\cup$ means disjoint union. Let $g(k)$ and $h(k)$ be the weighted count of the sets B_k and H_k, respectively, namely,

$$g(k) = \sum_{\alpha\in B_k} w(\alpha) \quad \text{and} \quad h(k) = \sum_{\alpha\in H_k} w(\alpha),$$

then the bijection (1.8) implies the relation

$$f(k) + h(k) = g(k) + h(k+1). \tag{1.9}$$

Just like the conditions for creative telescoping [6,9,12], we suppose that $H_0 = \emptyset$ and H_k vanishes for sufficiently large k. Summing (1.9) over k gives the following relation

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} g(k), \tag{1.10}$$

which is equivalent to a recurrence relation of the sum (1.7).

Combining all the bijections ϕ_k in (1.8), we get a correspondence

$$\phi \colon A \cup H \longrightarrow B \cup H, \tag{1.11}$$

given by $\phi(\alpha) = \phi_k(\alpha)$ if $\alpha \in A_k \cup H_k$, where

$$A = \bigcup_{k=0}^{\infty} A_k, \quad B = \bigcup_{k=0}^{\infty} B_k, \quad H = \bigcup_{k=0}^{\infty} H_k.$$

By the method of cancelation, see Feldman and Propp [4], the above bijection ϕ implies a bijection

$$\psi \colon A \longrightarrow B.$$

More specifically, we can define the bijection $\psi \colon A \to B$ by setting $\psi(a)$ to be the first element b that falls into B while iterating the action of ϕ on $a \in A$.

For the purpose of this paper, we shall express A_k as a sum over n, namely,

$$A_k = \bigcup_{n=0}^{\infty} A_{n,k}.$$

It should be noted that our bijections do not require an explicit formula for $A_{n,k}$. Roughly speaking, our idea is to use bijections to establish a telescoping relation involving $A_{n,k}$ possibly with coefficients depending only on n.

For any n and k, we aim to find bijections $\phi_{n,k}$ for a given integer r:

$$\phi_{n,k} \colon \bigcup_{i=0}^{r} \{a_i(n)\} \times A_{n-i,k} \cup H_{n,k} \to \bigcup_{i=0}^{r} \{b_i(n)\} \times A_{n-i,k} \cup H_{n,k+1}, \tag{1.12}$$

where the leading coefficients of $a_i(n)$ and $b_i(n)$ are positive, and $\{0\} \times A_{n-i,k}$ is considered as the empty set. Let

$$F_{n,k} = \sum_{\alpha \in A_{n,k}} w(\alpha)$$

be a weighted count of the set $A_{n,k}$, and let

$$F_n = \sum_{k=0}^{\infty} F_{n,k}.$$

Indeed, the motivation to find the bijections given in (1.12) is to obtain a recurrence relation of F_n. Once the relation (1.3) is established, we immediate get (1.1).

This paper is organized as follows. In Section 2, we illustrate our method of combinatorial telescoping for sums of positive terms by giving a telescoping proof of an identity of MacMahon [8, p.41]. In Section 3, we provide a solution to Problem 12 of Andrews [2] by using the idea of combinatorial telescoping to construct the recurrence relation (1.3) for the following equivalent form of (1.1):

$$\sum_{k=0}^{n} \frac{(q^{n-k+1};q)_{2k}}{(q^2;q^2)_k} q^{\binom{n-k}{2}} = (-1)^n q^{n^2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \tag{1.13}$$

which can be obtained by multiplying both sides of (1.1) by $q^{\binom{n}{2}}$.

2 MacMahon's identity

In this section, we use MacMahon's identity on partitions to illustrate the idea of combinatorial telescoping for sums of positive terms.

Let us recall some notation and definitions in [1]. A *partition* is a nonincreasing finite sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The integers λ_i are called the parts of λ. The sum of parts and the number of parts are denoted by $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ and $\ell(\lambda) = l$, respectively. The special partition with no parts is denoted by $\emptyset$. Denote by D the set of partitions with distinct parts, and denote by E the set of partitions with even parts. We shall use diagrams to represent partitions and use rows to represent parts.

