Context-free Grammars for Permutations and Increasing Trees

William Y.C. Chen[1] and Amy M. Fu[2]

[1]Center for Applied Mathematics
Tianjin University
Tianjin 300072, P.R. China

[1,2]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

Email: chenyc@tju.edu.cn, fu@nankai.edu.cn

Abstract. In this paper, we introduce the notion of a grammatical labeling to describe a recursive process of generating combinatorial objects based on a context-free grammar. For example, by labeling the ascents and descents of a Stirling permutation, we obtain a grammar for the second-order Eulerian polynomials. By using the grammar for 0-1-2 increasing trees given by Dumont, we obtain a grammatical derivation of the generating function of the André polynomials obtained by Foata and Schützenberger, without solving a differential equation. We also find a grammar for the number $T(n,k)$ of permutations of $[n]=\{1,2,\ldots,n\}$ with k exterior peaks, which was independently discovered by Ma. We demonstrate that Gessel's formula for the generating function of $T(n,k)$ can be deduced from this grammar. Moreover, by using grammars we show that the number of the permutations of $[n]$ with k exterior peaks equals the number of increasing trees on $[n]$ with $2k+1$ vertices of even degree. A combinatorial proof of this fact is also presented.

Keywords: Context-free grammar, Eulerian grammar, grammatical labeling, increasing tree, exterior peak of a permutation, Stirling permutation

AMS Classification: 05A15, 05A19

1 Introduction

A context-free grammar G over an alphabet A is defined as a set of substitution rules replacing a letter in A by a formal function over A. Chen [2] introduced the notion of the formal derivative of a context-free grammar, and used this approach to derive combinatorial identities including identities on generating functions and the Lagrange inversion formula. The formal derivative with respect to a context-free grammar satisfies the relations just like the derivative,

$$D(u+v)=D(u)+D(v),$$

$$D(uv)=D(u)v+uD(v).$$

So the Leibniz rule is valid,

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

As a consequence, we see that

$$D(w^{-1}) = -w^{-2} D(w),$$

since $D(ww^{-1}) = 0$.

The formal derivatives are also connected with the exponential generating functions. Let

$$\mathrm{Gen}(w,t) = \sum_{n \geq 0} D^n(w) \frac{t^n}{n!}$$

for any formal function w. Then we have the following relations

$$\mathrm{Gen}'(w,t) = \mathrm{Gen}(D(w),t), \tag{1.1}$$

$$\mathrm{Gen}(u+v,t) = \mathrm{Gen}(u,t) + \mathrm{Gen}(v,t), \tag{1.2}$$

$$\mathrm{Gen}(uv,t) = \mathrm{Gen}(u,t)\mathrm{Gen}(v,t), \tag{1.3}$$

where u, v and w are formal functions and $\mathrm{Gen}'(w,t)$ means the derivative of $\mathrm{Gen}(w,t)$ with respect to t.

Dumont [3] introduced the following grammar

$$G: \quad x \rightarrow xy, \quad y \rightarrow xy \tag{1.4}$$

and showed that it generates the Eulerian polynomials $A_n(x)$. For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, the index $i \in [n-1]$ is an ascent of π if $\pi_i < \pi_{i+1}$, a descent if $\pi_i > \pi_{i+1}$. Let $asc(\pi)$ be the number of ascents of π and S_n denote the set of permutation on $[n] = \{1, 2, \ldots, n\}$. The Eulerian polynomial $A_n(x)$ is defined by

$$A_n(x) = \sum_{\pi \in S_n} x^{asc(\pi)+1}. \tag{1.5}$$

To give a grammatical interpretation of $A_n(x)$, Dumont defined bivariate polynomials $A_n(x,y)$ based on cyclic permutations on $[n]$. For a cyclic permutation σ, an index i $(1 \leq i \leq n)$ is an ascent if $i < \sigma(i)$ and a descent if $i > \sigma(i)$. Let $asc_c(\sigma)$ be the number of ascents of σ, and let $des_c(\sigma)$ be the number of descents of σ. We assume that a cyclic permutation is oriented clockwise. For example, Figure 1.1 is a cyclic permutation on [6]. Let C_n denote the set of cyclic permutations on $[n]$. For $n \geq 1$, Dumont defined a the polynomial $A_n(x,y)$ as follows,

$$A_n(x,y) = \sum_{\sigma \in C_{n+1}} x^{asc_c(\sigma)} y^{des_c(\sigma)}. \tag{1.6}$$

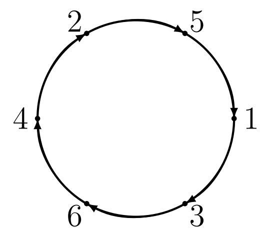


Figure 1.1: a cyclic permutation of [6]

It should be noted that Dumont used the notation $A_{n+1}(x, y)$ instead of $A_n(x, y)$ for the above polynomial. We choose the above notation for the reason of consistency with the notation that we shall use in the next section.

Setting $y = 1$ in (1.6), we get that for $n \geq 1$

$$A_n(x, y)|_{y=1} = A_n(x). \tag{1.7}$$

For $n \geq 1$, to obtain $A_{n+1}(x, y)$ from $A_n(x, y)$, Dumont observed that the insertion of $n + 1$ into a cyclic permutation of $[n]$ after i leads to a replacement of the arc $(i, \sigma(i))$ with $(i, n + 1)$ followed by $(n + 1, \sigma(i))$. If i is an ascent, $(i, \sigma(i))$ corresponds to x with respect to the definition of $A_n(x, y)$ and the insertion of $n + 1$ corresponds to substitution of x by xy. If i is a descent, the insertion of $n + 1$ corresponds to substitution of y by xy. Thus, $A_{n+1}(x, y)$ can be obtained from $A_n(x, y)$ by applying the substitution rules of the grammar G, namely,

$$A_{n+1}(x, y) = D(A_n(x, y)).$$

It follows that

$$D^n(x) = A_n(x, y).$$

To demonstrate how to use a context-free grammar to generate combinatorial objects, we introduce the concept of a grammatical labeling. This idea is implicit in the partition argument with respect to the grammar $f_i \to f_{i+1}g_1$, $g_i \to g_{i+1}$ to generate partitions as given by Chen [2]. It turns out that a grammatical labeling serves a concrete connection between a grammar and the corresponding combinatorial structure.

This paper is organized as follows. In Section 2, we use examples to illustrate the notion of a grammatical labeling. We give an explanation of relation (1.7) by labeling ascents and descents of a permutation instead of a cyclic permutation. Similarly, by labeling ascents, descents and plateaux of a Stirling permutation, we obtain a grammatical interpretation of the second-order Eulerian polynomials. As another example, we give a grammatical explanation of the Lah numbers by labeling the ascents and descents of a partition into lists. We also demonstrate how to use the formal derivative with respect to the grammar $x \to xy$, $y \to xy$ to deduce an identity on the Eulerian polynomials.

Section 3 is devoted to the applications of the grammar $x \to xy$, $y \to x$ found by Dumont [3] for the André polynomials defined in terms of 0-1-2 increasing trees. As shown in Chen [2], a context-free grammar can be rigorously used to derive combinatorial identities in the sense that a formal derivative plays a role analogous to the derivative in calculus. We shall demonstrate how to use the grammar for 0-1-2 increasing trees given by Dumont [3] to give a grammatical derivation of the generating function of the André polynomials obtained by Foata and Schützenberger, without solving a differential equation.

In Section 4, we use the grammatical labeling to concern permutations with exterior peaks. We find that the following grammar

$$G: \quad x \to xy, \quad y \to x^2$$

can be used to generate permutations with respect to exterior peaks. This grammar was independently discovered by Ma [11]. We show that Gessel's formula for the generating function of permutations on exterior peaks can be derived by using this grammar.

