Factors of the Gaussian Coefficients

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Abstract. There are many reasons for the Gaussian coefficients (or the q-binomial coefficients) to be polynomials. We show that the divisibility directly follows from the basic properties of cyclotomic polynomials. Writing the Gaussian coefficient with numerator n and denominator k in a form such that $2k \leq n$ by the symmetry in k, we find the coefficient has exactly k factors if one carries out the divisibility computation without further factorization (or as done by Maple). We further deduce the fact that the Gaussian coefficents have no multiple roots. For the n-th q-Catalan number, we show that it has exactly $n-1$ factors after the divisibility computation.

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The *q-multinomial coefficients* are defined by

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix} = \frac{(q;q)_n}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_r}},$$

where $n_1 + n_2 + \cdots + n_r = n$ and

$$(q;q)_m = (1-q)(1-q^2)\cdots(1-q^m).$$

For $r=2$, they are usually called the *q-binomial coefficients* or the *Gaussian coefficients* and are written as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{(1-q^{n-k+1})(1-q^{n-k+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)}. \qquad (1)$$

The factorization of q-binomial coefficients plays an important role in the study of divisibility properties of generalized Euler numbers [1, 2, 8]. While there are many reasons for the Gaussian coefficients to be polynomials, we find that there is still more to be said. From the point of view of cyclotomic polynomials, the divisibility for the Gaussian coefficients turns out to be a rather natural fact. The only point that seems to be neglected is the condition $2k \le n$ under which the Gaussian coefficients have k factors after the straightforward computation (as done by Maple).

Let $\Phi_n(x)$ be the n-th cyclotomic polynomial defined by

$$\Phi_n(x) = \prod_{\substack{1\le j\le n\\ \gcd(j,n)=1}} (x-\zeta_n^j),$$

where $\zeta_n = e^{2\pi\sqrt{-1}/n}$ is the n-th root of unity and $\gcd(j, n)$ denotes the great common divisor of j and n. It is well-known that $\Phi_n(x) \in \mathbb{Z}[x]$ is the irreducible polynomial for ζ_n (see, for example, [9]). The polynomial $x^n - 1$ has the following factorization into irreducible polynomials over $\mathbb{Z}$:

$$x^n - 1 = \prod_{j\,|\,n} \Phi_j(x). \tag{2}$$

We give the following factorization of q-multinomial coefficients, where the notation $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Lemma 1 *The q-multinomial cofficients $\begin{bmatrix} n \\ n_1,n_2,\ldots,n_r\end{bmatrix}$ are polynomials in q and can be factored as*

$$\prod_{i=1}^{n} \left(\Phi_i(q)\right)^{\lfloor \frac{n}{i}\rfloor - \lfloor \frac{n_1}{i}\rfloor - \lfloor \frac{n_2}{i}\rfloor - \cdots - \lfloor \frac{n_r}{i}\rfloor}. \tag{3}$$

Proof. By Equation (2), we have

$$(-1)^m (q,q)_m = \prod_{j=1}^{m}\prod_{i\,|\,j}\Phi_i(q) = \prod_{i=1}^{m}\Phi_i^{\lfloor \frac{m}{i}\rfloor}(q) = \prod_{i=1}^{\infty}\Phi_i^{\lfloor \frac{m}{i}\rfloor}(q).$$

Therefore,

$$\begin{aligned}\begin{bmatrix} n \\ n_1,n_2,\ldots,n_r\end{bmatrix} &= \frac{\prod_{i=1}^{n}\Phi_i^{\lfloor \frac{n}{i}\rfloor}(q)}{\prod_{i=1}^{\infty}\Phi_i^{\lfloor \frac{n_1}{i}\rfloor}(q)\cdot\prod_{i=1}^{\infty}\Phi_i^{\lfloor \frac{n_2}{i}\rfloor}(q)\cdots\prod_{i=1}^{\infty}\Phi_i^{\lfloor \frac{n_r}{i}\rfloor}(q)}\\ &= \prod_{i=1}^{\infty}\left(\Phi_i(q)\right)^{\lfloor \frac{n}{i}\rfloor - \lfloor \frac{n_1}{i}\rfloor - \lfloor \frac{n_2}{i}\rfloor - \cdots - \lfloor \frac{n_r}{i}\rfloor}.\end{aligned}$$

Since $\sum_{j=1}^{r} n_j = n$ and $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a+b \rfloor$, all the power indices in (3) are non-negative, which implies that the q-multinomial coefficients are polynomials in q. ∎

We have the following observation.