We shall adopt the common notation and terminology on basic hypergeometric series in [5]. The q-shifted factorials and the q-binomial coefficients, or the Gaussian coefficients, are defined by

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

In his classical treatise [7], MacMahon gave combinatorial proof of the following identity, see also Pak [8, p 41]:

$$\sum_{k=-m}^{n} z^k q^{k^2} \begin{bmatrix} m+n \\ m+k \end{bmatrix}_{q^2} = (-q/z, q^2)_m (-zq;q^2)_n. \tag{2.1}$$

It is easily seen that as $m, n \to \infty$, MacMahon's identity reduces to Jacobi's triple product identity [5].

To prove the identity (2.1), we first give a combinatorial telescoping argument for the following recurrence

$$F_{n,m}(q) = (1 + q^{2m-1}/z) F_{n,m-1}(q), \tag{2.2}$$

where $F_{n,m}(q)$ denotes the sum on the left hand side of (2.1). To compute $F_{n,m}(q)$, we still need the initial value $F_{n,0}(q)$.

Again, by combinatorial telescoping we get the following recurrence for $F_{n,0}(q)$:

$$F_{n,0}(q) = (1 + zq^{2n-1}) F_{n-1,0}(q). \tag{2.3}$$

Now we construct bijections for the recurrence relation (2.2). For a positive integer k, we denote the square partition with k rows by S_k, namely, the partition with k occurrences of the part k. For $k = 0$, S_k is considered as the empty partition. Moreover, we define S_{-k} to be the square partition with k rows associated with a minus sign. We call S_k a positive square partition, and call S_{-k} a negative square partition.

To give a combinatorial interpretation of the left hand side of (2.1), for $-m \leq k \leq n$, we define the following set of pairs of partitions

$$P_{n,m,k} = \{(\lambda, \mu)\colon \lambda = S_k,\ \mu_1 \leq 2m + 2k,\ \ell(\mu) \leq n - k,\ \mu \in E\}\,.$$

In other words, $\lambda = S_k$ is a square partition, μ is a partition with at most $n-k$ even parts but no odd parts such that the largest part does not exceed $2m+2k$. It can be easily verified that the k-th summand of the left hand side of (2.1) can be viewed as a weighted count of $P_{n,m,k}$, that is,

$$\sum_{(\lambda,\mu)\in P_{n,m,k}} z^k q^{\,|\lambda|+|\mu|} = z^k q^{k^2} \begin{bmatrix} m+n \\ m+k \end{bmatrix}_{q^2}.$$

Let

$$G_{n,m,k} = \{(\lambda, \mu) \in P_{n,m,k}\colon \mu_1 = 2m + 2k\}.$$

By definition, $G_{n,m,k} = \emptyset$ for $k < -m$ or $k \geq n$. For integers $m, n \geq 0$ and $-m \leq k \leq n$, we shall construct a bijection

$$\phi_{n,m,k}\colon P_{n,m,k} \cup G_{n,m,k-1} \longrightarrow P_{n,m-1,k} \cup \{2m-1\} \times P_{n,m-1,k} \cup G_{n,m,k}.$$

This bijection can be easily deduced from the following classification of

$$P_{n,m,k} \cup G_{n,m,k-1}.$$

Let (λ, μ) be a pair of partitions in $P_{n,m,k} \cup G_{n,m,k-1}$.

1. For $(\lambda, \mu) \in P_{n,m,k}$, if $\mu_1 = 2m + 2k$, then $(\lambda, \mu) \in G_{n,m,k}$. We set $\phi_{n,m,k}(\lambda, \mu) = (\lambda, \mu)$.

2. For $(\lambda, \mu) \in P_{n,m,k}$, if $\mu_1 < 2m + 2k$, we have $\mu_1 \leq 2m + 2k - 2$, which implies that $(\lambda, \mu) \in P_{n,m-1,k}$. We set $\phi_{n,m,k}(\lambda, \mu) = (\lambda, \mu)$.