In Section 5, by specializing a grammar of Dumont [3] for increasing trees, we find that this grammar also generates increasing trees with respect to the number of vertices with even degree. To be more specific, the degree of a vertex in a rooted tree is meant to be the number of its children. As a consequence, we obtain that the number of permutations of $[n]$ with k exterior peaks equals the number of increasing trees on $[n]$ with $2k+1$ vertices of even degree.

We conclude this paper with a bijection between permutations and increasing trees which connects these two statistics. This bijection is an extension of a correspondence between alternating permutations and even increasing trees given by Kuznetsov, Pak and Postnikov [10].

2 Grammatical Labelings

In order to connect a context-free grammar to a combinatorial structure, we associate the elements of a combinatorial structure with letters in a grammar. Such a labeling scheme of a combinatorial structure is called a grammatical labeling.

For example, consider the following grammar given by Dumont [3],

$$G: \quad x \to xy, \quad y \to xy. \tag{2.1}$$

We shall use a grammatical labeling on permutations to show that the Eulerian polynomial $A_n(x)$ can be expressed in terms of the formal derivative with respect to the grammar G. This labeling can be easily extended to Stirling permutations and partitions into lists.

Denote by $A(n,m)$ the number of permutations of $[n]$ with $m-1$ ascents. The generating function

$$A_n(x)=\sum_{m=1}^{n}A(n,m)x^m$$

is known as the Eulerian polynomial.

We now give a grammatical labeling on permutations to generate the Eulerian polynomials. Let π be a permutation of $[n]$. An index i $(1\le i\le n-1)$, is called an ascent if $\pi_i<\pi_{i+1}$, a descent if $\pi_i>\pi_{i+1}$. Set $\pi_0=\pi_{n+1}=0$. For $0\le i\le n$, if $\pi_i<\pi_{i+1}$, we label i by x, and if $\pi_i>\pi_{i+1}$, we label i by y. With this labeling, the weight of π is defined as the product of the labels, that is,

$$w(\pi)=x^{asc(\pi)+1}y^{des(\pi)+1},$$

where $asc(\pi)$ denotes the number of ascents in π and $des(\pi)$ denotes the number of descents in π. As will been seen, the polynomial $A_n(x,y)$ defined in terms of the descent number and the ascent number of a cyclic polynomial as given in (1.6) can also be expressed in terms of the descent number and the ascent number of a permutation, that is, for $n\ge 1$,

$$A_n(x,y)=\sum_{\pi\in S_n}x^{asc(\pi)+1}y^{des(\pi)+1}.$$

A grammatical labeling plays a role of establishing a connection between the action of the formal derivative D and the insertion of the element $n+1$ into a combinatorial structure on $[n]$. For example, let $n=6$ and $\pi=325641$. The grammatical labeling of π is given below

$$\begin{array}{ccccccccccccc} & 3 & & 2 & & 5 & & 6 & & 4 & & 1 & \\ x & & y & & x & & x & & y & & y & & y. \end{array}$$

If we insert 7 after 5, the resulting permutation and its grammatical labeling are as follows,

$$\begin{array}{ccccccccccccccc} & 3 & & 2 & & 5 & & 7 & & 6 & & 4 & & 1 & \\ x & & y & & x & & x & & y & & y & & y & & y. \end{array}$$

It can be seen that the insertion of 7 after 5 corresponds to the differentiation on the label x associated with 5. The same argument applies to the case when the new element is inserted after an element labeled by y. Hence the action of the formal derivative D on the set of weights of permutations in S_n gives the set of weights of permutations in S_{n+1}. This yields the following grammatical expression for $A_n(x,y)$.

Theorem 2.1 *Let D be the formal derivative with respect to grammar* (2.1). *For $n\geqslant 1$, we have*

$$D^n(x)=\sum_{m=1}^{n}A(n,m)x^my^{n+1-m}.$$

From Theorem 2.1, it follows that $D^n(x)|_{y=1}=A_n(x)$. Here we give a grammatical proof of the following classical recurrence for the Eulerian polynomials $A_n(x)$.

Proposition 2.2 *For $n \geq 1$, we have*

$$A_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(x)(x-1)^{n-1-k}, \tag{2.2}$$

where $A_0(x) = 1$.

Proof. By the definition of grammar (2.1), we have $D(x^{-1}) = -x^{-2}D(x) = -x^{-1}y$. Hence

$$D(x^{-1}y) = x^{-1}D(y) + yD(x^{-1}) = x^{-1}y(x-y). \tag{2.3}$$

Since $(x-y)$ is a constant with respect to D, we see that

$$D^n(x^{-1}y) = x^{-1}y(x-y)^n. \tag{2.4}$$

By the Leibniz formula, we have for $n \geq 1$,

$$D^n(x) = D^n(y) = D^n(xx^{-1}y) = \sum_{k=0}^{n} \binom{n}{k} D^k(x)D^{n-k}(x^{-1}y). \tag{2.5}$$

Substituting (2.4) into (2.5) we get

$$(x-y)x^{-1}D^n(x) = \sum_{k=0}^{n-1} \binom{n}{k} x^{-1}yD^k(x)(x-y)^{n-k}.$$

Setting $y = 1$, we arrive at (2.2). ∎

Next, we introduce a grammar to generate Stirling permutations. Let $[n]_2$ denote the multiset $\{1^2, 2^2, \ldots, n^2\}$, where i^2 stands for two occurrences of i. A Stirling permutation is a permutation π of the multiset $[n]_2$ such that for each $1 \leq i \leq n$ the elements between two occurrences of i are larger than i, see Gessel and Stanley [8]. For example, 123321455664 is a Stirling permutation on $[6]_2$.

For a Stirling permutation $\pi = \pi_1\pi_2 \cdots \pi_{2n}$, an index i $(1 \leq i \leq 2n-1)$, is called an ascent if $\pi_i < \pi_{i+1}$, a descent if $\pi_i > \pi_{i+1}$ and a plateaux if $\pi_i = \pi_{i+1}$. We shall show that the following grammar

$$G\colon \quad x \to xy^2, \quad y \to xy^2 \tag{2.6}$$

can be used to generate Stirling permutations. We now give a grammatical labeling on Stirling permutations. Let $\pi = \pi_1\pi_2 \cdots \pi_{2n}$ be a Stirling permutation on $[n]_2$. First, we add a zero at the beginning and a zero at the end of π. Then we label an ascent of $0\pi_1\pi_2 \cdots \pi_{2n}0$ by x and label a descent or a plateau by y. For example, let $\pi = 244215566133$. The grammatical labeling of π is given below

$$\begin{array}{ccccccccccccccccccccccccc} & 2 & & 4 & & 4 & & 2 & & 1 & & 5 & & 5 & & 6 & & 6 & & 1 & & 3 & & 3 & \\ x & & x & & y & & y & & y & & x & & y & & x & & y & & y & & x & & y & & y. \end{array}$$

If we insert 77 after the first occurrence of 4, we get

$$\begin{array}{cccccccccccccccc} & 2 & 4 & 7 & 7 & 4 & 2 & 1 & 5 & 5 & 6 & 6 & 1 & 3 & 3 \\ x & x & x & y & y & y & y & x & y & x & y & y & x & y & y. \end{array}$$

If we insert 77 after the second occurrence of 1, we get

$$\begin{array}{cccccccccccccccc} & 2 & 4 & 4 & 2 & 1 & 5 & 5 & 6 & 6 & 1 & 7 & 7 & 3 & 3 \\ x & x & y & y & y & x & y & x & y & y & x & y & y & y & y. \end{array}$$

Notice that each Stirling permutation on $[n]_2$ can be obtained by inserting nn into a Stirling permutation on $[n-1]_2$. Thus, we get a grammatical interpretation of generating function of Stirling permutations with respect to the number of ascents.

Theorem 2.3 *Let D be the formal derivative with respect to grammar* (2.6). *Then we have*

$$D^n(x) = \sum_{m=1}^{n} C(n,m) x^m y^{2n+1-m}, \tag{2.7}$$

where $C(n,m)$ denotes the number of Stirling permutations of $[n]_2$ with $m-1$ ascents.

