A Class of Kazhdan-Lusztig $R$-Polynomials and $q$-Fibonacci Numbers

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Abstract

Let $S_n$ denote the symmetric group on $\{1, 2, \ldots, n\}$. For two permutations $u, v \in S_n$ such that $u \leq v$ in the Bruhat order, let $R(u, v)(q)$ and $R(u, v)(q)$ denote the Kazhdan-Lusztig $R$-polynomial and $R$-polynomial, respectively. Let $v_n = 34 \cdots n 12$, and let $\sigma$ be a permutation such that $\sigma \leq v_n$. We obtain a formula for the $R$-polynomials $R_{\sigma, v_n}(q)$ in terms of the $q$-Fibonacci numbers depending on a parameter determined by the reduced expression of $\sigma$. When $\sigma$ is the identity permutation $e$, this reduces to a formula obtained by Pagliacci. In another direction, we obtain a formula for the $R$-polynomial $R_{e, v_n}(q)$, where $v_{n,i} = 34 \cdots i n (i + 1) \cdots (n - 1) 12$. In a more general context, we conjecture that for any two permutations $\sigma, \tau \in S_n$ such that $\sigma \leq \tau \leq v_n$, the $R$-polynomial $R_{\sigma, \tau}(q)$ can be expressed as a product of $q$-Fibonacci numbers multiplied by a power of $q$.

Keywords: Kazhdan-Lusztig $R$-polynomial, $q$-Fibonacci number, symmetric group

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1 Introduction

Let $S_n$ denote the symmetric group on $\{1, 2, \ldots, n\}$. For two permutations $u, v \in S_n$ such that $u \leq v$ in the Bruhat order, let $R(u, v)(q)$ be the Kazhdan-Lusztig $R$-polynomial and $R(u, v)(q)$ be the Kazhdan-Lusztig $R$-polynomial. Let $v_n = 34 \cdots n 12$, and let $\sigma$ be a permutation such that $\sigma \leq v_n$. The main result of this paper is a formula for the $R$-polynomials $R_{\sigma, v_n}(q)$ in terms of the $q$-Fibonacci numbers depending on a parameter determined by the reduced expression of $\sigma$. When $\sigma$ is the identity permutation $e$, a formula for the $R$-polynomials has been given by Pagliacci [6, Theorem 4.1].

We also derive a formula for the $R$-polynomials $R_{e, v_{n,i}}(q)$, where $v_{n,i} = 34 \cdots i n (i + 1) \cdots (n - 1) 12$, which can be viewed as a generalization of Pagliacci’s formula [6, Theorem 4.1] in another direction. We conclude this paper with a conjecture that for any two permutations $\sigma, \tau \in S_n$ such that $\sigma \leq \tau \leq v_n$, the $R$-polynomial $R_{\sigma, \tau}(q)$ can be expressed as a product of $q$-Fibonacci numbers and a power of $q$.

Let us give an overview of some notation and background. For each permutation $\pi$ in $S_n$, it is known that $\pi$ can be expressed as a product of simple transpositions $s_i = (i, i+1)$ subject
to the following braid relations
\[ s_is_j = s_js_i, \quad \text{for } |i - j| > 1; \]
\[ s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad \text{for } 1 \leq i \leq n - 2. \]

An expression \( \omega \) of \( \pi \) is said to be reduced if the number of simple transpositions appearing in \( \omega \) is minimum. The following word property is due to Tits, see Björner and Brenti [1, Theorem 3.3.1].

**Theorem 1.1** (Word Property) Let \( \pi \) be a permutation of \( S_n \), and \( \omega_1 \) and \( \omega_2 \) be two reduced expressions of \( \pi \). Then \( \omega_1 \) and \( \omega_2 \) can be obtained from each other by applying a sequence of braid relations.

Let \( \ell(\pi) \) denote the length of \( \pi \), that is, the number of simple transpositions in a reduced expression of \( \pi \). Write \( D_R(\pi) \) for the set of right descents of \( \pi \), namely,
\[ D_R(\pi) = \{ s_i : 1 \leq i \leq n - 1, \ell(\pi s_i) < \ell(\pi) \}. \]

The exchange condition gives a characterization for the (right) descents of a permutation in terms of reduced expressions, see Humphreys [4, Section 1.7].

**Theorem 1.2** (Exchange Condition) Let \( \pi = s_is_{i_2}\cdots s_{i_k} \) be a reduced expression of \( \pi \). If \( s_i \in D_R(\pi) \), then there exists an index \( i_j \) for which \( \pi s_i = s_i s_{i_2}\cdots s_{i_j} s_{i_2}\cdots s_{i_k} \), where \( s_{i_2} \) means that \( s_{i_2} \) is missing. In particular, \( \pi \) has a reduced expression ending with \( s_i \) if and only if \( s_i \in D_R(\pi) \).