3. For $(\lambda, \mu) \in G_{n,m,k-1}$, λ is the square partition S_{k-1}, we set $\lambda' = S_k$. Removing the first row of μ, we obtain μ'. It is easy to check that the resulting pair of partitions (λ', μ') belongs to $P_{n,m-1,k}$. Set $\phi_{n,m,k}(\lambda, \mu) = (2m - 1, (\lambda', \mu'))$.

Define the weight function w on $P_{n,m,k}$ and $(2m - 1) \times P_{n,m-1,k}$ as follows

$$\begin{aligned} w(\lambda, \mu) &= z^k q^{\,|\lambda|+|\mu|}, \\ w(2m-1, (\lambda, \mu)) &= \frac{q^{2m-1}}{z} z^k q^{\,|\lambda|+|\mu|}. \end{aligned}$$

It can be verified that $\phi_{n,m,k}$ is a weight preserving bijection. This yields recurrence relation (2.2).

We now turn to the evaluation of the initial value $F_{n,0}(q)$. To prove the identity

$$\sum_{k=0}^{n} z^k q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = (-zq; q^2)_n, \tag{2.4}$$

we consider the set of pairs of partitions

$$Q_{n,k} = \{(\lambda,\mu)\colon \lambda = S_k,\ \ell(\mu) \le k,\ \mu_1 \le 2n-2k,\ \mu \in E\}.$$

Notice that the k-th summand of the left hand side of (2.4) can be viewed as a weighted count of $Q_{n,k}$, that is,

$$\sum_{(\lambda,\mu)\in Q_{n,k}} z^{\ell(\lambda)} q^{|\lambda|+|\mu|}.$$

Let

$$H_{n,k} = \{(\lambda,\mu) \in Q_{n,k}\colon \mu_1 = 2n-2k\}.$$

By definition, $H_{n,k} = \emptyset$ for $k = 0$ or $k \ge n$. For any integers $n, k \ge 0$, we shall construct a bijection

$$\psi_{n,k}\colon Q_{n,k} \cup H_{n,k+1} \longrightarrow Q_{n-1,k} \cup \{2n-1\} \times Q_{n-1,k} \cup H_{n,k}.$$

This bijection can be easily deduced from the following classification of

$$Q_{n,k} \cup H_{n,k+1}.$$

Let (λ,μ) be a pair of partitions in $Q_{n,k} \cup H_{n,k+1}$.

1. For $(\lambda,\mu) \in Q_{n,k}$, if $\mu_1 = 2n-2k$, then $(\lambda,\mu) \in H_{n,k}$. We set $\psi_{n,k}(\lambda,\mu) = (\lambda,\mu)$.

2. For $(\lambda,\mu) \in Q_{n,k}$, if $\mu_1 < 2n-2k$, we have $\mu_1 \le 2n-2k-2$, which implies that $(\lambda,\mu) \in Q_{n-1,k}$. We set $\psi_{n,k}(\lambda,\mu) = (\lambda,\mu)$.

3. For $(\lambda,\mu) \in H_{n,k+1}$, λ is the square partition S_{k+1}, we set $\lambda' = S_k$. Removing the first row from μ, we obtain μ'. Clearly, resulting pair of partitions (λ',μ') belongs to $Q_{n-1,k}$. Set $\psi_{n,k}(\lambda,\mu) = (2n-1,(\lambda',\mu'))$.

Define the weight function w on $Q_{n,k}$ and $(2n-1) \times Q_{n-1,k}$ as follows

$$\begin{aligned} w(\lambda,\mu) &= z^{\ell(\lambda)} q^{|\lambda|+|\mu|}, \\ w(2n-1,(\lambda,\mu)) &= zq^{2n-1} z^{\ell(\lambda)} q^{|\lambda|+|\mu|}. \end{aligned}$$

One sees that $\psi_{n,k}$ is a weight preserving bijection. So we get the recurrence relation (2.3)

$$F_{n,0}(q) = (1 + zq^{2n-1}) F_{n-1,0}(q),$$

where $F_{n,0}(q)$ denotes the sum on the left hand side of (2.4), with the initial value $F_{0,0}(q) = 1$. Since $F_{0,0} = 1$, combining the recurrence relations (2.2) and (2.3), we arrive at MacMahon's identity (2.1).