We use the notation $C_n(x)$ as used in Bóna [1] to denote the second-order Eulerian polynomials

$$C_n(x) = \sum_{m=1}^{n} C(n,m) x^m.$$

From Theorem 2.3, we see that $D^n(x)|_{y=1} = C_n(x)$.

In general, we can use the grammar

$$G\colon \quad x \to xy^r, \quad y \to xy^r$$

to generate r-Stirling permutations. An r-Stirling permutation is a permutation on $[n]_r = \{1^r, 2^r, \ldots, n^r\}$ such that the elements between two occurrences of i are not smaller than i.

To conclude this section, we give the following grammar

$$G\colon \quad z \to xyz, \quad x \to xy, \quad y \to xy, \tag{2.8}$$

and we show that this grammar can be used to generate partitions of $[n]$ into lists. We call the above grammar the Lah grammar. Recall that a partition of $[n]$ into lists is a partition of $[n]$ for which the elements of each block are linearly ordered. For a partition into lists, label the partition itself by z. Express a list $\sigma_1\sigma_2\cdots\sigma_m$ by $0\sigma_1\sigma_2\cdots\sigma_m 0$ and label an ascent and a descent of $0\sigma_1\sigma_2\cdots\sigma_m 0$ by x and y respectively. For example, let $\pi = \{325, 614, 7\}$. Below is the labeling of π:

$$\begin{array}{cccccccccccccc} & & & 3 & & 2 & & 5 & & & 6 & & 1 & & 4 & & & 7 & \\ z & & x & & y & & x & & y & x & & y & & x & & y & x & & y \end{array}$$

Using this labeling, it can be easily seen that grammar (2.8) generates partitions into lists.

Theorem 2.4 *Let $C(n,k,m)$ be the number of partitions of $[n]$ into k lists with m ascents. Then, we have*

$$D^n(z) = \sum_{k=1}^{n}\sum_{m=k}^{n} C(n,k,m)x^m y^{k+n-m} z.$$

In particular, setting $y = x$, we get the grammar

$$G: \quad z \rightarrow x^2 z, \quad x \rightarrow x^2, \tag{2.9}$$

which generates the signless Lah numbers

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}.$$

Corollary 2.5 *Let D be the formal derivative with respect to grammar* (2.9). *Then*

$$D^n(z) = x^n z \sum_{k=1}^{n} L(n,k)x^k. \tag{2.10}$$

3 The André Polynomials

In this section, we use the grammar found by Dumont [3] to give a proof of the generating function formula for the André polynomials without solving a differential equation. This formula was first obtained by Foata and Schützenberger [7].

Recall that the André polynomials are defined in terms of 0-1-2 increasing trees. An increasing tree on $[n]$ is a rooted tree with vertex set $\{0, 1, 2, \ldots, n\}$ in which the labels of the vertices are increasing along any path from the root. Note that 0 is the root. A 0-1-2 increasing tree is an increasing tree in which the degree of any vertex is at most two. Recall that in this paper, the degree of a vertex in a rooted tree is meant to be the number of its children. Given a 0-1-2 increasing tree T, let $l(T)$ denote the number of leaves of T, and $u(T)$ denote the number of vertices of T with degree 1. Then the André polynomial is defined by

$$E_n(x,y) = \sum_{T} x^{l(T)} y^{u(T)},$$

where the sum ranges over 0-1-2 increasing trees on $[n-1]$.

Setting $x = y = 1$, $E_n(x,y)$ reduces to the n-th Euler number E_n, which counts both 0-1-2 increasing trees on $[n-1]$ and alternating permutations of $[n]$, see [5,7,10].

Foata and Schützenberger obtained the generating function of the André polynomials in [7] by solving a differential equation. Later, Foata and Han [6] found a way to compute the generating function of $E_n(x, 1)$ without solving a differential equation, or equivalently, the generating function of $E_n(x, y)$.

Dumont [3] introduced the grammar

$$G: \quad x \to xy, \quad y \to x \tag{3.1}$$

and showed that it generates the André polynomials $E_n(x, y)$. This fact can be justified intuitively in terms of the following grammatical labeling. Given a 0-1-2 increasing tree T, a leaf of T is labeled by x, a vertex of degree 1 in T is labeled by y and a vertex of degree 2 in T is labeled by 1. The following figure illustrates the labeling of a 0-1-2 increasing tree on $\{1, 2, 3, 4, 5\}$.

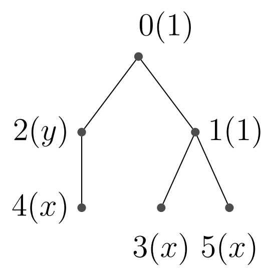
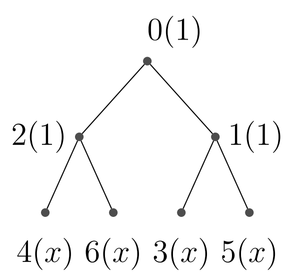


Figure 3.2: The labeling of a 0-1-2 increasing tree on $\{1, 2, 3, 4, 5\}$

If we add 6 as a child of 2, the resulting tree is as follows.



After the vertex 6 is added, the label of 2 is changed from y to 1, and the vertex 6 gets a label x. This corresponds to the rule $y \to x$ of the grammar G. Similarly, adding the vertex 6 to a leaf of the increasing tree in Figure 3.2 corresponds to the rule $x \to xy$. Let D be the formal derivative with respect to the grammar in (3.1). The above grammatical labeling leads to the following relation

$$D^n(x) = E_n(x, y).$$

Now we demonstrate that one can use the grammar G in (3.1) to derive the generating function of $E_n(x, y)$ without solving a differential equation.

Theorem 3.1 (Foata and Schützenberger) *We have*

$$\sum_{n=0}^{\infty} \frac{E_n(x,y)}{n!} t^n$$
$$= \frac{x\sqrt{2x-y^2} + y(2x-y^2)\sin(t\sqrt{2x-y^2}) - (x-y^2)\sqrt{2x-y^2}\cos(t\sqrt{2x-y^2})}{(x-y^2)\sin(t\sqrt{2x-y^2}) + y\sqrt{2x-y^2}\cos(t\sqrt{2x-y^2})}. \tag{3.2}$$

Setting $x = y = 1$, we get

$$\sum_{n=0}^{\infty} \frac{E_n}{n!} t^n = \sec t + \tan t. \tag{3.3}$$

Proof. By the Leibniz rule, we have

$$\mathrm{Gen}(x^{-1}y, t) = \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}(y, t). \tag{3.4}$$

Differentiating both sides of (3.4) with respect to t yields

$$\mathrm{Gen}'(x^{-1}y, t) = \mathrm{Gen}'(x^{-1}, t)\mathrm{Gen}(y, t) + \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}'(y, t). \tag{3.5}$$

Since $D(x^{-1}) = -x^{-1}y$, we have

$$\mathrm{Gen}'(x^{-1}, t) = \mathrm{Gen}(D(x^{-1}), t) = -\mathrm{Gen}(x^{-1}y, t). \tag{3.6}$$

Using $D(y) = x$, we get

$$\mathrm{Gen}(x^{-1}, t)\mathrm{Gen}'(y, t) = \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}(D(y), t) = \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}(x, t) = 1. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we deduce that

$$\mathrm{Gen}'(x^{-1}y, t) = 1 - \mathrm{Gen}(x^{-1}y, t)\mathrm{Gen}(y, t),$$

and hence

$$\mathrm{Gen}(y, t) = \frac{1 - \mathrm{Gen}'(x^{-1}y, t)}{\mathrm{Gen}(x^{-1}y, t)}. \tag{3.8}$$