The following subword property serves as a definition of the Bruhat order. For other equivalent definitions of the Bruhat order, see Björner and Brenti [1]. For a reduced expression \( \omega = s_is_{i_2}\cdots s_{i_k} \), we say that \( s_{i_{j_1}} s_{i_{j_2}}\cdots s_{i_{j_m}} \) is a subword of \( \omega \) if \( 1 \leq j_1 < j_2 < \cdots < j_m \leq k \).

**Theorem 1.3** (Subword Property) Let \( u \) and \( v \) be two permutation in \( S_n \). Then \( u \leq v \) in the Bruhat order if and only if every reduced expression of \( v \) has a subword that is a reduced expression of \( u \).

The Bruhat order satisfies the following lifting property, see Björner and Brenti [1, Proposition 2.2.7].

**Theorem 1.4** (Lifting Property) Suppose that \( u \) and \( v \) are two permutations in \( S_n \) such that \( u < v \). For any simple transposition \( s_i \) in \( D_R(v) \setminus D_R(u) \), we have \( u \leq vs_i \) and \( us_i \leq v \).

The Kazhdan-Lusztig \( R \)-polynomials, which were introduced by Kazhdan and Lusztig [5], can be recursively determined by the following properties, see also Humphreys [4, Section 7.5].

**Theorem 1.5** For any \( u, v \in S_n \),

(i) \( R_{u,v}(q) = 0 \), if \( u \not\leq v \);
(ii) \( R_{u,v}(q) = 1 \), if \( u = v \);

(iii) If \( u < v \) and \( s \in D_R(v) \),

\[
R_{u,v}(q) = \begin{cases} 
R_{us,vs}(q), & \text{if } s \in D_R(u); \\
qR_{us,vs}(q) + (q-1)R_{u,vs}(q), & \text{if } s \notin D_R(u).
\end{cases}
\]

While \( R \)-polynomials may contain negative coefficients, a variant of the \( R \)-polynomials introduced by Dyer [3], which has been called the \( \bar{R} \)-polynomials, has only nonnegative coefficients. For an alternative definition of the \( \bar{R} \)-polynomials for the symmetric group, see Brenti [2]. The following two theorems are due to Dyer [3], see also Brenti [2].

**Theorem 1.6** Let \( u, v \in S_n \) with \( u \leq v \). Then, for \( s \in D_R(v) \),

\[
\bar{R}_{u,v}(q) = \begin{cases} 
\bar{R}_{us,vs}(q), & \text{if } s \in D_R(u); \\
\bar{R}_{us,vs}(q) + q\bar{R}_{u,vs}(q), & \text{if } s \notin D_R(u).
\end{cases}
\]

(1.1)

**Theorem 1.7** Let \( u, v \in S_n \) with \( u \leq v \). Then

\[
R_{u,v}(q) = q^{\frac{\ell(v) - \ell(u)}{2}} \bar{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).
\]

Recall that \( v_n = 34 \cdots n \ 12 \) and \( v_{n,i} = 34 \cdots i \ n \ (i + 1) \cdots (n - 1) \ 12 \). We shall use the recurrence relations in Theorem 1.6 to deduce a formula for the \( \bar{R} \)-polynomials \( \bar{R}_{\sigma,v_n}(q) \), from which we also find a formula for the \( \bar{R} \)-polynomial \( \bar{R}_{e,v_{n,i}}(q) \).

## 2 Main result

The main result of this paper is an equation for \( \bar{R}_{\sigma,v_n}(q) \), where \( v_n = 34 \cdots n \ 12 \) and \( \sigma \leq v_n \) in the Bruhat order. Combining this equation with a formula of Pagliacci [6], we obtain an expression of \( \bar{R}_{\sigma,v_n}(q) \) in terms of \( q \)-Fibonacci numbers. To describe our result, we need the following reduced expression of \( v_n \).

**Proposition 2.1** For \( n \geq 3 \),

\[
\Omega_n = s_2s_1s_3s_2 \cdots s_{n-1}s_{n-2}
\]

is a reduced expression of \( v_n \).

Let \( \sigma \) be a permutation of \( S_n \) such that \( \sigma \leq v_n \). By the subword property in Theorem 1.3, \( \sigma \) can be expressed as a reduced subword of \( \Omega_n \). We introduce two statistics of a reduced subword of \( \Omega_n \).

Let \( \omega = s_{i_1}s_{i_2} \cdots s_{i_k} \) be a reduced subword of \( \Omega_n \). Define

\[
D(\omega) = \{1 \leq t < k: i_t - i_{t+1} = 1\}.
\]
We use \( d(\omega) \) to denote the cardinality of \( D(\omega) \), and let
\[
h(\omega) = n - \ell(\omega) + d(\omega). \tag{2.1}
\]
For example, for a reduced subword \( \omega = s_2s_3s_4s_5s_6s_7s_8 \) of \( \Omega_9 \), we have \( D(\omega) = \{3, 5\} \), and thus \( d(\omega) = 2 \) and \( h(\omega) = 4 \). Note that \( h(\omega) \) depends on both \( \omega \) and \( n \). This causes no confusion since the index \( n \) is always clear from the context.