3 An Open Problem of Andrews

In this section, we provide a solution to Problem 12 of Andrews [2] by using the idea of combinatorial telescoping. Define

$$P_{n,k} = \left\{ (\tau,\lambda,\mu) \;\middle|\; \begin{array}{l} \tau = (n-k-1, n-k-2, \ldots, 2, 1, 0), \\ n-k+1 \le \lambda_i \le n+k,\ (i = 1,2,\ldots,\ell(\lambda)),\ \lambda \in D, \\ \mu_1 \le 2k,\ \mu \in E. \end{array} \right\}.$$

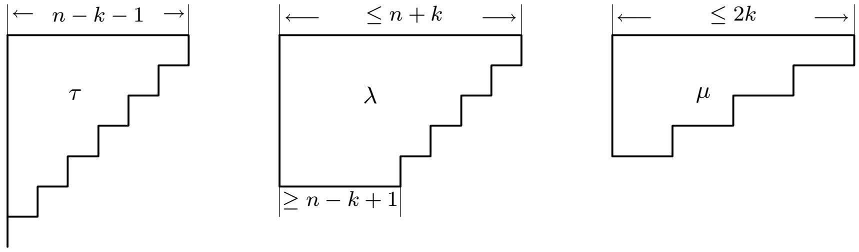

Figure 3.1: The diagram $(\tau, \lambda, \mu) \in P_{n,k}$.

Figure 3.1 gives an illustration of an element of $P_{n,k}$.

In other words, τ is a triangular partition containing a zero part, λ is a partition with distinct parts, each part of λ is smaller than $n+k$ and greater than $n-k+1$, μ is a partition with each part even and with the largest part not exceeding $2k$. As will be seen, we have a reason to include the zero in a triangular partition. For $k=0$, we have $P_{n,0} = \{(\tau, \emptyset, \emptyset)\}$, where $\tau = (n-1, n-2, \ldots, 2, 1, 0)$, and for $k > n$, we set $P_{n,k} = \emptyset$. For $k = n-1$ and $k = n$, we have

$$\begin{aligned} P_{n,n-1} &= \{(\tau, \lambda, \mu) \colon \tau = (0),\ 2 \leq \lambda_i \leq 2n-1,\ \lambda \in D,\ \mu_1 \leq 2n-2,\ \mu \in E\}, \\ P_{n,n} &= \{(\tau, \lambda, \mu) \colon \tau = \emptyset,\ 1 \leq \lambda_i \leq 2n,\ \lambda \in D,\ \mu_1 \leq 2n,\ \mu \in E\}. \end{aligned}$$

Notice that we have imposed the distinction between the partition of zero and the empty partition. Under this convention, one sees that $\bigcup_{k\geq 0} P_{n,k}$ is a disjoint union of $P_{n,k}$. Moreover, the k-th summand $F_{n,k}$ of the left hand side of (1.13) can be viewed as a weighted count of $P_{n,k}$, that is,

$$F_{n,k} = \sum_{(\tau,\lambda,\mu)\in P_{n,k}} (-1)^{\ell(\lambda)} q^{\,|\tau|+|\lambda|+|\mu|}\,.$$

Notice that the summand term $F_{n,k}$ does not contain the factor $(-1)^k$ as in an alternating sum. So the summation (1.13) should be viewed as a sum of positive terms.

Now we give a combinatorial telescoping relation for $P_{n,k}$.