We now compute the generating function $\mathrm{Gen}(x^{-1}y, t)$. It is easily verified that

$$D^{2m+1}(x^{-1}y) = (1 - x^{-1}y^2)(y^2 - 2x)^m \tag{3.9}$$

and

$$D^{2m}(x^{-1}y) = x^{-1}y(y^2 - 2x)^m. \tag{3.10}$$

Using (3.9) and (3.10), we have

$$\begin{aligned}
\operatorname{Gen}(x^{-1}y,t) &= \sum_{n=0}^{\infty} \frac{D^n(x^{-1}y)}{n!}t^n \\
&= (1-x^{-1}y^2)\sum_{n=0}^{\infty} \frac{(y^2-2x)^n}{(2n+1)!}t^{2n+1} + x^{-1}y\sum_{n=0}^{\infty}\frac{(y^2-2x)^n}{(2n)!}t^{2n} \\
&= \frac{1-x^{-1}y^2}{\sqrt{2x-y^2}}\sum_{n=0}^{\infty}\frac{(-1)^n(t\sqrt{2x-y^2})^{2n+1}}{(2n+1)!} + x^{-1}y\sum_{n=0}^{\infty}\frac{(-1)^n(t\sqrt{2x-y^2})^{2n}}{(2n)!} \\
&= \frac{1-x^{-1}y^2}{\sqrt{2x-y^2}}\sin(t\sqrt{2x-y^2}) + x^{-1}y\cos(t\sqrt{2x-y^2}). \qquad (3.11)
\end{aligned}$$

Plugging (3.11) into (3.8), we arrive at (3.2), and hence the proof is complete. ∎

4 Permutations with k Exterior Peaks

In this section, we introduce the following grammar

$$G: \quad x \to xy, \quad y \to x^2 \qquad (4.1)$$

and we show that G generates the number $T(n,k)$ of permutations of $[n]$ with k exterior peaks. Let

$$T_n(x) = \sum_{k\geq 0} T(n,k)x^k.$$

The grammar G also leads to a recurrence relation of $T_n(x)$. Moreover, we give a grammatical proof of the formula for the generating function of $T_n(x)$ due to Gessel, see [12].

Recall that for a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, the index i is called an exterior peak if $1 < i < n$ and $\pi_{i-1} < \pi_i > \pi_{i+1}$, or $i = 1$ and $\pi_1 > \pi_2$. We shall use a grammatical labeling of permutations to show that grammar (4.1) generates the polynomial $T_n(x)$.

For a permutation π of $[n]$, we give a labeling as follows. First, we add an element 0 at the end of the permutation. If i is an exterior peak, then we label i and $i+1$ by x. In addition, the element 0 is labeled by x, and all other elements are labeled by y. The weight w of a permutation is defined to be the product of all the labels. For a permutation π with k exterior peaks, its weight is given by

$$w(\pi) = x^{2k+1}y^{n-2k}.$$

For example, let $\pi = 325641$. The labeling of π is as follows

$$\begin{array}{ccccccc} 3 & 2 & 5 & 6 & 4 & 1 & 0 \\ x & x & y & x & x & y & x, \end{array}$$

and the weight of π is x^5y^2. If we insert 7 before 3, then the labeling of the resulting permutation is

$$\begin{array}{cccccccc} 7 & 3 & 2 & 5 & 6 & 4 & 1 & 0 \\ x & x & y & y & x & x & y & x. \end{array}$$

We see that the label of 2 changes from x to y and the label of 7 is x. So this insertion corresponds to the rule $x \to xy$. If we insert 7 before 0, we get

$$\begin{array}{cccccccc} 3 & 2 & 5 & 6 & 4 & 1 & 7 & 0 \\ x & x & y & x & x & y & y & x, \end{array}$$

where the label x of 0 remains the same and the label of 7 is y. In this case, the insertion corresponds to the rule $x \to xy$. If we insert 7 before 5, we get

$$\begin{array}{cccccccc} 3 & 2 & 7 & 5 & 6 & 4 & 1 & 0 \\ x & x & x & x & x & x & y & x, \end{array}$$

where the label of 5 changes from y to x and the label of 7 is x. This corresponds to the rule $y \to x^2$ in grammar (4.1). In general, the above labeling leads to the following theorem.

Theorem 4.1 *Let D be the formal derivative with respect to grammar* (4.1). *For $n \geq 1$,*

$$D^n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k) x^{2k+1} y^{n-2k}. \tag{4.2}$$

The grammar (4.1) and relation (4.2) were announced at the International Conference on Designs, Matrices and Enumerative Combinatorics held at National Taiwan University in 2011. Ma [11] independently discovered grammar (4.1) and gave an inductive proof of relation (4.2).

By Theorem 4.1, we obtain the following recurrence relation.

Proposition 4.2 *For $n \geq 1$,*

$$T_n(x) = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} (1-x)^{\lfloor j/2 \rfloor} T_{n-j}(x). \tag{4.3}$$

Proof. Note that

$$D(x^{-1}) = -x^{-1}y, \; D(-x^{-1}y) = x^{-1}(y^2 - x^2), \; D(y^2 - x^2) = 0.$$

Hence

$$D^{2m+1}(x^{-1}) = -x^{-1}y(y^2 - x^2)^m \tag{4.4}$$

and

$$D^{2m}(x^{-1}) = x^{-1}(y^2 - x^2)^m. \tag{4.5}$$

Setting $y = 1$ in (4.4) and (4.5), we get

$$D^j(x^{-1})|_{y=1} = (-1)^j x^{-1}(1-x^2)^{\lfloor j/2 \rfloor}.$$

By the Leibniz rule we have

$$\begin{aligned} D^n(x^{-1}x)|_{y=1} = 0 &= \sum_{j=0}^{n} \binom{n}{j} D^j(x^{-1})|_{y=1} D^{n-j}(x)|_{y=1} \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j x^{-1}(1-x^2)^{\lfloor j/2 \rfloor} D^{n-j}(x)|_{y=1}. \end{aligned} \tag{4.6}$$

According to Theorem 4.1, we see that

$$D^n(x)|_{y=1} = xT_n(x^2).$$

Hence (4.3) follows from (4.6). ∎

With the aid of grammar (4.1), we give a derivation of the following generating function of $T_n(x)$ due to Gessel, see [12].

Theorem 4.3 (Gessel) *We have*

$$\sum_{n=0}^{\infty} \frac{T_n(x)}{n!} t^n = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(\sqrt{1-x}t) - \sinh(\sqrt{1-x}t)}. \tag{4.7}$$

To prove Theorem 4.3, we need the following generating function. As will be seen, this generating function is related to the generating function of $T(n,k)$.

Theorem 4.4 *For the the following grammar*

$$G: \quad u \to v^2, \quad v \to v, \tag{4.8}$$

we have

$$\mathrm{Gen}(u^{-1}v, t) = \frac{v}{u\cosh(t) + (v^2 - u)\sinh(t)}. \tag{4.9}$$

Proof. Let D be the formal derivative with respect to G. Since $D(v) = v$, we have

$$\mathrm{Gen}(v, t) = ve^t.$$

By (1.3), we find that

$$\mathrm{Gen}(u^{-1}v^2, t) = \mathrm{Gen}(v, t)\mathrm{Gen}(u^{-1}v, t) = ve^t\mathrm{Gen}(u^{-1}v, t). \tag{4.10}$$

We proceed to compute $(\mathrm{Gen}(u^{-1}v^2, t))'$ in two ways. It is easily checked that

$$D(u^{-1}v^2) = -(u^{-1}v)^2(v^2 - 2u).$$

Thus, from (1.1) and (1.3) we deduce that

$$(\mathrm{Gen}(u^{-1}v^2,t))' = \mathrm{Gen}\left(D(u^{-1}v^2),t\right) = -\mathrm{Gen}^2(u^{-1}v,t)\mathrm{Gen}(v^2-2u,t). \tag{4.11}$$