The main result in this paper is the following equation for the \( \tilde{R} \)-polynomials \( \tilde{R}_{\sigma, v_n}(q) \).

**Theorem 2.2** For \( n \geq 3 \), let \( \sigma \) be a permutation in \( S_n \) such that \( \sigma \leq v_n \), and let \( \omega \) be any reduced expression of \( \sigma \) that is a subword of \( \Omega_n \). Then we have
\[
\tilde{R}_{\sigma, v_n}(q) = q^{\ell(\omega) - 2d(\omega)} \tilde{R}_{e, v_{h(\omega)}}(q). \tag{2.2}
\]

Let \( F_n(q) \) be the \( q \)-Fibonacci numbers, that is, \( F_0(q) = F_1(q) = 1 \) and for \( n \geq 2 \),
\[
F_n(q) = F_{n-1}(q) + qF_{n-2}(q).
\]

Pagliacci [6, Theorem 4.1] has shown that
\[
\tilde{R}_{e, v_n}(q) = q^{2n^2 - 2}F_{n-2}(q). \tag{2.3}
\]
As a consequence of Theorem 2.2 and formula (2.3), we obtain an expression of \( \tilde{R}_{\sigma, v_n}(q) \) in terms of \( q \)-Fibonacci numbers.

**Corollary 2.3** For \( n \geq 3 \), let \( \sigma \) be a permutation in \( S_n \) such that \( \sigma \leq v_n \), and let \( \omega \) be any reduced expression of \( \sigma \) that is a subword of \( \Omega_n \). Then we have
\[
\tilde{R}_{\sigma, v_n}(q) = q^{2n^2 - 2d(\sigma) - 4}F_{h(\omega) - 2}(q). \tag{2.4}
\]

To give an inductive proof of Theorem 2.2, we need three lemmas. Assume that \( \omega \) is a reduced subword of \( \Omega_n \). When \( \omega s_{n-1} \leq \Omega_n \), the first two lemmas are concerned with the existence of a reduced expression \( \omega' \) of \( \omega s_{n-1} \) such that \( d(\omega') = d(\omega) \). When \( \omega s_{n-1} \not\leq \Omega_n \), the third lemma shows that \( h(\omega) = 2 \).

**Lemma 2.4** Let \( \omega \) be a reduced subword of \( \Omega_n \). If \( \omega s_{n-1} \leq \Omega_n \) and \( s_{n-1} \in D_R(\omega) \), then there exists a reduced expression \( \omega' \) of \( \omega s_{n-1} \) such that \( \omega' \) is a subword of \( \Omega_n \) and \( d(\omega') = d(\omega) \).

**Proof.** We use induction on \( n \). It is easy to check that the lemma holds for \( n \leq 3 \). Assume that \( n > 3 \) and the assertion holds for \( n - 1 \). We now consider the case for \( n \). By definition, we have \( \Omega_n = \Omega_{n-1}s_{n-1}s_{n-2} \). Since \( \omega \) is a subword of \( \Omega_n \), we can write \( \omega = \omega_1\omega_2 \), where \( \omega_1 \) is a subword of \( \Omega_{n-1} \) and \( \omega_2 \) is a subword of \( s_{n-1}s_{n-2} \). Because \( s_{n-1} \in D_R(\omega) \), we have the following two cases.

Case 1: \( \omega = \omega_1s_{n-1} \). Set \( \omega' = \omega_1 \). Clearly, \( \omega' \) is a reduced expression of \( \omega s_{n-1} \). Moreover, it is easy to check that \( D(\omega) = D(\omega') \), and thus \( d(\omega') = d(\omega) \), that is, \( \omega' \) is a desired reduced expression of \( \omega s_{n-1} \).

Case 2: \( \omega = \omega_1s_{n-1}s_{n-2} \). Since \( s_{n-1} \in D_R(\omega) \), by Theorem 1.2, there exists a reduced expression of \( \omega \) ending with \( s_{n-1} \). Hence the word property in Theorem 1.1 ensures the existence
of a reduced expression of $\omega_1$ ending with $s_{n-2}$. This implies that $s_{n-2}$ belongs to $D_R(\omega_1)$.

By the induction hypothesis, there exists a reduced expression $\omega'_1$ of $\omega_1s_{n-2}$ such that $\omega'_1$ is a subword of $\Omega_{n-1}$ and $d(\omega'_1) = d(\omega_1)$.