Theorem 2 *The Gaussian coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ have no multiple roots.*

Proof. It is sufficient to prove that the factorization of $\begin{bmatrix} n \\ k \end{bmatrix}$ into irreducible factors contains no repeated factors. Using the following inequality for real numbers a and b

$$\lfloor a \rfloor + \lfloor b \rfloor + 1 \geq \lfloor a+b \rfloor ,$$

and the factorization (3), we obtain

$$\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{k}{i} \right\rfloor - \left\lfloor \frac{n-k}{i} \right\rfloor \leq 1, \quad \text{for} \quad 1 \leq i \leq n.$$

Since Φ_i are pair-wise relatively prime, it follows that $\begin{bmatrix} n \\ k \end{bmatrix}$ have no multiple roots. ∎

Combining Lemma 1 and Theorem 2, we have

Corollary 3 *The total number of irreducible factors of $\begin{bmatrix} n \\ k \end{bmatrix}$ is given by*

$$\sum_{\ell=1}^{k} \big(\nu(n-\ell+1) - \nu(\ell)\big)$$

where $\nu(m)$ is the number of divisors of m.

The number of irreducible factors of $\begin{bmatrix} n \\ k \end{bmatrix}$ has the following bounds:

Theorem 4 *The Gaussian coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ has at most $n-1$ irreducible factors. It has at least k irreducible factors if $n \geq 2k$.*

Proof. It is obvious that $\Phi_1(q) = q-1$ is not a divisor of $\begin{bmatrix} n \\ k \end{bmatrix}$. Hence, (3) implies that $\begin{bmatrix} n \\ k \end{bmatrix}$ has at most $n-1$ irreducible factors. Assume that $n \geq 2k$ and $n-k+1 \leq i \leq n$. Then we have $2i \geq 2n-n+2 > n$, $i \geq 2k-k+1 = k+1$ and $i \geq n-k+1$. Hence,

$$\left\lfloor \frac{n}{i} \right\rfloor = 1 \quad \text{and} \quad \left\lfloor \frac{k}{i} \right\rfloor = \left\lfloor \frac{n-k}{i} \right\rfloor = 0,$$

which implies that $\Phi_i(q)$ is an irreducible factor of $\begin{bmatrix} n \\ k \end{bmatrix}$. Thus, $\begin{bmatrix} n \\ k \end{bmatrix}$ has at least k irreducible factors: $\Phi_{n-k+1}, \Phi_{n-k+2}, \ldots, \Phi_n$. ∎

Remark. Theorem 4 implies that the Gaussian coefficient can be written as the product of exactly k non-trivial factors if one carries out the divisibility computation without further factorization (or as done by Maple). In fact, we may factorize $\begin{bmatrix} n \\ k \end{bmatrix}$ into k factors by the following procedure. Let $S_i = \{j : j \text{ divides } n-i+1\}, i = 1, \ldots, k$. Then

$$(1-q^{n-k+1})(1-q^{n-k+2})\cdots(1-q^n) = (-1)^k \prod_{i=1}^{k} \prod_{j \in S_i} \Phi_j(q).$$

Similarly,

$$(1-q)(1-q^2)\cdots(1-q^k) = (-1)^k \prod_{i=1}^{k} \prod_{j \in T_i} \Phi_j(q),$$

where $T_i = \{j : j \text{ divides } i\}, i = 1, \ldots, k$. Cancelling the common elements in S_i and T_j, we get subsets R_i of S_i such that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{k} \prod_{j \in R_i} \Phi_j(q).$$

Note that $n-i+1 \in S_i$, but it does not belong to any T_j. It follows that $n-i+1 \in R_i$, which implies that $\prod_{j \in R_i} \Phi_j(q)$ are non-constant polynomials in q.

The irreducible factors of $\begin{bmatrix} n \\ k \end{bmatrix}$ can be characterized as follows, where $\{x\}$ denotes the fractional part of x.