On the other hand, since

$$D(u^{-1}v) = u^{-1}v(1-u^{-1}v^2),$$

from (4.10) we find that

$$\begin{aligned}(\mathrm{Gen}(u^{-1}v^2,t))' &= (ve^t\mathrm{Gen}(u^{-1}v,t))' \\ &= ve^t\mathrm{Gen}(u^{-1}v,t) + ve^t\mathrm{Gen}(D(u^{-1}v),t) \\ &= ve^t\mathrm{Gen}(u^{-1}v,t) + ve^t\mathrm{Gen}(u^{-1}v,t)\mathrm{Gen}(1-u^{-1}v^2,t). \end{aligned} \tag{4.12}$$

Comparing (4.11) with (4.12), we obtain that

$$\begin{aligned}&-\mathrm{Gen}^2(u^{-1}v,t)\mathrm{Gen}(v^2-2u,t) \\ &\quad= ve^t\mathrm{Gen}(u^{-1}v,t) + ve^t\mathrm{Gen}(u^{-1}v,t)\mathrm{Gen}(1-u^{-1}v^2,t),\end{aligned}$$

or, equivalently,

$$-\mathrm{Gen}(u^{-1}v,t)\mathrm{Gen}(v^2-2u,t) = ve^t + ve^t\mathrm{Gen}(1-u^{-1}v^2,t). \tag{4.13}$$

Since $D(v^2-2u)=0$, we get

$$\mathrm{Gen}(v^2-2u,t) = v^2-2u.$$

Clearly, $\mathrm{Gen}(1-u^{-1}v^2,t) = 1-\mathrm{Gen}(u^{-1}v^2,t)$. Thus (4.13) can be simplified to

$$-(v^2-2u)\mathrm{Gen}(u^{-1}v,t) = 2ve^t - ve^t\mathrm{Gen}(u^{-1}v^2,t). \tag{4.14}$$

Plugging (4.10) into (4.14), we arrive at

$$\mathrm{Gen}(u^{-1}v,t) = \frac{2v}{v^2e^t-(v^2-2u)e^{-t}},$$

which can be written in the form of (4.9), and so the proof is complete. ∎

We proceed to show that $\mathrm{Gen}(u^{-1}v,t)$ can be used to derive the generating function of $T_n(x)$ as given in Theorem 4.3. To this end, we consider the following grammar

$$G\colon\quad x\to xy,\quad y\to wx^2. \tag{4.15}$$

For a permutation π of $[n]$, we give a labeling which is essentially the same as the labeling given before. First, add an element 0 at the end of the permutation. If i is an exterior peak, then we label i by wx and $i+1$ by x. In addition, the element 0 is labeled by x, and all other elements are labeled by y. For example, let $\pi = 325641$. The labeling of π is as follows

$$\begin{array}{ccccccc} 3 & 2 & 5 & 6 & 4 & 1 & 0 \\ wx & x & y & wx & x & y & x. \end{array}$$

For the grammar in (4.15), we have

$$D^n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k) x^{2k+1} y^{n-2k} w^k. \tag{4.16}$$

Proof of Theorem 4.3. For the grammar (4.8) in Theorem 4.4, notice the relations

$$\begin{aligned} D(u^{-1}v) &= u^{-1}v(1 - u^{-1}v^2), \\ D(1 - u^{-1}v^2) &= (u^{-1}v)^2(v^2 - 2u), \\ D(v^2 - 2u) &= 0. \end{aligned}$$

Comparing the above relations with the rules of the grammar in (4.15) and making the substitutions $x = u^{-1}v,\ y = 1 - u^{-1}v^2,\ w = v^2 - 2u$, we get the rules as in grammar (4.15), namely, $D(x) = xy$, $D(y) = wx^2$ and $D(w) = 0$. Hence relation (4.16) implies that

$$D^n(u^{-1}v) = \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(u^{-1}v)^{2k+1}(1 - u^{-1}v^2)^{n-2k}(v^2 - 2u)^k,$$

that is,

$$\mathrm{Gen}(u^{-1}v, t) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(u^{-1}v)^{2k+1}(1 - u^{-1}v^2)^{n-2k}(v^2 - 2u)^k. \tag{4.17}$$

Comparing (4.9) with (4.17), we get

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(u^{-1}v)^{2k}(1 - u^{-1}v^2)^{n-2k}(v^2 - 2u)^k = \frac{u}{u\cosh(t) + (v^2 - u)\sinh(t)}.$$

Since the above relation is valid for indeterminates u and v, we can set $v = \sqrt{u-1}$ to deduce the following relation

$$\sum_{n \geq 0} \frac{t^n u^{-n}}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(1 - u^2)^k = \frac{u}{u\cosh(t) - \sinh(t)}. \tag{4.18}$$

Substituting t by ut in (4.18), we get

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(1 - u^2)^k = \frac{u}{u\cosh(ut) - \sinh(ut)}. \tag{4.19}$$

Finally, by setting $x = 1 - u^2$ in (4.19), we reach (4.7). This completes the proof. ∎

5 Peaks in permutations and increasing trees

In this section, we use a grammatical approach to establish the following theorem on a connection between permutations with a given number of exterior peaks and increasing trees with a given number of vertices of even degree. Then we give a combinatorial interpretation of this fact.

Theorem 5.1 *The number of permutations σ of $[n]$ with k exterior peaks equals the number of increasing trees T_σ on $[n]$ with $2k+1$ vertices which have even degree.*

To prove the above theorem by using grammars, we first recall a grammar given by Dumont [3],

$$G: \quad x_i \to x_0 x_{i+1}. \tag{5.1}$$

Let D be the formal derivative with respect to G. Dumont [3] showed that

$$D^n(x_0) = \sum_T x_0^{m_0(T)} x_1^{m_1(T)} x_2^{m_2(T)} \cdots, \tag{5.2}$$

where the sum ranges over increasing trees T on $[n]$ and $m_i(T)$ denotes the number of vertices of degree i in T.

Relation (5.2) can be justified by labeling a vertex of degree i with x_i in an increasing tree. Here is an example.

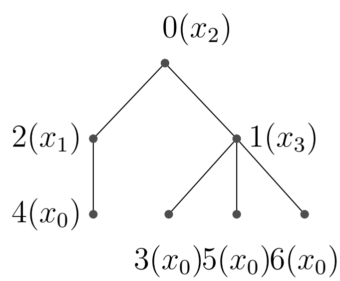
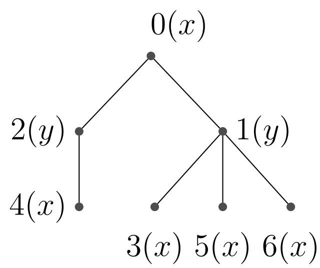


Figure 5.3: A labeling on an increasing tree

Let T be an increasing tree on $[n]$ with the above labeling. When adding the vertex $n+1$ to T as the child of a vertex v of degree i, the label of v changes from x_i to x_{i+1} and the label of $n+1$ is x_0. This corresponds to the rule $x_i \to x_0 x_{i+1}$. Since the increasing trees on $[n+1]$ can be generated by adding $n+1$ to the increasing trees on $[n]$, the above labeling leads to (5.2).

By setting $x_{2i} = x$ and $x_{2i+1} = y$, we see that the grammar (5.1) becomes the grammar (4.1) that generates the polynomial $T_n(x)$ for permutations with a given number of exterior peaks. Intuitively, this leads to a grammatical reasoning of Theorem 5.1. Next we give a rigorous proof of this observation by a grammatical labeling with respect to the parities of the vertices in an increasing tree.

Grammatical Proof of Theorem 5.1. We give the following grammatical labeling of an increasing tree. We label a vertex of even degree with x and a vertex of odd degree with y. For example, the labeling of the increasing tree in Figure 5.3 is given below.