Set 
$$\omega' = \omega'_1s_{n-1}s_{n-2}.$$ 

We deduce that $\omega'$ is a desired reduced subword. Since 
$$\omega' = \omega'_1s_{n-1}s_{n-2} = \omega_1s_{n-2}s_{n-1}s_{n-2} = \omega_1s_{n-1}s_{n-2}s_{n-1} = \omega s_{n-1},$$ 
we see that $\omega'$ is an expression of $\omega s_{n-1}$. On the other hand, since $\omega'$ consists of $\ell(\omega'_1) + 2$ simple transpositions and 
$$\ell(\omega'_1) + 2 = \ell(\omega_1) + 1 = \ell(\omega) - 1 = \ell(\omega s_{n-1}),$$ 
we conclude that $\omega'$ is a reduced expression of $\omega s_{n-1}$. By the construction of $\omega'$, we have 
$$d(\omega') = d(\omega'_1) + 1 = d(\omega_1) + 1 = d(\omega).$$ 

This completes the proof. 

The next lemma deals with the case $s_{n-1} \notin D_R(\omega)$.

**Lemma 2.5** Let $\omega$ be a reduced subword of $\Omega_n$. If $\omega s_{n-1} \leq \Omega_n$ and $s_{n-1} \notin D_R(\omega)$, then there exists a reduced expression $\omega'$ of $\omega s_{n-1}$ such that $\omega'$ is a subword of $\Omega_n$ and $d(\omega') = d(\omega)$.

**Proof.** We use induction on $n$. It is easily checked that the lemma holds for $n \leq 3$. Assume that $n > 3$ and the assertion holds for $n - 1$. We now consider the case for $n$. Let $\omega = \omega_1\omega_2$, where $\omega_1$ is a subword of $\Omega_{n-1}$ and $\omega_2$ is a subword of $s_{n-1}s_{n-2}$. Since $s_{n-1} \notin D_R(\omega)$, we have the following three cases.

Case 1: $\omega = \omega_1$. Set $\omega' = \omega_1s_{n-1}$. It is easily seen that $\omega'$ is a desired reduced expression.

Case 2: $\omega = \omega_1s_{n-2}$. We claim that $\omega = \omega_1s_{n-2} \leq \Omega_{n-1}$. Note that $s_{n-1}$ does not appear in $\omega_1$. Since $\omega s_{n-1} = \omega_1s_{n-2}s_{n-1}$, by Theorem 1.1, there does not exist any reduced expression of $\omega s_{n-1}$ ending with $s_{n-2}$. This implies that $s_{n-2}$ does not belong to $D_R(\omega s_{n-1})$. Thus, by the lifting property in Theorem 1.4, we deduce that 
$$\omega s_{n-1} \leq \Omega_{n}s_{n-2} = \Omega_{n-1}s_{n-1}.$$ 

This implies that $\omega = \omega_1s_{n-2} \leq \Omega_{n-1}$, as claimed.

Since $\omega = \omega_1s_{n-2}$ is reduced, we see that $s_{n-2} \notin D_R(\omega_1)$. By the induction hypothesis, there exists a reduced expression $\omega'_1$ of $\omega_1s_{n-2}$ such that $\omega'_1$ is a subword of $\Omega_{n-1}$ and $d(\omega'_1) = d(\omega_1)$. Set $\omega' = \omega'_1s_{n-1}$. We find that $\omega'$ is a reduced expression of $\omega s_{n-1}$ such that $d(\omega') = d(\omega)$.

Case 3: $\omega = \omega_1s_{n-1}s_{n-2}$. We claim that $s_{n-2} \notin D_R(\omega_1)$. Suppose to the contrary that $s_{n-2} \in D_R(\omega_1)$. By Theorem 1.2, there exists a reduced expression of $\omega_1$ ending with $s_{n-2}$. Write $\omega_1 = \mu s_{n-2}$, where $\mu$ is a reduced expression. Then we get 
$$\omega = \mu s_{n-2}s_{n-1}s_{n-2} = \mu s_{n-1}s_{n-2}s_{n-1},$$ 

contradicting the assumption that $s_{n-1} \notin D_R(\omega)$. So the claim is proved.
On the other hand, since
\[ \omega_1 s_{n-2} s_{n-1} s_{n-2} = \omega_1 s_{n-1} s_{n-2} s_{n-1} = \omega s_{n-1} \leq \Omega_n, \]
we have \( \omega_1 s_{n-2} \leq \Omega_{n-1} \). It follows from the induction hypothesis that there exists a reduced expression \( \omega' \) of \( \omega s_{n-2} \) such that \( \omega' \) is a subword of \( \Omega_{n-1} \) and \( d(\omega') = d(\omega) \).

Let
\[ \omega' = \omega_1 s_{n-1} s_{n-2}. \]
Since
\[ \omega' = \omega_1 s_{n-1} s_{n-2} = \omega_1 s_{n-2} s_{n-1} s_{n-2} = \omega_1 s_{n-1} s_{n-2} s_{n-1} = \omega s_{n-1}, \]
we deduce that \( \omega' \) is a reduced expression of \( \omega s_{n-1} \). By the construction of \( \omega' \), we obtain that
\[ d(\omega') = d(\omega'_1) + 1 = d(\omega_1) + 1 = d(\omega), \]
as required.

