The Method of Combinatorial Telescoping

William Y.C. Chen[1], Qing-Hu Hou[2], and Lisa H. Sun[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

[1]chen@nankai.edu.cn, [2]hou@nankai.edu.cn, [3]sunhui@nankai.edu.cn

Abstract. We present a method for proving q-series identities by combinatorial telescoping in the sense that one can transform a bijection or classification of combinatorial objects into a telescoping relation. We shall demonstrate this idea by giving a combinatorial reasoning of Watson's identity which implies the Rogers-Ramanujan identities.

Keywords. Watson's identity, Schur's identity, Rogers-Ramanujan identities, combinatorial telescoping

AMS Subject Classification. 05A17; 11P83

1 Introduction

The main objective of this paper is to present the method of combinatorial telescoping for proving q-series identities. The benchmark of this method is the classical identity of Watson which implies Rogers-Ramanujan identities.

There have been many combinatorial proofs of the Rogers-Ramanujan identities. Schur [9] provided an involution for the following identity which is equivalent to the first Rogers-Ramanujan identity:

$$\prod_{k=1}^{\infty}(1-q^k)\left(1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}\right)=\sum_{k=-\infty}^{\infty}(-1)^k q^{k(5k-1)/2}.$$

Andrews [1] proved the Rogers-Ramanujan identities by introducing the notion of k-partitions. Garsia and Milne [7] gave a bijection by using the involution principle. Bressoud and Zeilberger [4, 5] provided a different involution principle proof based on an algebraic proof by Bressoud [3]. Boulet and Pak [2] found a combinatorial proof which relies on the symmetry properties of a generalization of Dyson's rank.

Let us consider a summation of the form

$$\sum_{k=0}^{\infty}(-1)^k f(k).$$

Suppose that for each k,

$$f(k) = \sum_{\alpha \in A_k} w(\alpha)$$

is the weighted count, or the weight, of a set A_k. Inspired by the idea of the creative telescoping of Zeilberger [12], we aim to find sets B_k and G_k such that there is a bijection

$$\phi_k \colon A_k \longrightarrow B_k \cup G_k \cup G_{k+1}, \tag{1.1}$$

which is weight preserving on $\phi_k^{-1}(G_k \cup G_{k+1})$. Throughout this paper, $\cup$ stands for the disjoint union. Since the weights of $\phi_k^{-1}(G_{k+1})$ and $\phi_{k+1}^{-1}(G_{k+1})$ are both equal to the weight of G_{k+1}, we obtain a telescoping relation. Suppose that $G_0 = \emptyset$ and G_k vanishes for sufficient large k. Let

$$A = \bigcup_{k=0}^{\infty} A_k, \quad \text{and} \quad B = \bigcup_{k=0}^{\infty} B_k.$$

Then the bijections $\{\phi_k\}$ altogether lead to a bijection between A and B after certain cancelations. More precisely, we have a bijection

$$\phi \colon A \setminus \bigcup_{k=0}^{\infty} \phi_k^{-1}(G_k \cup G_{k+1}) \longrightarrow B$$

and an involution

$$\psi \colon \bigcup_{k=0}^{\infty} \phi_k^{-1}(G_k \cup G_{k+1}) \longrightarrow \bigcup_{k=0}^{\infty} \phi_k^{-1}(G_k \cup G_{k+1}),$$

given by $\phi(\alpha) = \phi_k(\alpha)$ if $\alpha \in A_k$ and

$$\psi(\alpha) = \begin{cases} \phi_{k-1}^{-1}\phi_k(\alpha), & \text{if } \alpha \in \phi_k^{-1}(G_k), \\ \phi_{k+1}^{-1}\phi_k(\alpha), & \text{if } \alpha \in \phi_k^{-1}(G_{k+1}). \end{cases}$$

We call the bijections $\{\phi_k\}$ a *combinatorial telescoping* for A. Once the combinatorial telescoping is established, we can deduce that

$$\sum_{k=0}^{\infty} (-1)^k f(k) = \sum_{k=0}^{\infty} (-1)^k \sum_{\beta \in B_k} w(\phi^{-1}(\beta)).$$

In Section 2, we apply this method to prove the following formulation of Watson's identity [11] (see also [8, Section 2.7])

$$\sum_{k=0}^{\infty} (-1)^k \frac{1 - aq^{2k}}{(q;q)_k (aq^k;q)_\infty} a^{2k} q^{k(5k-1)/2} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n}, \tag{1.2}$$

where

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), \quad \text{and} \quad (a;q)_\infty = \prod_{i=0}^{\infty} (1-aq^i).$$

When $a=1$, Watson's identity becomes Schur's identity [2]

$$\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^k q^{k(5k-1)/2}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}.$$

Applying Jacobi's triple product identity to the left hand side, we arrive at the first Rogers-Ramanujan identity.