Theorem 3.1 *For any nonnegative integer n and $0 \leq k \leq n-2$, there is a bijection*

$$\phi_{n,k} \colon P_{n,k} \cup \{2n-1\} \times P_{n-1,k-1} \to P_{n-1,k-1} \cup \{2n-3\} \times P_{n-2,k}. \qquad (3.1)$$

Proof. For $k=0$, as $P_{n-1,k-1}$ is the empty set, the bijection $\phi_{n,0}$ is defined by

$$\phi_{n,0} \colon (\tau, \emptyset, \emptyset) \mapsto (2n-3, (\tau', \emptyset, \emptyset)),$$

where τ' is obtained from τ by removing the first two parts. For example, when $n=2$, $\tau = (1,0)$ and the triple of partitions is mapped to $(1, (\emptyset, \emptyset, \emptyset))$ belonging to the set $\{2n-3\} \times P_{n-2,k}$. Because of the zero part, it is always possible to remove two parts of τ.

For positive integer k, the bijection $\phi_{n,k}$ is essentially a classification of the set $P_{n,k}$ into four classes, namely,

$$P_{n,k} = A_{n,k} \cup B_{n,k} \cup C_{n,k} \cup P_{n-1,k-1},$$

where

$$A_{n,k} = \{(\tau, \lambda, \mu) \in P_{n,k} \colon \lambda_1 \le n + k - 2,\ \mu_1 = 2k\},$$
$$B_{n,k} = \{(\tau, \lambda, \mu) \in P_{n,k} \colon \text{either } n + k \text{ or } n + k - 1 \text{ appears in } \lambda, \text{ but not both}\},$$
$$C_{n,k} = \{(\tau, \lambda, \mu) \in P_{n,k} \colon \lambda_1 = n + k,\ \lambda_2 = n + k - 1\}.$$

We also need the following classification

$$P_{n-2,k} = A'_{n,k} \cup B'_{n,k} \cup C'_{n,k} \cup D_{n,k},$$

where

$$A'_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon \lambda_\ell \ge n - k + 1\},$$
$$B'_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon \text{either } n - k \text{ or } n - k - 1 \text{ appears in } \lambda, \text{ but not both}\},$$
$$C'_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon \lambda_\ell = n - k - 1,\ \lambda_{\ell-1} = n - k,\ \mu_1 = 2k\},$$
$$D_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon \lambda_\ell = n - k - 1,\ \lambda_{\ell-1} = n - k,\ \mu_1 < 2k\}.$$

Now we are ready to describe the bijection $\phi_{n,k}$. Assume that (τ, λ, μ) is a triple of partitions in $P_{n,k}$.

Case 1: $(\tau, \lambda, \mu) \in P_{n-1,k-1}$. Set $\phi_{n,k}(\tau, \lambda, \mu)$ to be (τ, λ, μ) itself.

Case 2: $(\tau, \lambda, \mu) \in A_{n,k}$. Removing the first two rows from τ and removing the first row from μ, we get τ' and μ', respectively. Let $\lambda' = \lambda$. Then we have $(\tau', \lambda', \mu') \in A'_{n,k}$ and

$$|\tau| + |\lambda| + |\mu| = 2n - 3 + |\tau'| + |\lambda'| + |\mu'|.$$

So we obtain a bijection $\varphi_A \colon A_{n,k} \to \{2n - 3\} \times A'_{n,k}$ as given by $(\tau, \lambda, \mu) \mapsto (2n - 3, (\tau', \lambda', \mu'))$. Figure 3.2 gives an illustration of the correspondence.