We now come to the third lemma.

**Lemma 2.6** Let \( \omega \) be a reduced subword of \( \Omega_n \). If \( \omega s_{n-1} \not\subseteq \Omega_n \), then we have \( h(\omega) = 2 \).

**Proof.** We proceed by induction on \( n \). It can be verified that the lemma holds for \( n \leq 3 \). Assume that \( n > 3 \) and the assertion holds for \( n - 1 \). Consider the case for \( n \). Write \( \omega = \omega_1 \omega_2 \), where \( \omega_1 \) is a subword of \( \Omega_{n-1} \) and \( \omega_2 \) is a subword of \( s_{n-1} s_{n-2} \). Since \( \omega s_{n-1} \not\subseteq \Omega_n \), we see that \( s_{n-1} \) is not a right descent of \( \omega \). We have the following two cases.

Case 1: \( \omega = \omega_1 s_{n-2} \). Since \( \omega s_{n-1} \not\subseteq \Omega_n \), we have \( \omega = \omega_1 s_{n-2} \not\subseteq \Omega_{n-1} \). Thus, by the induction hypothesis, we get \( h(\omega_1) = 2 \). Noticing that \( \ell(\omega_1) = \ell(\omega) - 1 \) and \( d(\omega_1) = d(\omega) \), we obtain that
\[ h(\omega) = n - \ell(\omega) + d(\omega) = n - 1 - \ell(\omega_1) + d(\omega_1) = h(\omega_1) = 2, \]
as required.

Case 2: \( \omega = \omega_1 s_{n-1} s_{n-2} \). We claim that \( \omega_1 s_{n-2} \not\subseteq \Omega_{n-1} \). Suppose to the contrary that \( \omega_1 s_{n-2} \leq \Omega_{n-1} \). Note that
\[ \omega s_{n-1} = \omega_1 s_{n-1} s_{n-2} s_{n-1} = \omega_1 s_{n-2} s_{n-1} s_{n-2}. \]
This yields \( \omega s_{n-1} \leq \Omega_n \), contradicting the assumption that \( \omega s_{n-1} \not\subseteq \Omega_n \). So the claim is verified.

By the induction hypothesis, we have \( h(\omega_1) = 2 \). Since \( \ell(\omega_1) = \ell(\omega) - 2 \) and \( d(\omega_1) = d(\omega) - 1 \), we find that
\[ h(\omega) = n - \ell(\omega) + d(\omega) = n - 1 - \ell(\omega_1) + d(\omega_1) = h(\omega_1) = 2, \]
as required.

Next we give a proof of Theorem 2.2 based on the above lemmas.

**Proof of Theorem 2.2.** Let
\[ T_n(q) = \overline{R}_{e,v_n}(q), \]
and
\[ g(\omega) = \ell(\omega) - 2d(\omega). \]

(2.5)
Then equation (2.2) can be rewritten as
\[ \tilde{R}_{\sigma, v_n}(q) = q^{g(\omega)}T_{h(\omega)}(q). \] (2.6)

We proceed to prove (2.6) by induction on \( n \). It can be checked that (2.6) holds for \( n \leq 3 \). Assume that \( n > 3 \) and (2.6) holds for \( n - 1 \). For the case for \( n \), let \( \omega = \omega_1 \omega_2 \), where \( \omega_1 \) is a subword of \( \Omega_{n-1} \) and \( \omega_2 \) is a subword of \( s_{n-1}s_{n-2} \). There are four cases.

Case 1: \( \omega = \omega_1 s_{n-2} \). It follows from (1.1) that
\[ \tilde{R}_{\sigma, v_n}(q) = \tilde{R}_{\omega_1 s_{n-2}, \Omega_{n-1} s_{n-1}s_{n-2}}(q) \]
\[ = \tilde{R}_{\omega_1, \Omega_{n-1}s_{n-1}}(q) \]
\[ = \tilde{R}_{\omega_1 s_{n-1}, \Omega_{n-1}}(q) + q\tilde{R}_{\omega_1, \Omega_{n-1}}(q). \] (2.7)

Since \( \omega_1 s_{n-1} \not\leq \Omega_{n-1} \), we see that the first term in (2.7) vanishes. Thus, (2.7) becomes
\[ \tilde{R}_{\sigma, v_n}(q) = q\tilde{R}_{\omega_1, \Omega_{n-1}}(q). \] (2.8)

By the induction hypothesis, we have
\[ \tilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^{g(\omega)}T_{h(\omega)}. \] (2.9)