Theorem 5 *$\Phi_i(q)$ is a factor of $\begin{bmatrix} n \\ k \end{bmatrix}$ if and only if $\left\{\frac{k}{i}\right\} > \left\{\frac{n}{i}\right\}$.*

Proof. By definition,

$$\begin{aligned} & \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{k}{i} \right\rfloor - \left\lfloor \frac{n-k}{i} \right\rfloor = 1 \\ \iff & \left\{ \frac{n}{i} \right\} - \left\{ \frac{k}{i} \right\} - \left\{ \frac{n-k}{i} \right\} = -1 \\ \iff & \left\{ \frac{k}{i} \right\} = \left\{ \frac{n}{i} \right\} + 1 - \left\{ \frac{n-k}{i} \right\} > \left\{ \frac{n}{i} \right\}. \end{aligned}$$

∎

Note that $\Phi_1(1) = 0$, and for $n > 1$,

$$\Phi_n(1) = \begin{cases} p, & \text{if } n = p^m \text{ for some prime number } p, \\ 1, & \text{otherwise,} \end{cases}$$

which follows immediately from the construction of $\Phi_n(x)$:

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} \quad \text{and} \quad \Phi_{np}(x) = \begin{cases} \frac{\Phi_n(x^p)}{\Phi_n(x)}, & \text{if } p \nmid n, \\ \Phi_n(x^p), & \text{if } p \mid n, \end{cases}$$

where p is a prime number (see [7]). We obtain Kummer's theorem from Theorem 5.

Corollary 6 (Kummer's Theorem) *The power of prime p dividing $\binom{n}{m}$ is given by the number of integers $j > 0$ for which $\{m/p^j\} > \{n/p^j\}$.*

Remark. Let $[n]! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. From the following factorization

$$[n]! = \prod_{i=2}^{n} (\Phi_i(q))^{\lfloor \frac{n}{i} \rfloor},$$

one obtains the power of a prime p dividing $n!$ by taking $q = 1$ (see [4]):

$$\epsilon_p(n) = \sum_{r \geq 0} \left\lfloor \frac{n}{p^r} \right\rfloor .$$

As a q-generalization of the Catalan numbers, the q-Catalan numbers have been extensively investigated (see [3, 5, 6]). Based on Theorem 4, we derive the following divisibility properties of the q-Catalan numbers.

Corollary 7 *The q-Catalan numbers $\frac{1-q}{1-q^{n+1}} \begin{bmatrix} 2n \\ n \end{bmatrix}$ are polynomials in q and have at least $n - 1$ irreducible factors.*

Proof. Since $\Phi_{n+2}, \Phi_{n+3}, \ldots, \Phi_{2n}$ are irreducible factors of $\begin{bmatrix} 2n \\ n \end{bmatrix}$ and are coprime with $1-q^{n+1}$, they are also irreducible factors of the q-Catalan number provided that it is a polynomial. In fact, for each factor $\Phi_i (i \geq 2)$ of $1-q^{n+1}$, we have $i \mid n+1$, namely, $n = ki - 1$. Therefore,

$$\left\{ \frac{n}{i} \right\} = \frac{i-1}{i} > \left\{ \frac{2n}{i} \right\} = \frac{i-2}{i}.$$

From Theorem 4, it follows that Φ_i is a factor of $\begin{bmatrix} 2n \\ n \end{bmatrix}$. ∎

As a generalization of Theorem 4, we have

Theorem 8 *Let $M = \max\{n_1, n_2, \ldots, n_r\}$. Then $\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix}$ has at least $n - M$ irreducible factors.*

Proof. For any $M+1 \le i \le n$, we have
$$\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n_1}{i} \right\rfloor - \left\lfloor \frac{n_2}{i} \right\rfloor - \cdots - \left\lfloor \frac{n_r}{i} \right\rfloor = \left\lfloor \frac{n}{i} \right\rfloor \ge 1.$$
Thus $\Phi_i(q)$ is an irreducible factor of $\left[{n \atop n_1, n_2, \ldots, n_r}\right]$ by (3). ∎

Except for the following special cases, the q-multinomial coefficient contains more irreducible factors.