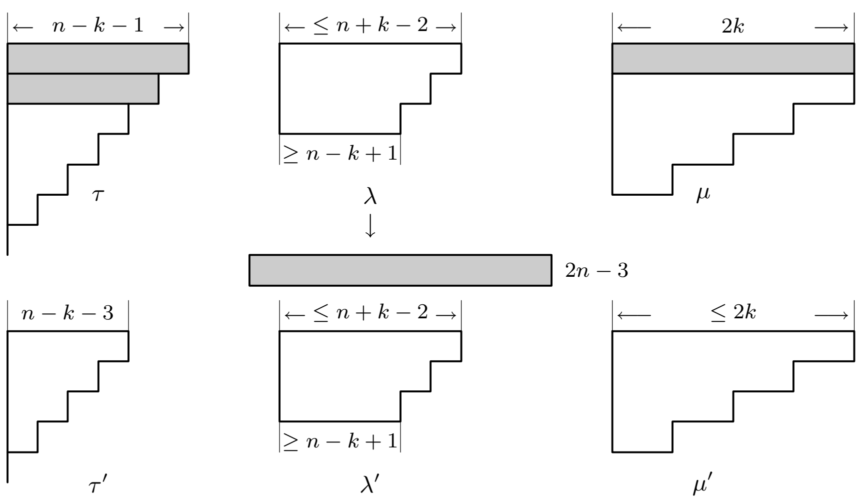

Figure 3.2: The bijection φ_A in Case 2.

Case 3: $(\tau, \lambda, \mu) \in B_{n,k}$. Removing the first two rows from τ, we get τ'. Subtracting $2k$ from the part λ_1 in λ, we get a partition λ'. Let $\mu' = \mu$. Then we have $(\tau', \lambda', \mu') \in B'_{n,k}$ and

$$|\tau| + |\lambda| + |\mu| = 2n - 3 + |\tau'| + |\lambda'| + |\mu'|.$$

Thus we obtain a bijection $\varphi_B \colon B_{n,k} \to \{2n-3\} \times B'_{n,k}$ defined by $(\tau, \lambda, \mu) \mapsto (2n-3, (\tau', \lambda', \mu'))$. See Figure 3.3 for an illustration.

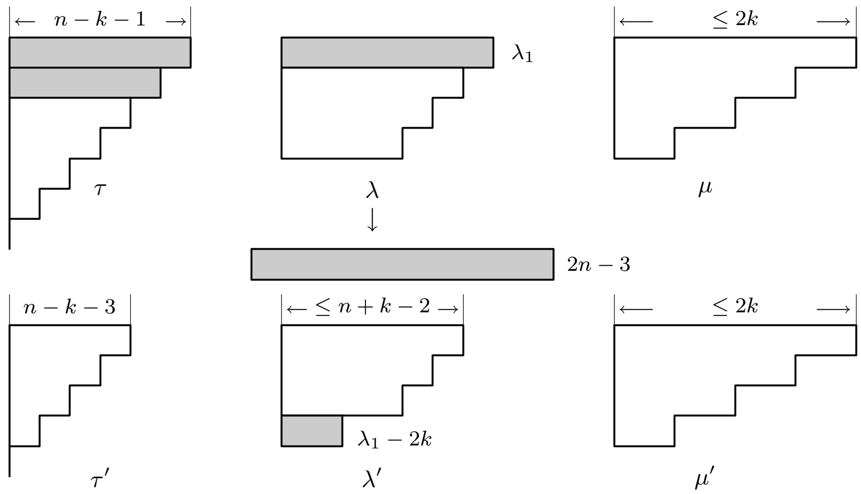

Figure 3.3: The bijection φ_B in Case 3.

Case 4: $(\tau, \lambda, \mu) \in C_{n,k}$. Removing first two rows from τ we get τ'. Subtracting $2k$ from the parts $n+k-1$ and $n+k$ in λ, we get a partition λ'. Adding $2k$ to μ as a new part, we get μ'. Then we have $(\tau', \lambda', \mu') \in C'_{n,k}$ and

$$|\tau| + |\lambda| + |\mu| = 2n - 3 + |\tau'| + |\lambda'| + |\mu'|.$$

Thus we obtain a bijection $\varphi_C \colon C_{n,k} \to \{2n-3\} \times C'_{n,k}$ as given by $(\tau, \lambda, \mu) \mapsto (2n-3, (\tau', \lambda', \mu'))$. This case is illustrated in Figure 3.4.