Since \( d(\omega_1) = d(\omega) \), we get
\[ g(\omega_1) = \ell(\omega_1) - 2d(\omega_1) = \ell(\omega) - 1 - 2d(\omega) = g(\omega) - 1 \] (2.10)
and
\[ h(\omega_1) = n - 1 - \ell(\omega_1) + d(\omega_1) = n - \ell(\omega) + d(\omega) = h(\omega). \] (2.11)

Plugging (2.10) and (2.11) into (2.9), we obtain
\[ \tilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^{g(\omega)}T_{h(\omega)} = q^{g(\omega)-1}T_{h(\omega)}(q), \]
which leads to
\[ \tilde{R}_{\sigma, v_n}(q) = q\tilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^{g(\omega)}T_{h(\omega)}(q). \]

Case 2: \( \omega = \omega_1 s_{n-1} \). By Theorem 1.1, there is no reduced expression of \( \sigma \) that ends with \( s_{n-2} \). This implies that \( s_{n-2} \) is not a right descent of \( \sigma \), so that
\[ \tilde{R}_{\sigma, v_n}(q) = \tilde{R}_{\omega_1 s_{n-1}s_{n-2}, \Omega_{n-1}s_{n-1}}(q) + q\tilde{R}_{\omega_1 s_{n-1}, \Omega_{n-1}s_{n-1}}(q) \]
\[ = \tilde{R}_{\omega_1 s_{n-1}s_{n-2}, \Omega_{n-1}s_{n-1}}(q) + q\tilde{R}_{\omega_1, \Omega_{n-1}}(q). \] (2.12)

We claim that the first term in (2.12) vanishes, or equivalently, \( \omega_1 s_{n-1}s_{n-2} \not\leq \Omega_{n-1}s_{n-1} \). Suppose to the contrary that \( \omega_1 s_{n-1}s_{n-2} \leq \Omega_{n-1}s_{n-1} \). By Theorem 1.3, there exists a subword \( \mu \) of \( \Omega_{n-1}s_{n-1} \) that is a reduced expression of \( \omega_1 s_{n-1} s_{n-2} \). Since \( s_{n-1} \) must appear in \( \mu \), we may write \( \mu \) in the following form
\[ \mu = s_{i_1}s_{i_2} \cdots s_{i_k}s_{n-1}, \]
where \( s_{i_1}s_{i_2} \cdots s_{i_k} \) is a reduced subword of \( \Omega_{n-1} \). By the word property in Theorem 1.1, \( \omega_1 s_{n-1}s_{n-2} \) can be obtained from \( \mu \) by applying the braid relations. However, this is impossible
since any simple transposition \( s_{n-2} \) appearing in \( \mu \) cannot be moved to the last position by applying the braid relations. So the claim is proved, and hence (2.12) becomes

\[
\tilde{R}_{\sigma, v_n}(q) = q\tilde{R}_{\omega_1, \Omega_{n-1}}(q).
\]

It is easily seen that

\[
g(\omega_1) = g(\omega) - 1 \quad \text{and} \quad h(\omega_1) = h(\omega).
\]

By the induction hypothesis, we deduce that

\[
\tilde{R}_{\sigma, v_n}(q) = q\tilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^g(\omega_1) + 1 T_{h(\omega_1)}(q) = q^g(\omega) T_{h(\omega)}(q).
\]

Case 3: \( \omega = \omega_1 s_{n-1} s_{n-2} \). It is clear from (1.1) that

\[
\tilde{R}_{\sigma, v_n}(q) = \tilde{R}_{\omega_1 s_{n-1}, \Omega_{n-2}}(q) = \tilde{R}_{\omega_1, \Omega_{n-1}}(q).
\]  \hspace{1cm} (2.13)

Noting that \( d(\omega) = d(\omega_1) + 1 \), we obtain

\[
g(\omega_1) = \ell(\omega_1) - 2d(\omega_1) = \ell(\omega_1) + 2 - 2d(\omega) = \ell(\omega) - 2d(\omega) = g(\omega)
\]

and

\[
h(\omega_1) = n - 1 - \ell(\omega_1) + d(\omega_1) = n - 2 - \ell(\omega_1) + d(\omega) = n - \ell(\omega) + d(\omega) = h(\omega).
\]

Thus, by (2.13) and the induction hypothesis, we find that

\[
\tilde{R}_{\sigma, v_n}(q) = \tilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^g(\omega_1) T_{h(\omega_1)}(q) = q^g(\omega) T_{h(\omega)}(q).
\]

Case 4: \( \omega = \omega_1 \). Here are two subcases.