The idea of the combinatorial telescoping for Waston's identity can be described as follows. Assume that the k-th summand (without the sign) of the left hand side of (1.2) is the weight of a set P_k consisting of certain combinatorial objects. We further divide P_k into a disjoint union of subsets $P_{n,k}, n=0,1,\ldots$ by considering the expansion of the summand in the parameter a. For each positive integer n, we can construct a combinatorial telescoping

$$\phi_{n,k}\colon P_{n,k}\to P_{n,k}\cup P_{n-1,k}\cup G_{n,k}\cup G_{n,k+1}. \tag{1.3}$$

The corresponding bijection ϕ_n leads to a recursion on

$$F_n(a,q)=\sum_{k=0}^{\infty}(-1)^k\sum_{\alpha\in P_{n,k}}w(\alpha)$$

as follows

$$F_n(a,q)=q^nF_n(a,q)+aq^{2n-1}F_{n-1}(a,q),\qquad n\geq 1.$$

By iteration of the above relation, we obtain that $F_n(a,q)=a^nq^{n^2}/(q;q)_n$ and hence (1.2) holds.

As another example, we consider Sylvester's identity [10]

$$\sum_{k=0}^{\infty}(-1)^k q^{k(3k+1)/2}x^k\frac{1-xq^{2k+1}}{(q;q)_k(xq^{k+1};q)_\infty}=1, \tag{1.4}$$

which has been investigated by Andrews [1]. It turns out that the combinatorial telescoping for Sylvester's identity is easy to find. It is our belief that combinatorial telescoping is a general phenomenon for q-series identities.

2 The combinatorial telescoping for Watson's identity

In this section, we shall use Watson's identity as a benchmark to illustrate the idea of combinatorial telescoping. In principle, we can translate the combinatorial telescoping into a bijection. However, this aspect will not be emphasized. Notice our telescoping approach can be considered purely combinatorial since no manipulations on q-series is really involved.

A *partition* is a non-increasing finite sequence of positive integers $\lambda=(\lambda_1,\ldots,\lambda_\ell)$. The integers λ_i are called the *parts* of λ. The sum of parts and the number of parts are denoted by $|\lambda|=\lambda_1+\cdots+\lambda_\ell$ and $\ell(\lambda)=\ell$, respectively. For a specific part, we call it a λ_i-part. The number of k-parts is denoted by $\ell_k(\lambda)$. The special partition with no parts is denoted by $\varnothing$ and we define $\ell_0(\varnothing)=+\infty$. We shall use diagrams to represent partitions with columns representing parts.

Let

$$P_k = \{(\tau, \lambda, \mu) \,|\, \tau = (k^{2k}, k-1, \ldots, 2, 1),\ \lambda_{\ell(\lambda)} \geq k,\ \lambda_i \neq 2k,\ \mu_1 \leq k\}, \tag{2.5}$$

where k^{2k} denotes $2k$ occurrences of the part k. In other words, τ is a trapezoid partition with $|\tau| = k(5k-1)/2$, λ is a partition with parts not less than k and not equal to $2k$, μ is a partition with parts not more than k. It is clear that the k-th summand of the left hand side of (1.2) without the sign can be viewed as the weight of P_k, that is,

$$\sum_{(\tau, \lambda, \mu) \in P_k} a^{\ell(\lambda)+2k} q^{|\tau|+|\lambda|+|\mu|}.$$

According to the exponent of a in the expansion, we divide P_k into a disjoint union of subsets

$$P_{n,k} = \{(\tau, \lambda, \mu) \in P_k \,|\, \ell(\lambda) = n - 2k\}. \tag{2.6}$$

The elements in $P_{n,k}$ are illustrated by Figure 1.

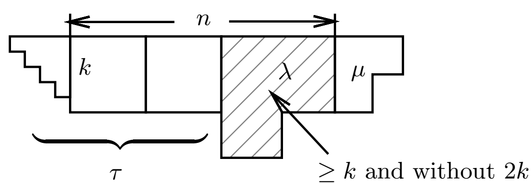

Figure 1: The diagram $(\tau, \lambda, \mu) \in P_{n,k}$

We have the following combinatorial telescoping relation for $P_{n,k}$.