Now we consider the quadruples

$$(2n-1, (\tau, \lambda, \mu)) \in \{2n-1\} \times P_{n-1,k-1}.$$

For any $(\tau, \lambda, \mu) \in P_{n-1,k-1}$, remove the first two rows of τ and add two parts $n-k$ and $n-k-1$ to λ to get τ' and λ'. Let $\mu' = \mu$. Then we see that $(\tau', \lambda', \mu') \in D_{n,k}$ and

$$2n - 1 + |\tau| + |\lambda| + |\mu| = 2n - 3 + |\tau'| + |\lambda'| + |\mu'|.$$

Thus we obtain a bijection

$$\varphi_D \colon \{2n-1\} \times P_{n-1,k-1} \to \{2n-3\} \times D_{n,k}$$

as given by $(2n-1, (\tau, \lambda, \mu)) \mapsto (2n-3, (\tau', \lambda', \mu'))$. This case is illustrated by Figure 3.5.

The proof is complete by combining the bijections φ_A, φ_B, φ_C and φ_D. ∎

The above theorem gives the bijections $\phi_{n,k}$ for $0 \leq k \leq n-2$. In the following theorem we consider the special cases $k = n-1$ and $k = n$.

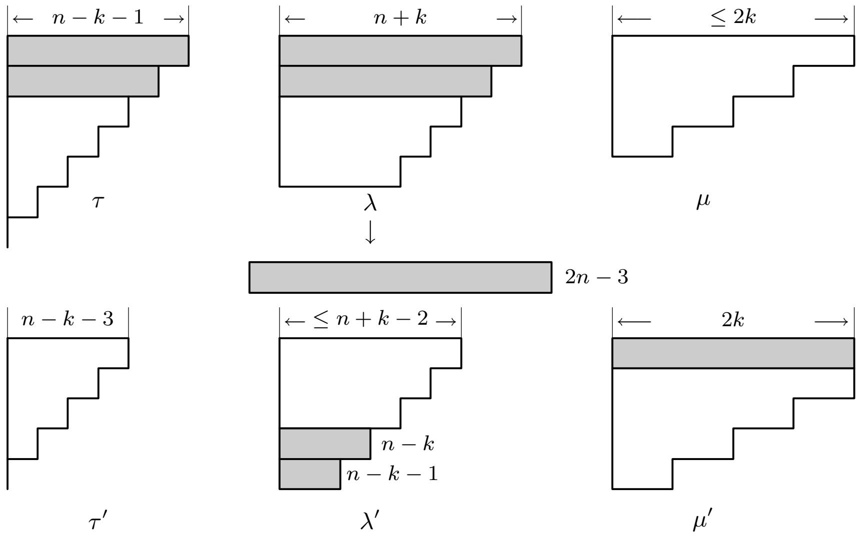

Figure 3.4: The bijection φ_C in Case 4.

Theorem 3.2 *For $n \geq 2$ and for $k = n-1$ or n, there is an involution*

$$I_{n,k}\colon P_{n,k} \cup \{2n-1\} \times P_{n-1,k-1} \to P_{n-1,k-1}. \tag{3.2}$$

Proof. We only give the description of the involution $I_{n,n}$ since $I_{n,n-1}$ can be constructed in the same manner.

Case 1. For $(\emptyset, \lambda, \mu) \in P_{n,n}$, if the first part of λ is $2n$, then move it to μ. Conversely, if μ contains a part $2n$ but λ does not, then move this part from μ back to λ.

Case 2. For $(\emptyset, \lambda, \mu) \in P_{n,n}$ with $\lambda_1 = 2n-1$ and $\mu_1 < 2n$, remove the first part $2n-1$ of λ to get λ', and set

$$I_{n,n}(\emptyset, \lambda, \mu) = (2n-1, (\emptyset, \lambda', \mu),$$

which belongs to $\{2n-1\} \times P_{n-1,n-1}$. Conversely, for

$$(2n-1, (\emptyset, \lambda, \mu)) \in \{2n-1\} \times P_{n-1,n-1},$$

add a part $2n-1$ to λ, we get λ' and set

$$I_{n,n}(2n-1, (\emptyset, \lambda, \mu)) = (\emptyset, \lambda', \mu),$$

which belongs to $P_{n,n}$.