Subcase 1: \( s_{n-2} \in D_R(\omega_1) \). By (1.1), we deduce that

\[
\tilde{R}_{\sigma, v_n}(q) = \tilde{R}_{\omega_1 s_{n-2}, \Omega_{n-1}}(q) = q\tilde{R}_{\omega_1 s_{n-2}, \Omega_{n-1}}(q).
\]  \hspace{1cm} (2.14)

By Lemma 2.4, there exists a reduced expression \( \omega_1' \) of \( \omega_1 s_{n-2} \) such that \( \omega_1' \) is a subword of \( \Omega_{n-1} \) and \( d(\omega_1') = d(\omega_1) \). Consequently,

\[
g(\omega_1') = \ell(\omega_1') - 2d(\omega_1') = \ell(\omega_1) - 1 - 2d(\omega_1) = g(\omega_1) - 1
\]

and

\[
h(\omega_1') = n - 1 - \ell(\omega_1') + d(\omega_1') = n - \ell(\omega_1) + d(\omega_1) = h(\omega_1) + 1.
\]

By the induction hypothesis, we obtain that

\[
\tilde{R}_{\omega_1 s_{n-2}, \Omega_{n-1}}(q) = \tilde{R}_{\omega_1', \Omega_{n-1}}(q) = q^g(\omega_1') T_{h(\omega_1')}(q) = q^g(\omega_1) T_{h(\omega_1)}(q) + 1(q).
\]  \hspace{1cm} (2.15)

But \( h(\omega) = h(\omega_1) + 1 \), substituting (2.15) into (2.14) gives

\[
\tilde{R}_{\sigma, v_n}(q) = q\tilde{R}_{\omega_1 s_{n-2}, \Omega_{n-1}}(q) = q^g(\omega) T_{h(\omega)}(q).
\]
Subcase 2: \( s_{n-2} \notin D_R(\omega_1) \). By (1.1), we see that
\[
\widetilde{R}_{\sigma, v_n}(q) = \widetilde{R}_{\omega_1 s_{n-2}, \Omega_n s_{n-1}(q)} + q \widetilde{R}_{\omega_1, \Omega_n s_{n-1}(q)}
= q \widetilde{R}_{\omega_1 s_{n-2}, \Omega_n} + q^2 \widetilde{R}_{\omega_1, \Omega_n}(q)
\] (2.16)

By the induction hypothesis, the second term in (2.16) equals
\[
q^2 \widetilde{R}_{\omega_1, \Omega_n}(q) = q^{{g(\omega_2)}} + 2 T_{h(\omega_1)}(q) = q^{{g(\omega_2)}} + 2 T_{h(\omega_1)}(q).
\] (2.17)

It remains to compute the first term in (2.16). To this end, we have the following two cases.

Subcase 2a: \( \omega_1 s_{n-2} \leq \Omega_{n-1} \). By Lemma 2.5, there exists a reduced expression \( \omega'_1 \) of \( \omega_1 s_{n-2} \) such that \( \omega'_1 \) is a subword of \( \Omega_{n-1} \) and \( d(\omega'_1) = d(\omega'_1) \). Hence
\[
g(\omega'_1) = \ell(\omega'_1) - 2d(\omega'_1) = \ell(\omega_1) + 1 - 2d(\omega_1) = g(\omega_1) + 1 = g(\omega) + 1
\]
and
\[
h(\omega'_1) = n - 1 - \ell(\omega'_1) + d(\omega'_1) = n - 2 - \ell(\omega_1) + d(\omega_1) = h(\omega_1) - 1 = h(\omega) - 2.
\]

By the induction hypothesis, we obtain that
\[
\widetilde{R}_{\omega_1 s_{n-2}, \Omega_n}(q) = q^{{g(\omega'_1)}} T_{h(\omega'_1)}(q) = q^{{g(\omega)}} T_{h(\omega)}(q).
\] (2.18)

Putting (2.17) and (2.18) into (2.16), we deduce that
\[
\widetilde{R}_{\sigma, v_n}(q) = q \widetilde{R}_{\omega_1 s_{n-2}, \Omega_n} + q^2 \widetilde{R}_{\omega_1, \Omega_n}(q)
= q^{{g(\omega)}} + 2 T_{h(\omega)}(q) + q^{{g(\omega)}} + 2 T_{h(\omega)}(q)
= q^{{g(\omega)}} (q^2 T_{h(\omega)}(q) + q^2 T_{h(\omega)}(q)).
\] (2.19)

In view of the following relation due to Pagliacci [6]
\[
T_n(q) = q^2 T_{n-2}(q) + q^2 T_{n-1}(q),
\]
(2.19) can be rewritten as
\[
\widetilde{R}_{\sigma, v_n}(q) = q^{{g(\omega)}} T_{h(\omega)}(q).
\]

Subcase 2b: \( \omega_1 s_{n-2} \not\leq \Omega_{n-1} \). In this case, we have
\[
\widetilde{R}_{\omega_1 s_{n-2}, \Omega_n}(q) = 0.
\] (2.20)

By Lemma 2.6, we find that \( h(\omega_1) = 2 \). Thus (2.17) reduces to
\[
q^2 \widetilde{R}_{\omega_1, \Omega_n}(q) = q^{{g(\omega)}} + 2 T_2(q) = q^{{g(\omega)}} + 2.
\] (2.21)