Let T be an increasing tree on $[n]$ with the above labeling. When adding the vertex $n+1$ to T as a child of a vertex v of even degree, the label of v changes from x to y and the label of $n+1$ is x. This corresponds to the rule $x \to xy$. Similarly, adding the vertex $n+1$ to T as a child of a vertex of odd degree corresponds to the rule $y \to x^2$. Thus, we obtain that

$$D^n(x) = \sum_T x^{m_e(T)} y^{m_o(T)}, \tag{5.3}$$

where the sum ranges over increasing trees T on $[n]$ and $m_e(T)$ denotes the number of vertices of even degree in T, $m_o(T)$ denotes the number of vertices of odd degree in T. Comparing (4.2) with (5.3), we deduce that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k) x^{2k+1} y^{n-2k} = \sum_T x^{m_e(T)} y^{m_o(T)},$$

where T ranges over increasing trees on $[n]$. This completes the proof. ∎

To conclude this paper, we give a combinatorial proof of Theorem 5.1. More precisely, we provide a bijection Φ between permutations and increasing trees such that a permutation of $[n]$ with k exterior peaks corresponds to an increasing tree on $[n]$ with $2k+1$ vertices of even degree. Recall that a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$ is called a up-down permutation if $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$. Similarly, σ is called a down-up permutation if $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$. When restricted to down-up permutations, Φ reduces to the bijection between down-up permutations and even increasing trees. An even increasing tree is meant to be an increasing tree such that each vertex possibly except for the root is of even degree. Kuznetsov, Pak and Postnikov [10] gave a bijection between up-down permutations and even increasing trees. So our bijection can be considered as an extension of the bijection given by Kuznetsov, Pak and Postnikov, since there is an obvious correspondence between up-down permutations and down-up permutations.

Before describing our bijection, we recall that the code of a permutation is defined as follows. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$, let $\mathrm{code}(\sigma) = (c_1, c_2, \ldots, c_n)$ denote the code of σ. In other words, c_i is the number of elements σ_j such that $j > i$ and $\sigma_i > \sigma_j$. It is clear that $c_n = 0$.

The increasing tree $\Phi(\sigma)$ can be constructed via n steps. At each step, a vertex is added to a forest of increasing trees. More precisely, at the k-th step, we obtain a forest of increasing trees with k vertices, and finally obtain an increasing tree $\Phi(\sigma)$ on $[n]$.

For $k = 1$, as the first step we start with an increasing tree F_1 with a single vertex $i_1 = n - c_1$. For $k > 1$, we assume that a forest F_{k-1} has been obtained at the $(k-1)$-th step. Denote by I_{k-1} and J_{k-1} the set of vertices and the set of roots of F_{k-1}. Let $\bar{I}_{k-1}$ be the complement of I_{k-1}, that is, $\bar{I}_{k-1} = [n] \setminus I_{k-1}$. The goal of the k-th step is to construct a forest F_k by adding an element from $\bar{I}_{k-1}$ to F_{k-1}.

Let $j_1, j_2, \ldots, j_l$ be the elements of J_{k-1} listed in decreasing order. For notational convenience, we assume that $j_0 = n + 1$, $j_{l+1} = 0$ and $c_0 = 0$. Let

$$U_k = \{m \in \bar{I}_{k-1} \mid j_{2p+2} < m < j_{2p+1} \text{ for some } p \geq 0\}, \tag{5.4}$$

$$V_k = \{m \in \bar{I}_{k-1} \mid j_{2p+1} < m < j_{2p} \text{ for some } p \geq 0\}. \tag{5.5}$$

It is clear that $U_k \cap V_k = \emptyset$ and $\bar{I}_{k-1} = U_k \cup V_k$.

Define M_k to be U_k if $c_{k-2} \leq c_{k-1} \leq c_k$ or $c_{k-2} > c_{k-1} > c_k$; otherwise, define M_k to be V_k. Let $m_1, \ldots, m_s$ be the elements of M_k listed in increasing order. We define i_k to be m_{c_k+1} if $c_{k-1} > c_k$, or $m_{n-k+1-c_k}$ if $c_{k-1} \leq c_k$. By the following lemma, it can be seen that it is feasible to choose such i_k, that is,

$$|M_k| \geq c_k + 1 \tag{5.6}$$

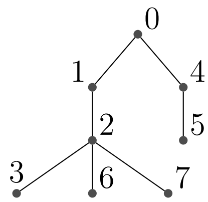
holds if $c_{k-1} > c_k$ and

$$|M_k| \geq n - k + 1 - c_k \tag{5.7}$$

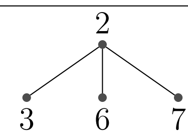
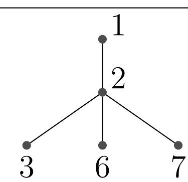
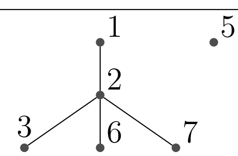
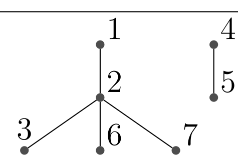
holds if $c_{k-1} \leq c_k$. Now, we add i_k to F_{k-1} by setting each $j_p \in J_{k-1}$ to be a child of i_k if $j_p > i_k$, and let the resulting forest by F_k.

When $k < n$, we may iterate the above process until we obtain a forest F_n on $[n]$. Setting each root of F_n to be a child of the vertex 0, we obtain an increasing tree T, which is set to be $\Phi(\sigma)$.

Here is an example for the above bijection. Let $\sigma = 5346721$. The code of σ is $\mathrm{code}(\sigma) = (4, 2, 2, 2, 2, 1, 0)$. The increasing tree $\Phi(\sigma)$ is as follows.



The values I_k, J_k, M_k, i_k and the forests F_k are given in the following table.

k	M_k	i_k	F_k	I_k, J_k
1	—	$i_1 = 3$		$I_1 = \{3\}$ $J_1 = \{3\}$
2	$M_2 = \{4, 5, 6, 7\}$	$i_2 = 6$		$I_2 = \{3, 6\}$ $J_2 = \{3, 6\}$
3	$M_3 = \{1, 2, 7\}$	$i_3 = 7$		$I_3 = \{3, 6, 7\}$ $J_3 = \{3, 6, 7\}$
4	$M_4 = \{1, 2\}$	$i_4 = 2$		$I_4 = \{2, 3, 6, 7\}$ $J_4 = \{2\}$
5	$M_5 = \{1\}$	$i_5 = 1$		$I_5 = \{1, 2, 3, 6, 7\}$ $J_5 = \{1\}$
6	$M_6 = \{4, 5\}$	$i_6 = 5$		$I_6 = \{1, 2, 3, 5, 6, 7\}$ $J_6 = \{1, 5\}$
7	$M_7 = \{4\}$	$i_7 = 4$		—

In the construction of Φ, at the k-th step ($2 \leq k \leq n$) conditions (5.6) and (5.7) are needed to ensure the existence of the element i_k. The following property implies conditions (5.6) and (5.7).

Lemma 5.2 *For $2 \leq k \leq n$, at the k-th step of the construction of Φ, if $c_{k-1} > c_k$, we have that*

$$|M_k| = c_{k-1}, \tag{5.8}$$

and if $c_{k-1} \leq c_k$,

$$|M_k| = n - k + 1 - c_{k-1}. \tag{5.9}$$

The proof of the above lemma is parallel to the construction of Φ. First, we show that Lemma 5.2 holds for $k = 2$. When Lemma 5.2 holds for k, where $k < n$, then (5.6) and (5.7) are valid for k, so the construction of Φ goes to the next step.