Theorem 9 *The q-multinomial coefficients $\left[{n \atop n_1, n_2, \ldots, n_r}\right]$ can be factored into exactly $n - M$ irreducible factors only for the following special cases:*
$$\begin{bmatrix} p \\ 1 \end{bmatrix} (p \text{ is prime}),\ \begin{bmatrix} 4 \\ 2 \end{bmatrix},\ \begin{bmatrix} 5 \\ 2 \end{bmatrix},\ \begin{bmatrix} 7 \\ 3 \end{bmatrix},\ \begin{bmatrix} 8 \\ 4 \end{bmatrix},\ \begin{bmatrix} 11 \\ 5 \end{bmatrix},\ \begin{bmatrix} 3 \\ 1,1,1 \end{bmatrix},\ \begin{bmatrix} 5 \\ 2,2,1 \end{bmatrix}.$$

Proof. Suppose $\left[{n \atop n_1, \ldots, n_r}\right]$ can be factored into exactly $n - M$ irreducible factors. Then
$$\left\lfloor \frac{n}{i} \right\rfloor = \sum_{j=1}^{r} \left\lfloor \frac{n_j}{i} \right\rfloor, \text{ for } 1 \le i \le M, \quad \text{and} \quad \left\lfloor \frac{n}{i} \right\rfloor = 1, \text{ for } M+1 \le i \le n.$$
Therefore $M < n < 2M+2$.

If $n = M+1$, we have $\left[{n \atop 1}\right] = \prod_{\substack{j \mid n \\ j \neq 1}} \Phi_j$. Hence it has only one factor if and only if n is prime. From now on, we may assume that $M+2 \le n \le 2M+1$.

Suppose $r \ge 3$ and $M = n_1 \ge n_2 \ge n_3 \ge \cdots$. If $n_2 + 1 \le M$, then for $i = n_2 + 1$,
$$\left\lfloor \frac{n_2}{i} \right\rfloor + \left\lfloor \frac{n_3}{i} \right\rfloor = 0 < 1 = \left\lfloor \frac{n_2 + n_3}{i} \right\rfloor,$$
which implies that Φ_i is an extra factor $\left[{n \atop n_1, \ldots, n_r}\right]$. Hence we have $n_2 + 1 > M$, which implies that $n_2 = M$ and $n_3 = 1$. Suppose $M = 2^a \cdot b$, where b is an odd integer.

(1) $b = 1$. For $i = 2^{a-1} + 1$, we have $2\left\lfloor \frac{M}{i} \right\rfloor = 2 < 3 = \left\lfloor \frac{2M+1}{i} \right\rfloor$ if $a > 1$. It follows that $a = 0$ or 1, leading to the two cases $\left[{3 \atop 1,1,1}\right]$ and $\left[{5 \atop 2,2,1}\right]$.

(2) $b > 1$. For $i = 2^{a+1}$, we have $2\left\lfloor \frac{M}{i} \right\rfloor = 2\left\lfloor \frac{b}{2} \right\rfloor < b = \left\lfloor \frac{2M+1}{i} \right\rfloor$, which implies that Φ_i is an extra factor.

It remains to consider the case $r = 2$. Suppose that $M = n_1 \ge n_2$. If $n_2 + 1 > M$, then $n_2 = M$. Using a similar argument for the above case $r \ge 3$, we get the special cases $\left[{4 \atop 2}\right]$, $\left[{8 \atop 4}\right]$. Otherwise, we have $n_2 + 1 \le M$ and
$$\left\lfloor \frac{M}{n_2+1} \right\rfloor + \left\lfloor \frac{n_2}{n_2+1} \right\rfloor = \left\lfloor \frac{M+n_2}{n_2+1} \right\rfloor,$$
which implies $(n_2+1) \mid M$. Finally, we are left with two cases.

(1) $M = 2m$. Taking $i = m+1$, we have $\left\lfloor \frac{M}{i} \right\rfloor = 1$ and

$$\left\lfloor \frac{n_2}{i} \right\rfloor = \begin{cases} 1, & \text{if } n_2 = M-1, \\ 0, & \text{if } n_2 = \frac{M}{r} - 1, r \geq 2. \end{cases}$$

Since $\left\lfloor \frac{M+n_2}{i} \right\rfloor \geq 2$ for $n_2 \geq 2$ and $\left\lfloor \frac{M+n_2}{i} \right\rfloor \geq 3$ for $n_2 = M-1$ and $m \geq 4$, we have $n_2 = M-1$ and $m = 2, 3$. So we get the two cases $\left[{7 \atop 3}\right], \left[{11 \atop 5}\right]$.

(2) $M = 2m+1$. Taking $i = m+1$, we get $\left[{5 \atop 2}\right]$ by a similar argument. ∎

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