Putting (2.20) and (2.21) into (2.16), we get
\[
\widetilde{R}_{\sigma, v_n}(q) = q^{{g(\omega)}} + 2.
\] (2.22)

Since \( T_3(q) = q^2 \) and \( h(\omega) = h(\omega_1) + 1 = 3 \), it follows from (2.22) that
\[
\widetilde{R}_{\sigma, v_n}(q) = q^{{g(\omega)}} T_3(q) = q^{{g(\omega)}} T_{h(\omega)}(q),
\]
and hence the proof is complete.
For $2 \leq i \leq n - 1$, let

$$v_{n,i} = \begin{cases} 
  n34 \cdots (n - 1)12, & \text{if } i = 2; \\
  34 \cdots i(n + 1) \cdots (n - 1)12, & \text{if } 3 \leq i \leq n - 1.
\end{cases}$$

We obtain the following formula for $\tilde{R}_{e,v_{n,i}}(q)$, which reduces to formula (2.3) due to Pagliacci in the case $i = n - 1$.

**Theorem 2.7** Let $n \geq 3$ and $2 \leq i \leq n - 1$. Then we have

$$\tilde{R}_{e,v_{n,i}}(q) = \sum_{k=0}^{n-i-1} q^{3n-i-2k-5} \binom{n-i-1}{k} F_{n-k-2}(q^{-2}).$$

(2.23)

**Proof.** Recall that

$$T_n(q) = \tilde{R}_{e,v_0}(q) = q^{2n-4}F_{n-2}(q^{-2}),$$

see (2.3). Hence (2.23) can be rewritten as

$$\tilde{R}_{e,v_{n,i}}(q) = \sum_{k=0}^{n-i-1} q^{n-i-1} \binom{n-i-1}{k} T_{n-k}(q).$$

(2.24)

Note that $\Omega_n s_n s_{n-4} \cdots s_i - 1$ is a reduced expression of the permutation $v_{n,i}$. By the defining relation (1.1) of $\tilde{R}$-polynomials, we obtain that

$$\tilde{R}_{e,v_{n,i}}(q) = \tilde{R}_{e,\Omega_n s_n s_{n-4} \cdots s_i - 1}(q)$$

$$= \tilde{R}_{e,\Omega_n s_n s_{n-4} \cdots s_i - 1}(q) + q \tilde{R}_{e,\Omega_n s_n s_{n-4} \cdots s_i - 1}(q)$$

$$= \left( \tilde{R}_{s_{i-1} s_i, \Omega_n s_n s_{n-4} \cdots s_i - 1}(q) + q \tilde{R}_{s_{i-1} s_i, \Omega_n s_n s_{n-4} \cdots s_i - 1}(q) \right)$$

$$+ q \left( \tilde{R}_{s_i, \Omega_n s_n s_{n-4} \cdots s_i - 1}(q) + q \tilde{R}_{e, \Omega_n s_n s_{n-4} \cdots s_i - 1}(q) \right)$$

$$= \cdots$$

$$= \sum_{i-1 \leq i_1 < \cdots < i_k \leq n-3} q^{n-i-1-k} \tilde{R}_{s_{i_1} \cdots s_{i_k}, \Omega_n}(q).$$

(2.25)

Observe that $s_{i_1} \cdots s_{i_k}$ is a reduced subword of $\Omega_n$ with $d(s_{i_1} \cdots s_{i_k}) = 0$. By Corollary 2.3, we find that

$$\tilde{R}_{s_{i_1} \cdots s_{i_k}, \Omega_n}(q) = q^k T_{n-k}(q).$$

(2.26)

Substituting (2.26) into (2.25), we get

$$\tilde{R}_{e,v_{n,i}}(q) = \sum_{i-1 \leq i_1 < \cdots < i_k \leq n-3} q^{n-i-1-k} \tilde{R}_{s_{i_1} \cdots s_{i_k}, \Omega_n}(q)$$

$$= \sum_{i-1 \leq i_1 < \cdots < i_k \leq n-3} q^{n-i-1} T_{n-k}(q)$$

$$= \sum_{k=0}^{n-i-1} q^{n-i-1} \binom{n-i-1}{k} T_{n-k}(q),$$

as required.

We conclude this paper with the following conjecture, which has been verified for $n \leq 9$. 
Conjecture 2.8 For $n \geq 2$ and $e \leq \sigma_1 \leq \sigma_2 \leq \Omega_n$, we have

$$\tilde{R}_{\sigma_1,\sigma_2}(q) = q^{g(\sigma_1,\sigma_2)} \prod_{i=1}^{k} F_{h_i(\sigma_1,\sigma_2)}(q^{-2}),$$

(2.27)

where $k$, $g(\sigma_1,\sigma_2)$ and $h_i(\sigma_1,\sigma_2)$ are integers depending on $\sigma_1$ and $\sigma_2$.

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References