Proof. For $2 \leq k \leq n$, we proceed to prove (5.8) and (5.9) step by step. It is clear that $|M_2| = c_1$ if $c_1 > c_2$, and $|M_2| = n - 1 - c_1$ if $c_1 \leq c_2$. In other words, (5.8) and (5.9) hold for $k = 2$. Assume that (5.8) and (5.9) hold for k. To compute $|M_{k+1}|$, we consider the following four cases:

Case 1: $c_{k-2} > c_{k-1} > c_k$. Let $j_1, j_2, \ldots, j_l$ be the elements of J_{k-1} listed in decreasing order, and let $j_0 = n + 1$ and $j_{l+1} = 0$. By the assumption $c_{k-2} > c_{k-1} > c_k$ and the definition of M_k, we get

$$M_k = U_k = \{m \in \bar{I}_{k-1} \mid j_{2p+2} < m < j_{2p+1} \text{ for some } p \geq 0\}.$$

Since $i_k \in M_k$, there exists $q \geq 0$ such that $j_{2q+2} < i_k < j_{2q+1}$. So the set of roots of F_k is given by

$$J_k = \{i_k, j_{2q+2}, \ldots, j_l\}.$$

It follows that

$$\begin{aligned} U_{k+1} &= \{m \in \bar{I}_k \mid m < i_k,\ j_{2p+2} < m < j_{2p+1} \text{ for some } p \geq q\} \\ &= \{m \in M_k \mid m < i_k\}. \end{aligned}$$

Since $c_{k-1} > c_k$, i_k is the $(c_k + 1)$-th smallest element in M_k. Hence

$$|U_{k+1}| = |\{m \in M_k \mid m < i_k\}| = c_k.$$

If $c_k > c_{k+1}$, by the assumption $c_{k-1} > c_k$, we have

$$|M_{k+1}| = |U_{k+1}| = c_k.$$

If $c_k \leq c_{k+1}$, we get

$$M_{k+1} = V_{k+1} = \bar{I}_k \setminus U_{k+1},$$

which implies that

$$|M_{k+1}| = n - k - c_k.$$

So we have verified that in this case (5.8) and (5.9) are also valid for M_{k+1}.

Case 2: $c_{k-2} > c_{k-1} \leq c_k$. In this case, we have

$$M_k = V_k = \{m \in \bar{I}_{k-1} \mid j_{2p+1} < m < j_{2p} \text{ for some } p \geq 0\}. \tag{5.10}$$

Since $i_k \in M_k$, there exists $q \geq 0$ such that $j_{2q+1} < i_k < j_{2q}$. It follows that

$$J_k = \{i_k, j_{2q+1}, \ldots, j_l\},$$

and hence

$$V_{k+1} = \{m \in \bar{I}_k \mid i_k < m < j_0 \text{ or } j_{2p+2} < m < j_{2p+1} \text{ for some } p \geq q\}.$$

Consequently,

$$
\begin{aligned}
V_{k+1} =& \{m \in \bar{I}_{k-1} \mid j_{2p+2} < m < j_{2p+1} \text{ for some } p \geq 0\} \\
&+ \{m \in \bar{I}_k \mid m > i_k,\ j_{2p+1} < m < j_{2p} \text{ for some } p \leq q\}.
\end{aligned}
\tag{5.11}
$$

Notice that the first subset on the right hand side of (5.11) is exactly U_k as defined by (5.4). To compute the cardinality of the second subset on the right hand side of (5.11), we observe that $j_{2q+1} < i_k < j_{2q}$. Hence we have

$$
\begin{aligned}
&\{m \in \bar{I}_k \mid m > i_k,\ j_{2p+1} < m < j_{2p} \text{ for some } p \leq q\} \\
&\quad = \{m \in \bar{I}_k \mid m > i_k,\ j_{2p+1} < m < j_{2p} \text{ for some } p \geq 0\} \\
&\quad = \{m \in M_k \mid m > i_k\}.
\end{aligned}
$$

So we obtain that

$$
|V_{k+1}| = |U_k| + |\{m \in M_k \mid m > i_k\}|. \tag{5.12}
$$

Using the hypothesis and (5.10), we find that

$$
|V_k| = |M_k| = n - k + 1 - c_{k-1}, \tag{5.13}
$$

so that

$$
|U_k| = |\bar{I}_{k-1} \setminus V_k| = c_{k-1}. \tag{5.14}
$$

Since $c_{k-1} \leq c_k$, i_k is the $(n-k+1-c_k)$-th smallest element in M_k, which implies that

$$
|\{m \in M_k \mid m \leq i_k\}| = n - k + 1 - c_k. \tag{5.15}
$$

From (5.13) and (5.15) we obtain that

$$
\begin{aligned}
&|\{m \in M_k \mid m > i_k\}| \\
&\quad = |M_k| - |\{m \in M_k \mid m \leq i_k\}| \\
&\quad = (n - k + 1 - c_{k-1}) - (n - k + 1 - c_k) \\
&\quad = c_k - c_{k-1}.
\end{aligned}
\tag{5.16}
$$

Substituting (5.14) and (5.16) into (5.11), we get $|V_{k+1}| = c_k$, and hence $|U_{k+1}| = n - k - c_k$.

If $c_k > c_{k+1}$, by the assumption $c_{k-1} \leq c_k$, we have $|M_{k+1}| = |V_{k+1}| = c_k$. If $\sigma_k < \sigma_{k+1}$, we get $|M_{k+1}| = |U_{k+1}| = n - k - c_k$. This proves that in this case (5.8) and (5.9) hold for M_{k+1}.

For the other two cases, $c_{k-2} \leq c_{k-1} \leq c_k$ and $c_{k-2} \leq c_{k-1} > c_k$, $|M_{k+1}|$ can be determined by the same argument. The details are omitted. Thus we have shown that (5.8) and (5.9) hold for $k+1$. Hence (5.8) and (5.9) hold for $2 \leq k \leq n$. This completes the proof. ∎

We now have shown that Φ is well-defined. To give a combinatorial proof of Theorem 5.1, we also need the following property.

Lemma 5.3 *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of $[n]$ and $\mathrm{code}(\sigma) = (c_1, c_2, \ldots, c_n)$. If $c_{n-1} = 1$, then the root of $\Phi(\sigma)$ is of even degree. If $c_{n-1} = 0$, then the root of $\Phi(\sigma)$ is of odd degree.*

Proof. Observe that for any rooted tree, there is an odd number of vertices of even degree. Clearly, for a permutation σ on $[n]$, c_{n-1} equals to 0 or 1. It is easily seen that $c_{n-1} = 1$ is equivalent to $\sigma_{n-1} > \sigma_n$ and $c_{n-1} = 0$ is equivalent to $\sigma_{n-1} < \sigma_n$. To prove the lemma, we proceed to show that there are an odd number of non-rooted vertices of even degree in $\Phi(\sigma)$ if $\sigma_{n-1} < \sigma_n$, whereas there are an even number of non-rooted vertices of even degree if $\sigma_{n-1} > \sigma_n$.

Recall that an index $2 \le k \le n-1$ is called a valley of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ if $\sigma_{k-1} > \sigma_k < \sigma_{k+1}$. It is clear that i_1 is a leaf of $\Phi(\sigma)$. Moreover, by the construction of Φ, for $2 \le k \le n$, i_k is a vertex of even degree if and only if $i_k \in V_k$. Also by the construction of Φ, it is easily seen that $i_k \in V_k$ if and only if $\sigma_{k-2} < \sigma_{k-1} > \sigma_k$ or $\sigma_{k-2} > \sigma_{k-1} < \sigma_k$. Hence, for $2 \le k \le n$, i_k is a vertex of even degree if and only if $k-1$ is either an exterior peak or a valley. From the above argument, we also see that i_1 does not correspond to any exterior peak or any valley.

We now consider the number of exterior peaks and the number of valleys in σ. Since $\sigma_0 = 0$, the elements of σ go up from σ_0, then go down to certain position, and go up, and so on. In other words, σ begins with an exterior peak, then the valleys and peaks occur alternately. If $\sigma_{n-1} < \sigma_n$, then σ ends up with a valley. Therefore, the number of exterior peaks equals the number of valleys in σ. This implies that the total number of exterior peaks and valleys is even. Since i_1 is a leaf of $\Phi(\sigma)$, we see that there are an odd number of non-rooted vertices in $\Phi(\sigma)$ that are of even degree. Hence the degree of 0 must be odd.