Theorem 2.1 *Let*

$$G_{n,k} = \{(\tau, \lambda, \mu) \in P_{n,k} \,|\, \ell_k(\lambda) \geq \ell_k(\mu) - 1\}. \tag{2.7}$$

Then for any positive integer n, there is a combinatorial telescoping for $P_{n,k}$,

$$\phi_{n,k} \colon P_{n,k} \longrightarrow P_{n,k} \cup P_{n-1,k} \cup G_{n,k} \cup G_{n,k+1}. \tag{2.8}$$

Proof. Let $(\tau, \lambda, \mu) \in P_{n,k}$. The bijection is essentially a classification of $P_{n,k}$ according to four cases.

Case 1. $\ell_k(\lambda) \geq \ell_k(\mu) - 1$. Then $(\tau, \lambda, \mu) \in G_{n,k}$. The image of (τ, λ, μ) remains unchanged.

Case 2. $\ell_k(\lambda) < \ell_k(\mu) - 1$ and $\ell_{2k+1}(\lambda) = 0$. Denote the set of all such elements by $U_{n,k}$. Since $\ell_k(\mu) \geq \ell_k(\lambda) + 2$, we can remove $(\ell_k(\lambda) + 2)$ k-parts from μ to obtain a partition μ'. In the meantime, we change each k-part of λ into a $2k$-part in order to obtain a partition λ' whose minimal part is strictly greater than k.

Next, we decrease each part of λ' by one in order to produce a partition λ'' whose minimal part is greater than or equal to k. Since λ contains no parts equal to $2k+1$, we see that

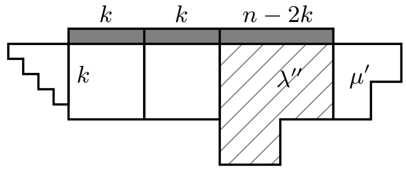

Figure 2: The resulting partition under the bijection φ_1.

λ'' contains no parts equal to $2k$. So we obtain a bijection $\varphi_1 \colon U_{n,k} \to P_{n,k}$ defined by $(\tau, \lambda, \mu) \mapsto (\tau, \lambda'', \mu')$. This case is illustrated by Figure 2.

Case 3. $\ell_k(\lambda) < \ell_k(\mu) - 1$, $\ell_{2k+1}(\lambda) > 0$ and $\ell_{k+1}(\lambda) + \ell_{2k+2}(\lambda) = 0$. Denote the set of all such elements by $V_{n,k}$. Let λ', μ' be given as in Case 2. We can remove one $(2k+1)$-part from λ' and decrease each of the rest parts by two in order to obtain λ''. This leads to a bijection $\varphi_2 \colon V_{n,k} \to P_{n-1,k}$ as given by $(\tau, \lambda, \mu) \mapsto (\tau, \lambda'', \mu')$. See Figure 3 for an illustration.

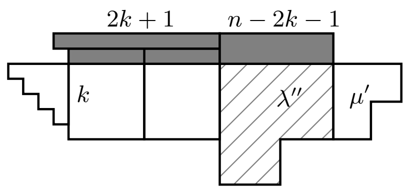

Figure 3: The resulting partition under the bijection φ_2.

Case 4. $\ell_k(\lambda) < \ell_k(\mu) - 1$, $\ell_{2k+1}(\lambda) > 0$ and $\ell_{k+1}(\lambda) + \ell_{2k+2}(\lambda) > 0$. Denote the set of all such triples of partitions by $W_{n,k}$. Let λ', μ' be given as in Case 2. We can change each $(2k+2)$-part of λ' into a $(k+1)$-part and add $\ell_{2k+2}(\lambda')$ $(k+1)$-parts to μ'. Denote the resulting partitions by λ'' and μ''. Then we have

$$\ell_{k+1}(\lambda'') = \ell_{k+1}(\lambda) + \ell_{(2k+2)}(\lambda) > 0, \quad \ell_{k+1}(\mu'') = \ell_{2k+2}(\lambda). \tag{2.9}$$

Remove one $(k+1)$-part and one $(2k+1)$-part from λ'' to obtain λ'''. By (2.9), we find

$$\ell_{k+1}(\lambda''') = \ell_{k+1}(\lambda'') - 1 \geq \ell_{k+1}(\mu'') - 1.$$

Moreover, it is clear that $|\lambda| + |\mu| = 2k + (k+1) + (2k+1) + |\lambda'''| + |\mu''|$. Let τ' be the trapezoid partition of size $k+1$. So we obtain a bijection $\varphi_3 \colon W_{n,k} \to G_{n,k+1}$ defied by $(\tau, \lambda, \mu) \mapsto (\tau', \lambda''', \mu'')$. This case is illustrated by Figure 4. ∎

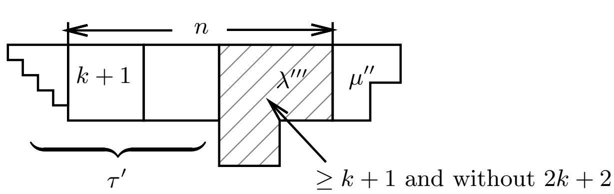

Figure 4: The resulting partition under the bijection φ_3.