Case 3. It can be seen that the set of triples $(\emptyset, \lambda, \mu) \in P_{n,n}$ with $\lambda_1 < 2n-1$ and $\mu_1 < 2n$ is exactly $P_{n-1,n-1}$. So we set $P_{n-1,n-1}$ to be the invariant set of the involution.

Thus we obtain an involution on $P_{n,n} \cup \{2n-1\} \times P_{n-1,n-1}$ with the invariant set $P_{n-1,n-1}$. ∎

Define a weight function w on $P_{n,k}$, $\{2n-1\} \times P_{n-1,k}$ and $\{2n-3\} \times P_{n-2,k}$ as given by

$$\begin{aligned} w(\tau, \lambda, \mu) &= (-1)^{\ell(\lambda)}\, q^{|\tau|+|\lambda|+|\mu|}, \\ w(2n-1, (\tau, \lambda, \mu)) &= q^{2n-1}\, (-1)^{\ell(\lambda)}\, q^{|\tau|+|\lambda|+|\mu|}, \\ w(2n-3, (\tau, \lambda, \mu)) &= q^{2n-3}\, (-1)^{\ell(\lambda)}\, q^{|\tau|+|\lambda|+|\mu|}. \end{aligned}$$

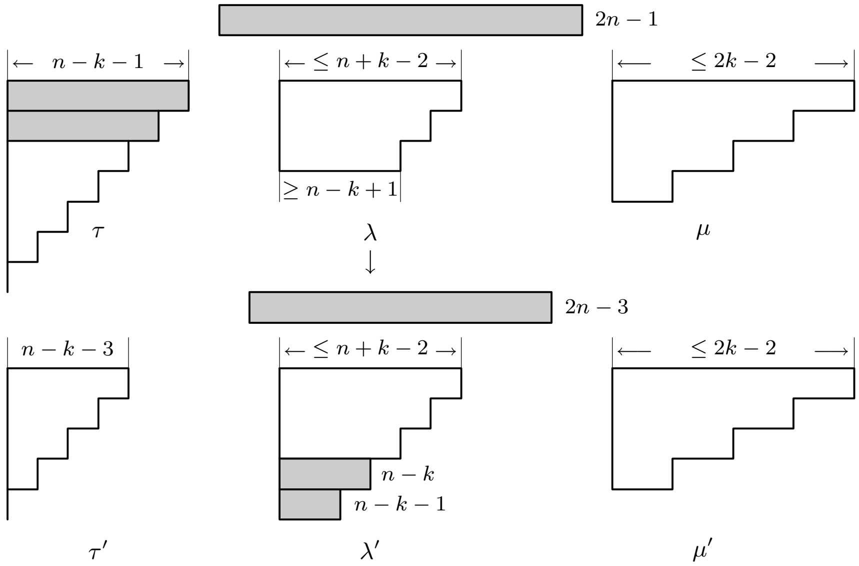

Figure 3.5: The bijection φ_D on $\{2n-1\} \times P_{n-1,k-1}$.

One sees that the bijections and involutions in Theorems 3.1 and 3.2 are weight preserving. Hence we get the following recurrence relation for

$$F_n(q) = \sum_{k \geq 0} F_{n,k}.$$

Corollary 3.3 *For $n \geq 2$, we have*

$$F_n(q) + (q^{2n-1} - 1)F_{n-1}(q) - q^{2n-3}F_{n-2}(q) = 0. \tag{3.3}$$

It is easy to verify that the right hand side of (1.13), namely, the sum

$$(-1)^n q^{n^2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \tag{3.4}$$

also satisfies the recurrence relation (3.3). Taking the initial values into consideration, we are led to the identity of Andrews.

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