When $\sigma_{n-1} > \sigma_n$, σ ends with a peak. In this case, the number of exterior peaks of σ exceeds the number of valleys of σ by one, so that the total number of exterior peaks and valleys is odd. Thus $\Phi(\sigma)$ has an even number of non-rooted vertices of even degree, since i_1 is a leaf. It follows that the degree of 0 is even, and hence the proof is complete. ∎

We are now ready to finish the combinatorial proof of Theorem 5.1.

Combinatorial Proof of Theorem 5.1. We have shown that Φ is well-defined. To show that Φ is a bijection, we construct the inverse map Ψ of Φ. Let T be an increasing tree on $[n]$. Start with T, we construct a sequence $(c_1, c_2, \ldots, c_n)$. Let σ be the permutation on $[n]$ such that $\mathrm{code}(\sigma) = (c_1, c_2, \ldots, c_n)$. Then we define $\Psi(T)$ to be σ.

First, let F_n be the forest obtained from T by deleting its root 0. Then from F_n, we construct a sequence of forests $F_{n-1}, \ldots, F_1$. For $k = n, n-1, \ldots, 2$, F_{k-1} is obtained by deleting a vertex from F_k. More precisely, for $k = n, n-1, \ldots, 2$, let i_k be the largest root of F_k, and let F_{k-1} be the forest obtained from F_k by deleting i_k. For $k = 1$, let i_1 be the largest root of F_1. For $1 \le k \le n$, let I_k denote the set of vertices in F_k and let J_k denote the set of roots in F_k. As before, let $\bar{I}_k$ denote the complement of I_k in

$[n]$. Given I_k and J_k, assume that U_k and V_k are defined the same as in (5.4) and (5.5), namely,

$$U_k = \{m \in \bar{I}_{k-1} \mid j_{2p+2} < m < j_{2p+1} \text{ for some } p \geq 0\},$$
$$V_k = \{m \in \bar{I}_{k-1} \mid j_{2p+1} < m < j_{2p} \text{ for some } p \geq 0\},$$

where $j_1, j_2, \ldots, j_l$ are the elements of J_{k-1} listed in decreasing order and $j_0 = n+1$, $j_{l+1} = 0$. Note that $i_k \in \bar{I}_{k-1}$ and $\bar{I}_{k-1}$ is the disjoint union of U_k and V_k. If $i_k \in U_k$, we set $M_k = U_k$. If $i_k \in V_k$, we set $M_k = V_k$.

Based on i_k and M_k, we can determine c_k for $1 \leq k \leq n$. For $k = n$, it is easily seen that $|M_n| = 1$. We set $c_n = 0$. For $k = n-1$, we set

$$c_{n-1} = \begin{cases} 1, & \text{if the degree of the root 0 in } T \text{ is even,} \\ 0, & \text{if the degree of the root 0 in } T \text{ is odd.} \end{cases} \tag{5.17}$$

Moreover, for $k = n-2, n-3, \ldots, 1$, we set

$$c_k = \begin{cases} |M_{k+1}|, & \text{if } M_{k+2} = U_{k+2} \text{ and } c_{k+1} > c_{k+2}, \\ n-k-|M_{k+1}|, & \text{if } M_{k+2} = U_{k+2} \text{ and } c_{k+1} \leq c_{k+2}, \\ n-k-|M_{k+1}|, & \text{if } M_{k+2} = V_{k+2} \text{ and } c_{k+1} > c_{k+2}, \\ |M_{k+1}|, & \text{if } M_{k+2} = V_{k+2} \text{ and } c_{k+1} \leq c_{k+2}. \end{cases} \tag{5.18}$$

In this way, we obtain $(c_1, c_2, \ldots, c_n)$. Next we aim to show that for $1 \leq k \leq n$,

$$0 \leq c_k \leq n-k. \tag{5.19}$$

Since for $2 \leq k \leq n$, $i_k \in M_k$ and $M_k \subseteq \bar{I}_{k-1}$, we have

$$1 \leq |M_k| \leq |\bar{I}_{k-1}|. \tag{5.20}$$

On the other hand, by the definition of I_{k-1}, we find that $|\bar{I}_{k-1}| = n-k+1$. It follows that for $2 \leq k \leq n$,

$$1 \leq |M_k| \leq n-k+1.$$

Clearly, for $1 \leq k \leq n-1$, c_k equals to $|M_{k+1}|$ or $n-k-|M_{k+1}|$. Thus, for $1 \leq k \leq n-1$, we have

$$0 \leq c_k \leq n-k.$$

Note that $c_n = 0$, and so (5.19) is proved.

Let σ be the permutation of $[n]$ with code $(c_1, c_2, \ldots, c_n)$. We define $\Psi(T)$ to be σ. By Lemma 5.2 and Lemma 5.3, it is straightforward to verify that every step of the construction of Ψ is the inverse of the corresponding step of Φ. Hence Φ is a bijection.

It remains to show that Φ maps a permutation of $[n]$ with m exterior peaks to an increasing tree on $[n]$ with $2m+1$ vertices of even degree. Let σ be a permutation on

$[n]$. Recall that in the proof of Lemma 5.3, we see that σ begins with an exterior peak, then the valleys and peaks occur alternately and each peak or valley corresponds to a vertex in $[n]$ of even degree. Suppose that σ has m exterior peaks. We shall show that $\Phi(\sigma)$ has $2m+1$ vertices of even degree.

If $\sigma_{n-1} < \sigma_n$, there are also m valleys in σ. These $2m$ indices correspond to $2m$ vertices in $[n]$ of even degree. As noted in the proof of Lemma 5.3, i_1 does not correspond to any peak or valley of σ. On the other hand, i_1 is a vertex of even degree since i_1 is a leaf of $\Phi(\sigma)$. Hence, there are $2m+1$ vertices in $[n]$ of even degree in $\Phi(\sigma)$. By Lemma 5.3, the degree of 0 is odd. So there are $2m+1$ vertices of even degree in $\Phi(\sigma)$.

If $\sigma_{n-1} > \sigma_n$, there are $m-1$ valleys in σ. These $2m-1$ indices correspond to $2m-1$ vertices in $[n]$ of even degree. Note that i_1 does not correspond to any peak or valley of σ, but i_1 is a vertex in $[n]$ of even degree. Hence there are $2m$ vertices in $[n]$ of even degree in $\Phi(\sigma)$. By Lemma 5.3, the degree of 0 is even. So there are $2m+1$ vertices of even degree in $\Phi(\sigma)$. This completes the proof. ∎

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References

[1] M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, SIAM J. Discrete Math. 23 (2008), 401–406.

[2] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993), 113–129.

[3] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, Sém. Lothar. Combin. 37 (1996) B37a, 21 pp.

[4] D. Dumont and A. Ramamonjisoa, Grammaire de Ramanujan et Arbres de Cayley, Electron. J. Combin. 3 (1996), R17.

[5] D. Foata, Groupes de réarrangements et nombers d'Euler, C. R. Acad. Sci. Paris 275 (1972), 1147–1150.

[6] D. Foata and G.-N. Han, Arbres minmax et polynômes d'André, Adv. in Appl. Math. 27 (2001), 367–389.

[7] D. Foata and M.-P. Schützenberger, Nombres d'Euler et permutations alternantes, in: A survey of combinatorial theory, ed. J.N. Srivastava et al., North-Holland, Amsterdam, 1973, 173–187.

[8] I. Gessel and R.P. Stanley, Stirling polynomials, J. Combin. Theory Ser. A 24 (1978), 24–33.

[9] J. Haglund and M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, European J. Combin. 33 (2012), 477–487.

[10] A.G. Kuznetsov, I.M. Pak and A.E. Postnikov, Increasing trees and alternating permutations, Russian Math. Surveys 49 (1994), 79–114.

[11] S.-M. Ma, Derivative polynomials and enumeration of permutations by number of interior and left peaks, Discrete Math. 312 (2012), 405–412.

[12] The OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.