Observe that the bijection φ_1 decreases $|\tau|+|\lambda|+|\mu|$ by n and φ_2 decreases $|\tau|+|\lambda|+|\mu|$ by $2n-1$. The above combinatorial telescoping immediately leads to a recurrence relation.

Corollary 2.2 *Let*

$$F_n(a,q)=\sum_{k=0}^{\infty}(-1)^k\sum_{(\tau,\lambda,\mu)\in P_{n,k}}a^nq^{|\tau|+|\lambda|+|\mu|}. \tag{2.10}$$

Then we have

$$F_n(a,q)=q^nF_n(a,q)+aq^{2n-1}F_{n-1}(a,q),\qquad n\geq 1. \tag{2.11}$$

Since $F_0(a,q)=1$, by iteration we deduce that

$$F_n(a,q)=\frac{aq^{2n-1}}{1-q^n}F_{n-1}(a,q)=\frac{a^2q^{4n-4}}{(1-q^n)(1-q^{n-1})}F_{n-2}(a,q)=\cdots=\frac{a^nq^{n^2}}{(q;q)_n}.$$

Summing over n, we arrive at Watson's identity (1.2).

As is well-known, taking $a=1$ and $a=q$ in Watson's identity and using Jacobi's triple product identity, one obtains the Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}(-q^2)^kq^{5\binom{k}{2}}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}(-q^4)^kq^{5\binom{k}{2}}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

3 The combinatorial telescoping for Sylvester's identity

In this section, we give the combinatorial telescoping for Sylvester's identity. Define

$$Q_{n,k}=\{(\tau,\lambda)\colon \tau=(k^{k+1},k-1,\ldots,2,1),\lambda_i\neq 2k+1,\ell_{>k}(\lambda)=n-k\},$$

where $\ell_{>k}(\lambda)$ denotes the number of parts of λ which are greater than k. It is straightforward to check that $Q_{n,k}$ is the disjoint union of three subsets:

$$\begin{aligned}
G_{n,k}&=\{(\tau,\lambda)\in Q_{n,k}\colon \ell_{k+1}(\lambda)\geq \ell_k(\lambda)\},\\
U_{n,k}&=\{(\tau,\lambda)\in Q_{n,k}\colon \ell_{k+1}(\lambda)<\ell_k(\lambda) \text{ and } \ell_{2k+2}(\lambda)=0\},\\
V_{n,k}&=\{(\tau,\lambda)\in Q_{n,k}\colon \ell_{k+1}(\lambda)<\ell_k(\lambda) \text{ and } \ell_{2k+2}(\lambda)>0\}.
\end{aligned}$$

Here we assume that $\ell_0(\lambda)=+\infty$. By an analogous argument to the proof of Theorem 2.1, we find that $U_{n,k}$ and $V_{n,k}$ are in one to one correspondence with $Q_{n,k}$ and $G_{n,k+1}$, respectively. Thus we have the combinatorial telescoping

$$\phi_{n,k}\colon Q_{n,k}\to Q_{n,k}\cup G_{n,k}\cup G_{n,k+1}.$$

Let

$$I_n(q) = \bigcup_{k=0}^{\infty} (-1)^k \sum_{(\tau,\lambda)\in Q_{n,k}} q^{|\tau|+|\lambda|}.$$

We see that $I_n(q) = q^n I_n(q)$, which implies that $I_n(q) = 0$ for $n \geq 1$. Clearly $I_0(q) = 1$ and hence Sylvester's identity holds.

To conclude this paper, we notice that both Watson's identity and Sylvester's identity can be verified by employing the q-Zeilberger algorithm for infinite q-series developed by Chen, Hou and Mu [6]. Let

$$f(a) = \sum_{k=0}^{\infty} (-1)^k \frac{(1-aq^{2k})}{(q;q)_k (aq^k;q)_\infty} a^{2k} q^{k(5k-1)/2}.$$

The q-Zeilberger algorithm gives that $f(a) = f(aq) + aqf(aq^2)$. It is easily checked that the right hand side of (1.2) satisfies the same recursion. By Theorem 3.1 of [6], one sees that (1.2) holds for arbitrary a provided that it holds for $a = 0$. Similarly, let

$$f(x) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k+1)/2} x^k \frac{1-xq^{2k+1}}{(q;q)_k (xq^{k+1};q)_\infty}.$$

The q-Zeilberger algorithm gives that $f(x) = f(xq)$, implying that $f(x) = 1$.

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